Geometric structures on moment-angle manifolds

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Topology of moment-angle manifolds and complexes

A convex polyhedron in $\mathbb{R}^n$ obtained by intersecting m halfspaces:

$$P = \bigl\{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\bigr\}.$$

Define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \bigl(\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x}\rangle + b_m\bigr).$$

If P has a vertex, then i_P is monomorphic, and $i_P(P)$ is the intersection of an n-plane with $\mathbb{R}^m_\geqslant = \{\boldsymbol{y} = (y_1, \ldots, y_m) \colon y_i \geqslant 0\}$.

Define the space $\mathcal{Z}_P$ from the diagram

$$\begin{array}{ccccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & \qquad & (z_1, \ldots, z_m) \\ \downarrow & & \downarrow \mu & & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant & & (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

$\mathcal{Z}_P$ has a $\mathbb{T}^m$-action, $\mathcal{Z}_P/\mathbb{T}^m = P$, and i_Z is a $\mathbb{T}^m$-equivariant inclusion.

Proposition 1. *If P is a simple polytope (more generally, if the presentation of P by inequalities is* generic*), then $\mathcal{Z}_P$ is a smooth manifold of dimension $m+n$.*

Proof. Write $i_P(\mathbb{R}^n)$ by $m-n$ linear equations in $(y_1,\ldots,y_m)\in\mathbb{R}^m$. Replace y_k by $|z_k|^2$ to obtain a presentation of $\mathcal{Z}_P$ by quadrics. □

$\mathcal{Z}_P$: polytopal moment-angle manifold corresponding to P.

Similarly, by considering the projection $\mu\colon\mathbb{R}^m\to\mathbb{R}^m_\geqslant$ instead of $\mu\colon\mathbb{C}^m\to\mathbb{R}^m_\geqslant$ we obtain the real moment-angle manifold $\mathcal{R}_P\subset\mathbb{R}^m$.

Example 1. $P=\{(x_1,x_2)\in\mathbb{R}^2\colon x_1\geqslant 0,\ x_2\geqslant 0,\ -\gamma_1x_1-\gamma_2x_2+1\geqslant 0\}$, $\gamma_1,\gamma_2>0$ (a 2-simplex). Then

$\mathcal{Z}_P=\{(z_1,z_2,z_3)\in\mathbb{C}^3\colon \gamma_1|z_1|^2+\gamma_2|z_2|^2+|z_3|^2)=1\}$ (a 5-sphere),

$\mathcal{R}_P=\{(u_1,u_2,u_3)\in\mathbb{R}^3\colon \gamma_1|u_1|^2+\gamma_2|u_2|^2+|u_3|^2)=1\}$ (a 2-sphere).

$\mathcal{K}$ an (abstract) simplicial complex on the set $[m] = \{1, \ldots, m\}$.

$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex. Always assume $\varnothing \in \mathcal{K}$.

Consider the unit polydisc in $\mathbb{C}^m$,

$$\mathbb{D}^m = \Big\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1, \quad i = 1, \ldots, m\Big\}.$$

Given $I \subset [m]$, set

$$B_I := \Big\{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon |z_j| = 1 \text{ for } j \notin I\Big\} \cong \prod_{i \in I} D^2 \times \prod_{i \notin I} S^1.$$

The moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} B_I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1\Big) \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the torus $\mathbb{T}^m$.

Example 2. $\mathcal{K} = 2$ points, then $\mathcal{Z}_{\mathcal{K}} = D^2 \times S^1 \cup S^1 \times D^2 \cong S^3$.

$\mathcal{K} = \Delta$, then $\mathcal{Z}_{\mathcal{K}} = (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2) \cong S^5$.

