Complex Geometry and Toric Topology

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1. Moment-angle complexes and manifolds.

\( \mathcal{K} \) an (abstract) simplicial complex on the set \([m] = \{1, \ldots, m\}\).

\( I = \{i_1, \ldots, i_k\} \in \mathcal{K} \) a simplex. Always assume \( \emptyset \in \mathcal{K} \).
Allow \( \{i\} \notin \mathcal{K} \) for some \( i \) (ghost vertices).

Consider the unit polydisc in \( \mathbb{C}^m \),

\[
\mathbb{D}^m = \{ (z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| \leq 1, \quad i = 1, \ldots, m \}.
\]

Given \( I \subset [m] \), set

\[
B_I := \{ (z_1, \ldots, z_m) \in \mathbb{D}^m : |z_j| = 1 \text{ for } j \notin I \}.
\]

Define the moment-angle complex

\[
\mathcal{Z}_\mathcal{K} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m
\]

It is invariant under the coordinatewise action of the standard torus \( \mathbb{T}^m = \{ (z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| = 1, \quad i = 1, \ldots, m \} \) on \( \mathbb{C}^m \).
**Constr 1** (polyhedral product). Given spaces \( W \subset X \) and \( I \subset [m] \), set

\[
(X, W)^I = \{(x_1, \ldots, x_m) \in X^m : x_j \in W \text{ for } j \notin I\} \cong \prod_{i \in I} X \times \prod_{i \notin I} W,
\]

and define the **polyhedral product** of \((X, W)\) as

\[
(X, W)^K = \bigcup_{I \in K} (X, W)^I \subset X^m.
\]

Then \( Z_K = (\mathbb{D}, \mathbb{T})^K \), where \( \mathbb{T} \) is the unit circle.

Another example is the complement of a coordinate subspace arrangement:

\[
U(K) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \ldots, i_k\} \notin K} \{z \in \mathbb{C}^m : z_{i_1} = \ldots = z_{i_k} = 0\},
\]

namely,

\[
U(K) = (\mathbb{C}, \mathbb{C}^\times)^K = \bigcup_{I \in K} \left( \prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \right),
\]

where \( \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \).

Clearly, \( Z_K \subset U(K) \). Moreover, \( Z_K \) is a \( \mathbb{T}^m \)-equivariant deformation retract of \( U(K) \) for every \( K \) [Buchstaber-P].
Prop 1. Assume $|\mathcal{K}| \cong S^{n-1}$ (a sphere triangulation with $m$ vertices). Then $Z_{\mathcal{K}}$ is a closed manifold of dimension $m + n$.

We refer to such $Z_{\mathcal{K}}$ as moment-angle manifolds.

If $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope $P$, then $Z_P = Z_{\mathcal{K}_P}$ embeds in $\mathbb{C}^m$ as a nondegenerate (transverse) intersection of $m - n$ real quadratic hypersurfaces. Therefore, $Z_P$ can be smoothed canonically.

Now assume $\mathcal{K}$ is the underlying complex of a complete simplicial fan $\Sigma$ (a starshaped sphere).
A fan is a finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of strongly convex cones in $\mathbb{R}^n$ such that every face of a cone in $\Sigma$ belongs to $\Sigma$ and the intersection of any two cones in $\Sigma$ is a face of each.

A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is complete if $\sigma_1 \cup \ldots \cup \sigma_s = \mathbb{R}^n$.

Let $\Sigma$ be a simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones generated by $a_1, \ldots, a_m$. Its underlying simplicial complex is

$$K_{\Sigma} = \{I \subset [m]: \{a_i: i \in I\} \text{ spans a cone of } \Sigma\}$$

Note: $\Sigma$ is complete iff $|K_{\Sigma}|$ is a triangulation of $S^{n-1}$. 
Given $\Sigma$ with 1-cones generated by $a_1, \ldots, a_m$, define a map

$$A: \mathbb{R}^m \to \mathbb{R}^n, \quad e_i \mapsto a_i,$$

where $e_1, \ldots, e_m$ is the standard basis of $\mathbb{R}^m$. Set

$$\mathbb{R}^m_+ = \{(y_1, \ldots, y_m) \in \mathbb{R}^m : y_i > 0\},$$

and define

$$R := \exp(\ker A) = \{(y_1, \ldots, y_m) \in \mathbb{R}^m_+ : \prod_{i=1}^m y_i^{\langle a_i, u \rangle} = 1 \text{ for all } u \in \mathbb{R}^n\},$$

$R \subset \mathbb{R}^m_+$ acts on $U(K_\Sigma) \subset \mathbb{C}^m$ by coordinatewise multiplications.

