1. Moment-angle complexes and manifolds.

\( \mathcal{K} \) an (abstract) simplicial complex on the set \([m] = \{1, \ldots, m\}\).

\( I = \{i_1, \ldots, i_k\} \in \mathcal{K} \) a simplex. Always assume \( \emptyset \in \mathcal{K} \).
Allow \( \{i\} \notin \mathcal{K} \) for some \( i \) (ghost vertices).

Consider the unit polydisc in \( \mathbb{C}^m \),

\[ \mathbb{D}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| \leq 1, \quad i = 1, \ldots, m\} \]

Given \( I \subset [m] \), set

\[ B_I := \{(z_1, \ldots, z_m) \in \mathbb{D}^m : |z_j| = 1 \text{ for } j \notin I\} \]

Define the moment-angle complex

\[ Z_\mathcal{K} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m \]

It is invariant under the coordinatewise action of the standard torus

\[ \mathbb{T}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| = 1, \quad i = 1, \ldots, m\} \]
on \( \mathbb{C}^m \).
Constr 1 (polyhedral product). Given spaces $W \subset X$ and $I \subset [m]$, set

$$(X, W)^I = \{ (x_1, \ldots, x_m) \in X^m : x_j \in W \text{ for } j \notin I \} \cong \prod_{i \in I} X \times \prod_{i \notin I} W,$$

and define the polyhedral product of $(X, W)$ as

$$(X, W)^K = \bigcup_{I \in K} (X, W)^I \subset X^m.$$ 

Then $Z_K = (\mathbb{D}, \mathbb{T})^K$, where $\mathbb{T}$ is the unit circle.

Another example is the complement of a coordinate subspace arrangement:

$$U(K) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \ldots, i_k\} \notin K} \{ z \in \mathbb{C}^m : z_{i_1} = \ldots = z_{i_k} = 0 \},$$

namely,

$$U(K) = (\mathbb{C}, \mathbb{C}^\times)^K = \bigcup_{I \in K} \left( \prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \right),$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Clearly, $Z_K \subset U(K)$. Moreover, $Z_K$ is a $\mathbb{T}^m$-equivariant deformation retract of $U(K)$ for every $K$ [Buchstaber-P].
Prop 1. Assume $|\mathcal{K}| \cong S^{n-1}$ (a sphere triangulation with $m$ vertices). Then $Z_{K}$ is a closed manifold of dimension $m + n$.

We refer to such $Z_{K}$ as moment-angle manifolds.

If $\mathcal{K} = \mathcal{K}_{P}$ is the dual triangulation of a simple convex polytope $P$, then $Z_{P} = Z_{K_{P}}$ embeds in $\mathbb{C}^{m}$ as a nondegenerate (transverse) intersection of $m - n$ real quadratic hypersurfaces. Therefore, $Z_{P}$ can be smoothed canonically.

Now assume $\mathcal{K}$ is the underlying complex of a complete simplicial fan $\Sigma$ (a starshaped sphere).
A fan is a finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of strongly convex cones in $\mathbb{R}^n$ such that every face of a cone in $\Sigma$ belongs to $\Sigma$ and the intersection of any two cones in $\Sigma$ is a face of each.

A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is complete if $\sigma_1 \cup \ldots \cup \sigma_s = \mathbb{R}^n$.

Let $\Sigma$ be a simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones generated by $a_1, \ldots, a_m$. Its underlying simplicial complex is

$$\mathcal{K}_\Sigma = \left\{ I \subset [m]: \{a_i: i \in I\} \text{ spans a cone of } \Sigma \right\}$$

Note: $\Sigma$ is complete iff $|\mathcal{K}_\Sigma|$ is a triangulation of $S^{n-1}$. 
Given $\Sigma$ with 1-cones generated by $a_1, \ldots, a_m$, define a map
\[ A : \mathbb{R}^m \to \mathbb{R}^n, \quad e_i \mapsto a_i, \]
where $e_1, \ldots, e_m$ is the standard basis of $\mathbb{R}^m$. Set
\[ \mathbb{R}^m_\succ = \{ (y_1, \ldots, y_m) \in \mathbb{R}^m : y_i > 0 \}, \]
and define
\[ R := \exp(\ker A) = \{ (y_1, \ldots, y_m) \in \mathbb{R}^m_\succ : \prod_{i=1}^m y_i^{\langle a_i, u \rangle} = 1 \text{ for all } u \in \mathbb{R}^n \}, \]
$R \subset \mathbb{R}^m_\succ$ acts on $U(K_\Sigma) \subset \mathbb{C}^m$ by coordinatewise multiplications.