More generally, let X a space, and $A \subset X$. Given $I \subset [m]$, set

$$(X, A)^I = \Big\{(x_1, \ldots, x_m) \in \prod_{i=1}^{m} X \colon x_j \in A \text{ for } j \notin I\Big\} \cong \prod_{i \in I} X \times \prod_{i \notin I} A.$$

The $\mathcal{K}$-polyhedral product of (X, A) is

$$\mathcal{Z}_{\mathcal{K}}(X, A) = \bigcup_{I \in \mathcal{K}} (X, A)^I \subset X^m.$$

Another important example is the complement of the coordinate subspace arrangement corresponding to $\mathcal{K}$:

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \ldots, i_k\} \notin \mathcal{K}} \{\boldsymbol{z} \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\},$$

namely,

$$U(\mathcal{K}) = \mathcal{Z}_{\mathcal{K}}(\mathbb{C}, \mathbb{C}^{\times}),$$

where $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$.

Theorem 1. *$\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$ is a $\mathbb{T}^m$-deformation retract of $U(\mathcal{K})$.*

Theorem 2. *If P is a simple polytope, $\mathcal{K}_P = \partial(P^*)$ (the dual triangulation), then $\mathcal{Z}_{\mathcal{K}_P} \cong \mathcal{Z}_P$ ($\mathbb{T}^m$-equivariantly homeomorphic).*

In particular, $\mathcal{Z}_{\mathcal{K}_P}$ is a manifold. More generally,

Proposition 2. *Assume $|\mathcal{K}| \cong S^{n-1}$ (a sphere triangulation with m vertices). Then $\mathcal{Z}_{\mathcal{K}}$ is a closed manifold of dimension $m+n$.*

The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \ldots, v_m] \big/ \Big(v_{i_1} \cdots v_{i_k} \colon \{i_1, \ldots, i_k\} \notin K \Big), \qquad \deg v_i = 2.$$

Theorem 3. *There is an isomorphism of (bi)graded algebras*

$$\begin{aligned} H^*(\mathcal{Z}_{\mathcal{K}}; \mathbb{Z}) &\cong \mathrm{Tor}^{*,*}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}) \\ &\cong H\Big[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}]; d \Big], \end{aligned}$$

where $du_i = v_i$, $dv_i = 0$ for $1 \leqslant i \leqslant m$. In particular,

$$H^p(\mathcal{Z}_{\mathcal{K}}) \cong \sum_{-i+2j=p} \mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}).$$

Corollary 1. $H^k(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{I\subset[m]} \widetilde{H}^{k-|I|-1}(\mathcal{K}_I),$

where $\mathcal{K}_I$ is the restriction of $\mathcal{K}$ to the subset $I \subset \{1,\ldots,m\}$.

If $\mathcal{K} = \mathcal{K}_P$, then can rewrite the above in terms of P instead of $\mathcal{K}$:

Corollary 2. $H^k(\mathcal{Z}_P) \cong \bigoplus_{I\subset[m]} \widetilde{H}^{k-|I|-1}(P_I),$

where P_I is the union of facets F_i of P with $i \in I$.

Remark 1. Integral version of Theorem 3 was proved independently by [Baskakov–Buchstaber–P] and [Franz].

2. The product in $H^*(\mathcal{Z}_\mathcal{K})$ given by Theorem 3 can be also described in terms of full subcomplexes $\mathcal{K}_I$ of Corollary 1 [Baskakov].

3. There is the stable decomposition $\Sigma\mathcal{Z}_\mathcal{K} \simeq \bigvee_{I\subset[m]} \Sigma^{|I|+2}|\mathcal{K}_I|$ behind the isomorphism of Corollary 1 [Bahri–Bendersky–Cohen–Gitler].

Geometric structures I. Lagrangian submanifolds

(M, ω) a symplectic Riemannian $2n$-manifold.

An immersion $i\colon N \looparrowright M$ of an n-manifold N is Lagrangian if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a Lagrangian submanifold of M.

A vector field ξ on M is Hamiltonian if the 1-form $\omega(\,\cdot\,, \xi)$ is exact.

A Lagrangian immersion $i\colon N \looparrowright M$ is Hamiltonian minimal (H-minimal) if the variations of the volume of $i(N)$ along all Hamiltonian vector fields with compact support are zero, i.e.

$$\frac{d}{dt}\mathrm{vol}(i_t(N))\Big|_{t=0} = 0,$$

where $i_0(N) = i(N)$, $i_t(N)$ is a Hamiltonian deformation of $i(N)$, and $\mathrm{vol}(i_t(N))$ is the volume of the deformed part of $i_t(N)$.