**Thm 1.** Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones, and let $K = K_\Sigma$ be its underlying simplicial complex. Then

(a) the group $R \cong \mathbb{R}^{m-n}$ acts on $U(K)$ freely and properly, so the quotient $U(K)/R$ is a smooth $(m+n)$-dimensional manifold;

(b) $U(K)/R$ is $\mathbb{T}^m$-equivariantly homeomorphic to $Z_K$.

Therefore, $Z_K$ can be smoothed canonically.
2. Complex-analytic structures.

We shall show that the even-dimensional moment-angle manifold $Z_K$ corresponding to a complete simplicial fan admits a structure of a complex manifold. The idea is to replace the action of $\mathbb{R}^{m-n}$ on $U(K)$ (whose quotient is $Z_K$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

**Rem 1.** Complex structures on polytopal moment-angle manifolds $Z_P$ were described by Bosio and Meersseman. They identified $Z_P$ with a class of complex manifolds known as LVM-manifolds (named after López de Medrano, Verjovsky and Meersseman).

Assume $m - n$ is even from now on. We can always achieve this by formally adding an ‘empty’ one-dimensional cone to $\Sigma$; this corresponds to adding a ghost vertex to $\mathcal{K}$, or multiplying $Z_K$ by a circle.

Set $\ell = \frac{m-n}{2}$. 

Constr 2. Choose a linear map $\Psi : \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying the two conditions:

(a) $\text{Re} \circ \Psi : \mathbb{C}^\ell \to \mathbb{R}^m$ is a monomorphism.

(b) $A \circ \text{Re} \circ \Psi = 0$.

The composite map of the top line in the following diagram is zero:

$$
\begin{align*}
\mathbb{C}^\ell & \xrightarrow{\Psi} \mathbb{C}^m & \mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
\downarrow \exp & & \downarrow \exp & & \downarrow \exp \\
(\mathbb{C}^\times)^m & \xrightarrow{|.|} \mathbb{R}^>_m & \xrightarrow{\exp A} & \mathbb{R}^>_n
\end{align*}
$$

where $|.|$ denotes the map $(z_1, \ldots, z_m) \mapsto (|z_1|, \ldots, |z_m|)$. Now set

$$
C = \exp \Psi(\mathbb{C}^\ell) = \left\{ (e^{\langle \psi_1, w \rangle}, \ldots, e^{\langle \psi_m, w \rangle}) \in (\mathbb{C}^\times)^m \right\}
$$

where $w = (w_1, \ldots, w_\ell) \in \mathbb{C}^\ell$, $\psi_i$ denotes the $i$th row of the $m \times \ell$-matrix $\Psi = (\psi_{ij})$.

Then $C \cong \mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(K)$ by holomorphic transformations.
**Ex 1.** Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $A: \mathbb{R}^2 \to 0$ is a zero map. Let $\Psi: \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$C = \{(e^z, e^{\alpha z})\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) of Constr 2 is void, while (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp \Psi: \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ with the natural complex structure is a complex torus $T^2_{\mathbb{C}}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha \mathbb{Z}) = T^2_{\mathbb{C}}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on $2\ell$ elements (so that $n = 0$, $m = 2\ell$), we may obtain any complex torus $T^{2\ell}_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C$. 
Thm 2. Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones, and let $K = K_\Sigma$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then

(a) the holomorphic action of the group $C \cong \mathbb{C}^\ell$ on $U(K)$ is free and proper, so the quotient $U(K)/C$ is a compact complex $(m - \ell)$-manifold;

(b) there is a $\mathbb{T}^m$-equivariant diffeomorphism $U(K)/C \cong \mathcal{Z}_K$ defining a complex structure on $\mathcal{Z}_K$ in which $\mathbb{T}^m$ acts holomorphically.
**Ex 2** (Hopf manifold). Let $\Sigma$ be the complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n + 1$ vectors $e_1, \ldots, e_n, -e_1 - \ldots - e_n$.

To make $m - n$ even we add one ‘empty’ 1-cone. We have $m = n + 2$, $\ell = 1$. Then $A: \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given by the matrix $(0 \ I \ -1)$, where $I$ is the unit $n \times n$ matrix, and $0, 1$ are the $n$-columns of zeros and units respectively.