**Thm 1.** Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones, and let $K = K_\Sigma$ be its underlying simplicial complex. Then

(a) the group $R \cong \mathbb{R}^{m-n}$ acts on $U(K)$ freely and properly, so the quotient $U(K)/R$ is a smooth $(m+n)$-dimensional manifold;

(b) $U(K)/R$ is $\mathbb{T}^m$-equivariantly homeomorphic to $Z_K$.

Therefore, $Z_K$ can be smoothed canonically.
2. Complex-analytic structures.

We shall show that the even-dimensional moment-angle manifold $\mathcal{Z}_K$ corresponding to a complete simplicial fan admits a structure of a complex manifold. The idea is to replace the action of $\mathbb{R}^{m-n}$ on $U(\mathcal{K})$ (whose quotient is $\mathcal{Z}_K$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

Rem 1. Complex structures on polytopal moment-angle manifolds $\mathcal{Z}_P$ were described by Bosio and Meersseman. They identified $\mathcal{Z}_P$ with a class of complex manifolds known as LVM-manifolds (named after López de Medrano, Verjovsky and Meersseman).

Assume $m - n$ is even from now on. We can always achieve this by formally adding an ‘empty’ one-dimensional cone to $\Sigma$; this corresponds to adding a ghost vertex to $\mathcal{K}$, or multiplying $\mathcal{Z}_K$ by a circle.

Set $\ell = \frac{m-n}{2}$. 
**Constr 2.** Choose a linear map $\Psi : \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying the two conditions:

(a) $\text{Re} \circ \Psi : \mathbb{C}^\ell \to \mathbb{R}^m$ is a monomorphism.

(b) $A \circ \text{Re} \circ \Psi = 0$.

The composite map of the top line in the following diagram is zero:

\[
\begin{array}{cccccc}
\mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\text{Re}} & \mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
\exp & & \downarrow \exp & & \downarrow \exp & & \\
(\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}^m & \xrightarrow{\exp A} & \mathbb{R}^n \\
\end{array}
\]

where $|\cdot|$ denotes the map $(z_1, \ldots, z_m) \mapsto (|z_1|, \ldots, |z_m|)$. Now set

$$C = \exp \Psi(\mathbb{C}^\ell) = \left\{ (e^{\langle \psi_1, w \rangle}, \ldots, e^{\langle \psi_m, w \rangle}) \in (\mathbb{C}^\times)^m \right\}$$

where $w = (w_1, \ldots, w_\ell) \in \mathbb{C}^\ell$, $\psi_i$ denotes the $i$th row of the $m \times \ell$-matrix $\Psi = (\psi_{ij})$.

Then $C \cong \mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.
Ex 1. Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $A: \mathbb{R}^2 \to 0$ is a zero map.

Let $\Psi: \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$C = \{(e^z, e^{\alpha z}) \in (\mathbb{C}^\times)^2\}.$$  

Condition (b) of Constr 2 is void, while (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp \Psi: \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ with the natural complex structure is a complex torus $T^2_\mathbb{C}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha \mathbb{Z}) = T^2_\mathbb{C}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on $2\ell$ elements (so that $n = 0$, $m = 2\ell$), we may obtain any complex torus $T^2_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C$. 
Thm 2. Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones, and let $K = K_{\Sigma}$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then

(a) the holomorphic action of the group $C \cong \mathbb{C}^\ell$ on $U(K)$ is free and proper, so the quotient $U(K)/C$ is a compact complex $(m - \ell)$-manifold;

(b) there is a $\mathbb{T}^m$-equivariant diffeomorphism $U(K)/C \cong Z_K$ defining a complex structure on $Z_K$ in which $\mathbb{T}^m$ acts holomorphically.
**Ex 2** (Hopf manifold). Let $\Sigma$ be the complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n + 1$ vectors $e_1, \ldots, e_n, -e_1 - \ldots - e_n$.

To make $m - n$ even we add one ‘empty’ 1-cone. We have $m = n + 2$, $\ell = 1$. Then $A : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given by the matrix $(0 \ I \ -1)$, where $I$ is the unit $n \times n$ matrix, and $0, 1$ are the $n$-columns of zeros and units respectively.