Recall: P a simple polytope

$$P = \Big\{ \boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m \Big\}.$$

The polytopal moment-angle manifold $\mathcal{Z}_P$,

$$\begin{array}{ccccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m & \qquad & (z_1, \ldots, z_m) \\ \big\downarrow & & \big\downarrow \mu & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant} & & (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

can be written as the intersection of $m - n$ real quadrics,

$$\mathcal{Z}_P = \Big\{ \boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \sum_{k=1}^{m} \gamma_{jk} |z_k|^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m - n \Big\}.$$

Also have the real moment-angle manifold,

$$\mathcal{R}_P = \Big\{ \boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \sum_{k=1}^{m} \gamma_{jk} u_k^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m-n \Big\}.$$

Set $\gamma_k = (\gamma_{1k}, \ldots, \gamma_{m-n,k}) \in \mathbb{R}^{m-n}$ for $1 \leqslant k \leqslant m$.

Assume that the polytope P is rational. Then have two lattices:

$$\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle \subset \mathbb{R}^n \quad \text{and} \quad L = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_m \rangle \subset \mathbb{R}^{m-n}.$$

Consider the $(m-n)$-torus

$$T_P = \Big\{ \Big(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle} \Big) \in \mathbb{T}^m \Big\},$$

i.e. $T_P = \mathbb{R}^{m-n}/L^*$, and set

$$D_P = \frac{1}{2} L^*/L^* \cong (\mathbb{Z}/2)^{m-n}.$$

Proposition 3. *The $(m-n)$-torus T_P acts on $\mathcal{Z}_P$ almost freely.*

Consider the map

$$f\colon \mathcal{R}_P \times T_P \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u}, \varphi) \mapsto \boldsymbol{u} \cdot \varphi = (u_1 e^{2\pi i \langle \gamma_1, \varphi \rangle}, \dots, u_m e^{2\pi i \langle \gamma_m, \varphi \rangle}).$$

Note $f(\mathcal{R}_P \times T_P) \subset \mathcal{Z}_P$ is the set of T_P-orbits through $\mathcal{R}_P \subset \mathbb{C}^m$.

Have an m-dimensional manifold

$$N_P = \mathcal{R}_P \times_{D_P} T_P.$$

Lemma 1. $f\colon \mathcal{R}_P \times T_P \to \mathbb{C}^m$ *induces an immersion* $j\colon N_P \looparrowright \mathbb{C}^m$*.*

Theorem 4 (Mironov)**.** *The immersion* $i_\Gamma\colon N_\Gamma \looparrowright \mathbb{C}^m$ *is H-minimal Lagrangian.*

When it is an embedding?

A simple rational polytope P is Delzant if for any vertex $v \in P$ the set of vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ normal to the facets meeting at v forms a basis of the lattice $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$:

$$\mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle = \mathbb{Z}\langle \boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n} \rangle \qquad \text{for any } v = F_{i_1} \cap \cdots \cap F_{i_n}.$$

Theorem 5. *The following conditions are equivalent:*

1) $j\colon N_P \to \mathbb{C}^m$ *is an* embedding *of an H-minimal Lagrangian submanifold;*
2) *the* $(m-n)$*-torus* T_P *acts on* $\mathcal{Z}_P$ freely*.*
3) P *is a* Delzant *polytope.*

Explicit constructions of families of Delzant polytopes are known in toric geometry and topology:
- simplices and cubes in all dimensions;
- products and face cuts;
- associahedra (Stasheff ptopes), permutahedra, and generalisations.