We have that $K$ is the boundary of an $n$-dim simplex with $n + 1$ vertices and 1 ghost vertex, $Z_K \cong S^1 \times S^{2n+1}$, and $U(K) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\Psi: \mathbb{C} \to \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \ldots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \not\in \mathbb{R}$. Then

$$C = \{(e^z, e^{\alpha z}, \ldots, e^{\alpha z}) : z \in \mathbb{C}\} \subset (\mathbb{C}^\times)^{n+2},$$

and $Z_K$ acquires a complex structure as the quotient $U(K)/C$:

$$\mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})/ \{(t, w) \sim (e^z t, e^{\alpha z} w)\} \cong (\mathbb{C}^{n+1} \setminus \{0\})/ \{w \sim e^{2\pi i \alpha} w\},$$

where $t \in \mathbb{C}^\times$, $w \in \mathbb{C}^{n+1} \setminus \{0\}$. The latter quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ is known as the Hopf manifold.
3. Holomorphic bundles over toric varieties.

Manifolds $\mathcal{Z}_K$ corresponding to complete regular simplicial fans are total spaces of holomorphic principal bundles over toric varieties with fibre a complex torus. This allows us to calculate invariants of the complex structures on $\mathcal{Z}_K$.

A toric variety is a normal algebraic variety $X$ on which an algebraic torus $(\mathbb{C}^\times)^n$ acts with a dense (Zariski open) orbit.

Toric varieties are classified by rational fans. Under this correspondence,

- complete fans $\leftrightarrow$ compact varieties
- normal fans of polytopes $\leftrightarrow$ projective varieties
- regular fans $\leftrightarrow$ nonsingular varieties
- simplicial fans $\leftrightarrow$ orbifolds
\( \Sigma \) complete, simplicial, rational; 
\( a_1, \ldots, a_m \) primitive integral generators of 1-cones; 
\( a_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n \).

Constr 3 (‘Cox construction’). Let \( A_{\mathbb{C}} : \mathbb{C}^m \to \mathbb{C}^n, \) \( e_i \mapsto a_i, \)

\[ \exp A_{\mathbb{C}} : (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n, \]

\[ (z_1, \ldots, z_m) \mapsto \left( \prod_{i=1}^{m} z_i^{a_{i1}}, \ldots, \prod_{i=1}^{m} z_i^{a_{in}} \right) \]

Set \( G = \text{Ker} \exp A_{\mathbb{C}}. \)

This is an \((m - n)\)-dimensional algebraic subgroup in \((\mathbb{C}^\times)^m\).

It acts almost freely (with finite isotropy subgroups) on \( U(K_\Sigma) \).

If \( \Sigma \) is regular, then \( G \cong (\mathbb{C}^\times)^{m-n} \) and the action is free.

\( V_\Sigma = U(K_\Sigma)/G \) the \textbf{toric variety} associated to \( \Sigma \).

The quotient torus \((\mathbb{C}^\times)^m/G \cong (\mathbb{C}^\times)^n\) acts on \( V_\Sigma \) with a dense orbit.
Observe that $\mathbb{C}^\ell \cong C \subset G_\Sigma \cong (\mathbb{C}^\times)^m$ as a complex subgroup.

**Prop 2.**

(a) The toric variety $V_\Sigma$ is homeomorphic to the quotient of $\mathcal{Z}_{K_\Sigma}$ by the holomorphic action of $G/C$.

(b) If $\Sigma$ is regular, then there is a holomorphic principal bundle $\mathcal{Z}_{K_\Sigma} \to V_\Sigma$ with fibre the compact complex torus $G/C$ of dimension $\ell$.

**Rem 2.** For singular varieties $V_\Sigma$ the quotient projection $\mathcal{Z}_{K_\Sigma} \to V_\Sigma$ is a holomorphic principal **Seifert bundle** for an appropriate orbifold structure on $V_\Sigma$. 
4. Submanifolds and analytic subsets.

The complex structure on $Z_K$ is determined by two pieces of data:
– the complete simplicial fan $\Sigma$ with primitive generators $a_1, \ldots, a_m$;
– the $\ell$-dimensional holomorphic subgroup $C \subset (\mathbb{C}^\times)^m$.

If this data is *generic* (in particular, the fan $\Sigma$ is not rational), then there is no holomorphic principal torus fibration $Z_K \to V_\Sigma$ over a toric variety $V_\Sigma$.

However, there still exists a holomorphic $\ell$-dimensional foliation $\mathcal{F}$ with a transverse Kähler form $\omega_\mathcal{F}$. This form can be used to describe submanifolds and analytic subsets in $Z_K$. 
Consider the complexified map $A_{\mathbb{C}}: \mathbb{C}^m \to \mathbb{C}^n$, $e_i \mapsto a_i$. and the following complex $(m - n)$-dimensional subgroup in $(\mathbb{C}^\times)^m$:

$$G = \exp(\ker A_{\mathbb{C}}) = \left\{(e^{z_1}, \ldots, e^{z_m}) \in (\mathbb{C}^\times)^m : (z_1, \ldots, z_m) \in \ker A_{\mathbb{C}}\right\}.$$ 

Note $C \subset G$.