We have that $\mathcal{K}$ is the boundary of an $n$-dim simplex with $n + 1$ vertices and 1 ghost vertex, $\mathcal{Z}_\mathcal{K} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\Psi : \mathbb{C} \to \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \ldots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \not\in \mathbb{R}$. Then

$$C = \{(e^z, e^{\alpha z}, \ldots, e^{\alpha z}) : z \in \mathbb{C}\} \subset (\mathbb{C}^\times)^{n+2},$$

and $\mathcal{Z}_\mathcal{K}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})/ \{(t, w) \sim (e^z t, e^{\alpha z} w)\} \cong (\mathbb{C}^{n+1} \setminus \{0\})/ \{w \sim e^{2\pi i\alpha} w\},$$

where $t \in \mathbb{C}^\times$, $w \in \mathbb{C}^{n+1} \setminus \{0\}$. The latter quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ is known as the Hopf manifold.
3. Holomorphic bundles over toric varieties.

Manifolds $\mathcal{Z}_K$ corresponding to complete regular simplicial fans are total spaces of holomorphic principal bundles over toric varieties with fibre a complex torus. This allows us to calculate invariants of the complex structures on $\mathcal{Z}_K$.

A toric variety is a normal algebraic variety $X$ on which an algebraic torus $(\mathbb{C}^\times)^n$ acts with a dense (Zariski open) orbit.

Toric varieties are classified by rational fans. Under this correspondence,

$$
\begin{align*}
\text{complete fans} & \leftrightarrow \text{compact varieties} \\
\text{normal fans of polytopes} & \leftrightarrow \text{projective varieties} \\
\text{regular fans} & \leftrightarrow \text{nonsingular varieties} \\
\text{simplicial fans} & \leftrightarrow \text{orbifolds}
\end{align*}
$$
Σ complete, simplicial, rational;  
\(a_1, \ldots, a_m\) primitive integral generators of 1-cones;  
\(a_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n\).

**Constr 3** (‘Cox construction’). Let \(A_{\mathbb{C}}: \mathbb{C}^m \to \mathbb{C}^n, e_i \mapsto a_i\),

\[
\exp A_{\mathbb{C}}: (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n,
\]

\[
(z_1, \ldots, z_m) \mapsto \left( \prod_{i=1}^{m} z_i^{a_{i1}}, \ldots, \prod_{i=1}^{m} z_i^{a_{in}} \right)
\]

Set \(G = \text{Ker } \exp A_{\mathbb{C}}\).

This is an \((m-n)\)-dimensional algebraic subgroup in \((\mathbb{C}^\times)^m\).

It acts almost freely (with finite isotropy subgroups) on \(U(K_\Sigma)\).

If \(\Sigma\) is regular, then \(G \cong (\mathbb{C}^\times)^{m-n}\) and the action is free.

\(V_\Sigma = U(K_\Sigma)/G\) the **toric variety** associated to \(\Sigma\).

The quotient torus \((\mathbb{C}^\times)^m/G \cong (\mathbb{C}^\times)^n\) acts on \(V_\Sigma\) with a dense orbit.
Observe that $\mathbb{C}^\ell \cong C \subset G_\Sigma \cong (\mathbb{C}^\times)^m$ as a complex subgroup.

**Prop 2.**

(a) The toric variety $V_\Sigma$ is homeomorphic to the quotient of $\mathbb{Z}_{K_\Sigma}$ by the holomorphic action of $G/C$.

(b) If $\Sigma$ is regular, then there is a holomorphic principal bundle $\mathbb{Z}_{K_\Sigma} \to V_\Sigma$ with fibre the compact complex torus $G/C$ of dimension $\ell$.

**Rem 2.** For singular varieties $V_\Sigma$ the quotient projection $\mathbb{Z}_{K_\Sigma} \to V_\Sigma$ is a holomorphic principal [Seifert bundle](#) for an appropriate orbifold structure on $V_\Sigma$. 
4. Submanifolds and analytic subsets.

The complex structure on $\mathcal{Z}_K$ is determined by two pieces of data:
– the complete simplicial fan $\Sigma$ with generators $a_1, \ldots, a_m$;
– the $\ell$-dimensional holomorphic subgroup $C \subset (\mathbb{C}^\times)^m$.

If this data is \textit{generic} (in particular, the fan $\Sigma$ is not rational), then there is no holomorphic principal torus fibration $\mathcal{Z}_K \to V_\Sigma$ over a toric variety $V_\Sigma$.

However, there still exists a holomorphic $\ell$-dimensional \textit{foliation} $\mathcal{F}$ with a transverse Kähler form $\omega_\mathcal{F}$. This form can be used to describe submanifolds and analytic subsets in $\mathcal{Z}_K$. 
Consider the complexified map $A_{\mathbb{C}}: \mathbb{C}^m \to \mathbb{C}^n$, $e_i \mapsto a_i$. and the following complex $(m - n)$-dimensional subgroup in $(\mathbb{C}^\times)^m$:

$$G = \exp(\text{Ker } A_{\mathbb{C}}) = \left\{(e^{z_1}, \ldots, e^{z_m}) \in (\mathbb{C}^\times)^m : (z_1, \ldots, z_m) \in \text{Ker } A_{\mathbb{C}} \right\}.$$ 

Note $\mathbb{C} \subset G$.