Example 3 (one quadric)**.** Let $P = \Delta^{m-1}$ (a simplex), i.e. $m - n = 1$ and $\mathcal{R}_{\Delta^{m-1}}$ is given by a single quadric

$$\gamma_1 u_1^2 + \cdots + \gamma_m u_m^2 = c \qquad (1)$$

with $\gamma_i > 0$, i.e. $\mathcal{R}_{\Delta^{m-1}} \cong S^{m-1}$. Then

$$N \cong S^{m-1} \times_{\mathbb{Z}/2} S^1 \cong \begin{cases} S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orient. of } S^{m-1}, \\ \mathcal{K}^m & \text{if } \tau \text{ reverses the orient. of } S^{m-1}, \end{cases}$$

where τ is the involution and $\mathcal{K}^m$ is an m-dimensional Klein bottle.

Proposition 4. *We obtain an H-minimal Lagrangian embedding of $N_{\Delta^{m-1}} \cong S^{n-1} \times_{\mathbb{Z}/2} S^1$ in $\mathbb{C}^m$ if and only if $\gamma_1 = \cdots = \gamma_m$ in (1). The topological type of $N_{\Delta^{m-1}} = N(m)$ depends only on the parity of m:*

$$N(m) \cong S^{m-1} \times S^1 \qquad \text{if } m \text{ is even,}$$
$$N(m) \cong \mathcal{K}^m \qquad \text{if } m \text{ is odd.}$$

The Klein bottle $\mathcal{K}^m$ with even m does *not* admit Lagrangian embeddings in $\mathbb{C}^m$ [Nemirovsky, Shevchishin].

Example 4 (two quadrics).

Theorem 6. *Let $m-n=2$, i.e. $P \simeq \Delta^{p-1} \times \Delta^{q-1}$.*
(a) $\mathcal{R}_P$ is diffeomorphic to $\mathcal{R}(p,q) \cong S^{p-1} \times S^{q-1}$ given by

$$\begin{aligned} u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \cdots + u_p^2 \qquad\qquad &= 1, \\ u_1^2 + \ldots + u_k^2 \qquad\qquad + u_{p+1}^2 + \cdots + u_m^2 &= 2, \end{aligned}$$

where $p+q=m$, $0<p<m$ and $0 \leqslant k \leqslant p$.
(b) If $N_P \to \mathbb{C}^m$ is an embedding, then N_P is diffeomorphic to

$$N_k(p,q) = \mathcal{R}(p,q) \times_{\mathbb{Z}/2\times\mathbb{Z}/2} (S^1 \times S^1),$$

where the two involutions act on $\mathcal{R}(p,q)$ by

$$\begin{aligned} \psi_1 &\colon (u_1,\ldots,u_m) \mapsto (-u_1,\ldots,-u_k,-u_{k+1},\ldots,-u_p,u_{p+1},\ldots,u_m), \\ \psi_2 &\colon (u_1,\ldots,u_m) \mapsto (-u_1,\ldots,-u_k,u_{k+1},\ldots,u_p,-u_{p+1},\ldots,-u_m). \end{aligned} \tag{2}$$

There is a fibration $N_k(p,q) \to S^{q-1} \times_{\mathbb{Z}/2} S^1 = N(q)$ with fibre $N(p)$ (the manifold from the previous example), which is trivial for $k=0$.

Example 5 (three quadrics).

In the case $m-n=3$ the topology of compact manifolds $\mathcal{R}_P$ and $\mathcal{Z}_P$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest P with $m-n=3$ is a (Delzant) pentagon, e.g.

$$P = \Big\{(x_1,x_2)\in\mathbb{R}^2\colon x_1\geqslant 0,\ x_2\geqslant 0,\ -x_1+2\geqslant 0,\ -x_2+2\geqslant 0,\ -x_1-x_2+3\geqslant 0\Big\}.$$

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3\times S^4$.

Get an H-minimal Lagrangian submanifold $N_P\subset\mathbb{C}^5$ which is the total space of a bundle over T^3 with fibre a surface of genus 5.

Proposition 5. *Let P be an m-gon. Then $\mathcal{R}_P$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.*

Get an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^m$ which is the total space of a bundle over T^{m-2} with fibre S_g. It is an aspherical manifold (for $m \geqslant 4$) whose fundamental group enters into the short exact sequence

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z}^{m-2} \longrightarrow 1.$$

For $n > 2$ and $m - n > 3$ the topology of $\mathcal{R}_P$ and $\mathcal{Z}_P$ is even more complicated.