The group $G$ acts on $U(K)$, and its orbits define a holomorphic foliation on $U(K)$. Since $G \subset (\mathbb{C}^\times)^m$, this action is free on open subset $(\mathbb{C}^\times)^m \subset U(K)$, so that the generic leaf of the foliation has complex dimension $m - n = 2\ell$.

The $\ell$-dimensional closed subgroup $C \subset G$ acts on $U(K)$ freely and properly by Theorem 2, so that $U(K)/C$ carries a holomorphic action of the quotient group $D = G/C$.

$\mathcal{F}$: the holomorphic foliation on $U(K)/C \cong Z_K$ by the orbits of $D$. 

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The subgroup $G \subset (\mathbb{C}^\times)^m$ is closed if and only if it is isomorphic to $(\mathbb{C}^\times)^{2\ell}$; in this case the subspace $\text{Ker} \ A \subset \mathbb{R}^m$ is rational. Then $\Sigma$ is a rational fan and $V_\Sigma$ is the quotient $U(\mathcal{K})/G$. The foliation $\mathcal{F}$ gives rise to a holomorphic principal Seifert fibration $\pi: Z_\mathcal{K} \to V_\Sigma$ with fibres compact complex tori $G/C$.

For a generic configuration of nonzero vectors $a_1, \ldots, a_m$, $G$ is biholomorphic to $\mathbb{C}^{2\ell}$ and $D = G/C$ is biholomorphic to $\mathbb{C}^\ell$. 
A $(1, 1)$-form $\omega_\mathcal{F}$ on the complex manifold $\mathcal{Z}_\mathcal{K}$ is called **transverse Kähler** with respect to the foliation $\mathcal{F}$ if
(a) $\omega_\mathcal{F}$ is closed, i.e. $d\omega_\mathcal{F} = 0$;
(b) $\omega_\mathcal{F}$ is nonnegative and the zero space of $\omega_\mathcal{F}$ is the tangent space of $\mathcal{F}$.

A complete simplicial fan $\Sigma$ in $\mathbb{R}^n$ is called **weakly normal** if there exists a (not necessarily simple) $n$-dimensional polytope $P$ such that $\Sigma$ is a simplicial subdivision of the normal fan $\Sigma_P$.

**Thm 3.** Assume that $\Sigma$ is a weakly normal fan. Then there exists an exact $(1, 1)$-form $\omega_\mathcal{F}$ on $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/\mathcal{C}$ which is transverse Kähler for the foliation $\mathcal{F}$ on the dense open subset $(\mathbb{C}^\times)^m/\mathcal{C} \subset U(\mathcal{K})/\mathcal{C}$. 
For each $J \subset [m]$, define the corresponding coordinate submanifold in $Z_K$ by

$$Z_{K,J} = \{(z_1, \ldots, z_m) \in Z_K : z_i = 0 \quad \text{for} \ i \notin J\}.$$  

Obviously, $Z_{K,J}$ is identified with the quotient of

$$U(K_J) = \{(z_1, \ldots, z_m) \in U(K) : z_i = 0 \quad \text{for} \ i \notin J\}$$

by $C \cong \mathbb{C}^\ell$. In particular, $U(K_J)/C$ is a complex submanifold in $Z_K = U(K)/C$.

Observe that the closure of any $(\mathbb{C}^\times)^m$-orbit of $U(K)$ has the form $U(K_J)$ for some $J \subset [m]$ (in particular, the dense orbit corresponds to $J = [m]$). Similarly, the closure of any $(\mathbb{C}^\times)^m/C$-orbit of $Z_K \cong U(K)/C$ has the form $Z_{K,J}$.
**Thm 4.** Assume that the data defining a complex structure on \( \mathcal{Z}_K = \mathcal{U}(K)/C \) is generic. Then any divisor of \( \mathcal{Z}_K \) is a union of coordinate divisors.

Furthermore, if \( \Sigma \) is a weakly normal fan, then any compact irreducible analytic subset \( Y \subset \mathcal{Z}_K \) of positive dimension is a coordinate submanifold.

**Cor 1.** Under generic assumptions, there are no non-constant meromorphic functions on \( \mathcal{Z}_K \).