The group $G$ acts on $U(K)$, and its orbits define a holomorphic foliation on $U(K)$. Since $G \subset (\mathbb{C}^\times)^m$, this action is free on open subset $(\mathbb{C}^\times)^m \subset U(K)$, so that the generic leaf of the foliation has complex dimension $m - n = 2\ell$.

The $\ell$-dimensional closed subgroup $\mathbb{C} \subset G$ acts on $U(K)$ freely and properly by Theorem 2, so that $U(K)/\mathbb{C}$ carries a holomorphic action of the quotient group $D = G/\mathbb{C}$.

$\mathcal{F}$: the holomorphic foliation on $U(K)/\mathbb{C} \cong \mathcal{Z}_K$ by the orbits of $D$. 
The subgroup \( G \subset (\mathbb{C}^\times)^m \) is closed if and only if it is isomorphic to \((\mathbb{C}^\times)^{2\ell}\); in this case the subspace \( \text{Ker} \ A \subset \mathbb{R}^m \) is rational. Then \( \Sigma \) is a rational fan and \( V_\Sigma \) is the quotient \( U(\mathcal{K})/G \). The foliation \( \mathcal{F} \) gives rise to a holomorphic principal Seifert fibration \( \pi: \mathcal{Z}_\mathcal{K} \to V_\Sigma \) with fibres compact complex tori \( G/C \).

For a generic configuration of nonzero vectors \( a_1, \ldots, a_m \), \( G \) is biholomorphic to \( \mathbb{C}^{2\ell} \) and \( D = G/C \) is biholomorphic to \( \mathbb{C}^\ell \).
A $(1, 1)$-form $\omega_F$ on the complex manifold $\mathcal{Z}_K$ is called transverse Kähler with respect to the foliation $\mathcal{F}$ if
(a) $\omega_F$ is closed, i.e. $d\omega_F = 0$;
(b) $\omega_F$ is nonnegative and the zero space of $\omega_F$ is the tangent space of $\mathcal{F}$.

A complete simplicial fan $\Sigma$ in $\mathbb{R}^n$ is called weakly normal if there exists a (not necessarily simple) $n$-dimensional polytope $P$ such that $\Sigma$ is a simplicial subdivision of the normal fan $\Sigma_P$.

**Thm 3.** Assume that $\Sigma$ is a weakly normal fan. Then there exists an exact $(1, 1)$-form $\omega_F$ on $\mathcal{Z}_K = U(K)/C$ which is transverse Kähler for the foliation $\mathcal{F}$ on the dense open subset $(\mathbb{C}^*)^m/C \subset U(K)/C$. 
For each $J \subset [m]$, define the corresponding coordinate submanifold in $\mathcal{Z}_K$ by

$$\mathcal{Z}_{K,J} = \{(z_1, \ldots, z_m) \in \mathcal{Z}_K : z_i = 0 \text{ for } i \notin J\}.$$ 

Obviously, $\mathcal{Z}_{K,J}$ is identified with the quotient of $U(K,J) = \{(z_1, \ldots, z_m) \in U(K) : z_i = 0 \text{ for } i \notin J\}$ by $C \cong \mathbb{C}^\ell$. In particular, $U(K,J)/C$ is a complex submanifold in $\mathcal{Z}_K \cong U(K)/C$.

Observe that the closure of any $(\mathbb{C}^\times)^m$-orbit of $U(K)$ has the form $U(K,J)$ for some $J \subset [m]$ (in particular, the dense orbit corresponds to $J = [m]$). Similarly, the closure of any $(\mathbb{C}^\times)^m/C$-orbit of $\mathcal{Z}_K \cong U(K)/C$ has the form $\mathcal{Z}_{K,J}$. 
**Thm 4.** Assume that the data defining a complex structure on $\mathcal{Z}_K = U(\mathcal{K})/C$ is generic. Then any divisor of $\mathcal{Z}_K$ is a union of coordinate divisors.

Furthermore, if $\Sigma$ is a weakly normal fan, then any compact irreducible analytic subset $Y \subset \mathcal{Z}_K$ of positive dimension is a coordinate submanifold.

**Cor 1.** Under generic assumptions, there are no non-constant meromorphic functions on $\mathcal{Z}_K$. 