Geometric structures II. Non-Kähler complex structures

Recall: if $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope P, then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ has a canonical smooth structure (e.g. as a nondegenerate intersection of Hermitian quadrics in $\mathbb{C}^m$).

Let $\mathcal{K}$ be a sphere triangulation, i.e. $|\mathcal{K}| \cong S^{n-1}$.
A realisation $|\mathcal{K}| \subset \mathbb{R}^n$ is starshaped if there is a point $\boldsymbol{x} \notin |\mathcal{K}|$ such that any ray from $\boldsymbol{x}$ intersects $|\mathcal{K}|$ in exactly one point.

A convex triangulation $\mathcal{K}_P$ is starshaped, but not vice versa!

$\mathcal{K}$ has a starshaped realisation if and only if it is the underlying complexes of a complete simplicial fan Σ.

Also recall $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\} \notin \mathcal{K}} \{\boldsymbol{z} \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\}$.

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$ be the generators of the 1-dimensional cones of Σ. Consider the linear map

$$\Lambda_{\mathbb{R}} \colon \mathbb{R}^m \to \mathbb{R}^n, \quad \boldsymbol{e}_i \mapsto \boldsymbol{a}_i,$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ is the standard basis of $\mathbb{R}^m$. Define

$$R_\Sigma := \exp(\operatorname{Ker} \Lambda_{\mathbb{R}}) = \Big\{ (y_1, \ldots, y_m) \in \mathbb{R}^m_{>} \colon \prod_{i=1}^{m} y_i^{\langle \boldsymbol{a}_i, \boldsymbol{u} \rangle} = 1 \text{ for all } \boldsymbol{u} \in \mathbb{R}^n \Big\},$$

$R_\Sigma \subset \mathbb{R}^m_{>}$ acts on $U(\mathcal{K}_\Sigma) \subset \mathbb{C}^m$ by coordinatewise multiplications.

Theorem 7. *Let $\mathcal{K}$ be the underlying complex of a complete simplicial fan Σ. Then*

(a) R_Σ acts on $U(\mathcal{K})$ freely and properly, so the quotient $U(\mathcal{K})/R_\Sigma$ has a canonical structure of a smooth $(m+n)$-manifold;

(b) $U(\mathcal{K})/R_\Sigma$ is $\mathbb{T}^m$-equivariantly homeomorphic to $\mathcal{Z}_{\mathcal{K}}$.

Therefore, $\mathcal{Z}_{\mathcal{K}}$ can be smoothed canonically.

Assume $m-n$ is even and set $\ell = \frac{m-n}{2}$.

Choose a linear map $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying the two conditions:

(a) $\mathrm{Re}\circ\Psi\colon \mathbb{C}^\ell \to \mathbb{R}^m$ is a monomorphism.

(b) $\Lambda_\mathbb{R} \circ \mathrm{Re}\circ\Psi = 0$.

Now set

$$C_{\Psi,\Sigma} = \exp\Psi(\mathbb{C}^\ell) = \Bigl\{\Bigl(e^{\langle\psi_1,\boldsymbol{w}\rangle},\ldots,e^{\langle\psi_m,\boldsymbol{w}\rangle}\Bigr) \in (\mathbb{C}^\times)^m\Bigr\}.$$

Then $C_{\Psi,\Sigma} \cong \mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup of $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

Theorem 8. *Let $\mathcal{K}$ be as before. Then*

(a) $C_{\Psi,\Sigma}$ acts on $U(\mathcal{K})$ freely and properly, so the quotient $U(\mathcal{K})/C_{\Psi,\Sigma}$ is a compact complex manifold of complex dimension $m-\ell$;

(b) there is a $\mathbb{T}^m$-equivariant diffeomorphism $U(\mathcal{K})/C_{\Psi,\Sigma} \cong \mathcal{Z}_\mathcal{K}$ defining a complex structure on $\mathcal{Z}_\mathcal{K}$ in which $\mathbb{T}^m$ acts holomorphically.

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