Homotopy types of moment-angle complexes

based on joint work with Jelena Grbic, Stephen Theriault and Jie Wu

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1. Problems

- identifying the homotopy type of the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ for certain simplicial complexes $\mathcal{K}$;
- describing the multiplication and higher Massey products in the Tor-algebra $H^*(\mathcal{Z}_{\mathcal{K}}) = \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$ of the face ring $\mathbf{k}[\mathcal{K}]$;
- describing the Yoneda algebra $\mathrm{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k},\mathbf{k})$ in terms of generators and relations;
- describing the structure of the Pontryagin algebra $H_*(\Omega DJ(\mathcal{K}))$ and its commutator subalgebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ via iterated and higher Whitehead (Samelson) products;
- identifying the homotopy type of the loop spaces $\Omega DJ(\mathcal{K})$ and $\Omega\mathcal{Z}_{\mathcal{K}}$.

2. Preliminaries

(X, A) a pair of spaces.

$\mathcal{K}$ a simplicial complex on $[m] = \{1, 2, \ldots, m\}$, $\varnothing \in \mathcal{K}$.

Given $I = \{i_1, \ldots, i_k\} \subset [m]$, set

$$(X, A)^I = Y_1 \times \cdots \times Y_m \qquad \text{where } Y_i = \begin{cases} X & \text{if } i \in I, \\ A & \text{if } i \notin I. \end{cases}$$

The $\mathcal{K}$-polyhedral product of (X, A) is

$$(X, A)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (X, A)^I = \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} X \times \prod_{i \notin I} A \Bigr).$$

Example

1. $(X, A) = (D^2, S^1)$,
   $\mathcal{Z}_\mathcal{K} := (D^2, S^1)^\mathcal{K}$ the moment-angle complex.
   It has an action of the torus T^m.
2. $(X, A) = (\mathbb{C}P^\infty, pt)$,
   $DJ(\mathcal{K}) := (\mathbb{C}P^\infty, pt)^\mathcal{K}$ the Davis–Januszkiewicz space.
3. $(X, A) = (\mathbb{C}, \mathbb{C}^\times)$,

$$U(\mathcal{K}) := (\mathbb{C}, \mathbb{C}^\times)^\mathcal{K} = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\}\notin\mathcal{K}} \{z_{i_1} = \cdots = z_{i_k} = 0\}$$

   the complement of a coordinate subspace arrangement.

Theorem

There exists a deformation retraction

$$\mathcal{Z}_\mathcal{K} \hookrightarrow U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_\mathcal{K}$$

There exists a homotopy fibration

$$\begin{array}{ccccc} \mathcal{Z}_{\mathcal{K}} & \longrightarrow & DJ(\mathcal{K}) & \longrightarrow & (\mathbb{C}P^\infty)^m \\ \| & & \| & & \| \\ (D^2, S^1)^{\mathcal{K}} & & (\mathbb{C}P^\infty, pt)^{\mathcal{K}} & & (\mathbb{C}P^\infty, \mathbb{C}P^\infty)^{\mathcal{K}} \end{array}$$

which splits after looping:

$$\Omega DJ(\mathcal{K}) \simeq \Omega \mathcal{Z}_{\mathcal{K}} \times T^m$$

Warning: this is not an H-space splitting

Proposition

There exists an exact sequence of noncommutative algebras

$$1 \longrightarrow H_*(\Omega \mathcal{Z}_{\mathcal{K}}) \longrightarrow H_*(\Omega DJ(\mathcal{K})) \xrightarrow{\mathrm{Ab}} \Lambda[u_1, \ldots, u_m] \longrightarrow 1$$

where $\Lambda[u_1, \ldots, u_m]$ *denotes the exterior algebra and* $\deg u_i = 1$.

Let $\mathbf{k}$ denote $\mathbb{Z}$ or a field.
The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is given by

$$\mathbf{k}[\mathcal{K}] := \mathbf{k}[v_1, \ldots, v_m] \big/ \big(v_{i_1} \cdots v_{i_k} = 0 \quad \text{if } \{i_1, \ldots, i_k\} \notin \mathcal{K} \big)$$

where $\deg v_i = 2$.

Theorem

$$\begin{aligned} H^*(DJ(\mathcal{K})) &\cong \mathbf{k}[\mathcal{K}] \\ H_*(\Omega DJ(\mathcal{K})) &\cong \mathrm{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k}, \mathbf{k}) && \mathbf{k} \text{ is a field} \\ H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \mathrm{Tor}_{\mathbf{k}[v_1, \ldots, v_m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \\ &\cong H\big[\Lambda[u_1, \ldots, u_m] \otimes \mathbf{k}[\mathcal{K}], d\big], && du_i = v_i, dv_i = 0 \\ &\cong \bigoplus_{I \in \mathcal{K}} \widetilde{H}^{*-|I|-1}(\mathcal{K}_I) && \mathcal{K}_I = \mathcal{K}|_I \end{aligned}$$

$\mathbf{k}[\mathcal{K}]$ is a Golod ring if the multiplication and all higher Massey operations in $\mathrm{Tor}_{\mathbf{k}[v_1, \ldots, v_m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ are trivial.

3. The case of a flag complex

A missing face of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but every proper subset of I is a simplex of $\mathcal{K}$.

$\mathcal{K}$ is a flag complex if each of its missing faces has two vertices.

Equivalently, $\mathcal{K}$ is flag if any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex.

$$\begin{array}{ccc} \{\text{flag complexes on } [m]\} & \stackrel{1-1}{\longleftrightarrow} & \{\text{simple graphs on } [m]\} \\ \mathcal{K} & \rightarrow & \mathcal{K}^1 \quad (\text{one-skeleton}) \\ \mathcal{K}(\Gamma) & \leftarrow & \Gamma \end{array}$$

where $\mathcal{K}(\Gamma)$ is the clique complex of Γ
(fill in each clique of Γ with a simplex).

A graph Γ is chordal if each of its cycles with $\geqslant 4$ edges has a chord.

Equivalently, Γ is chordal if there are no induced cycles of length $\geqslant 4$.

Theorem (Fulkerson–Gross)

A graph is chordal if and only if its vertices can be ordered in such a way that, for each vertex i, the lesser neighbours of i form a clique.

(perfect elimination ordering)

Theorem (Grbic-P-Theriault-Wu)

$\mathcal{K}$ is flag, $\mathbf{k}$ *a field. The following are equivalent:*

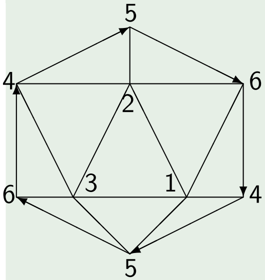
1. $\mathbf{k}[\mathcal{K}]$ *is a Golod ring;*
2. *the multiplication in* $H^*(\mathcal{Z}_\mathcal{K})$ *is trivial;*
3. $\Gamma = \mathcal{K}^1$ *is a chordal graph;*
4. $\mathcal{Z}_\mathcal{K}$ *has homotopy type of a wedge of spheres.*

The equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) was proved by Berglund and Jöllenbeck in 2007.

Implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are valid for arbitrary $\mathcal{K}$.

However, (3) $\Rightarrow$(4) fails in the non-flag case.

Example



The nonzero cohomology groups of $\mathcal{Z}_\mathcal{K}$ are

$$H^0 = \mathbb{Z}, \quad H^5 = \mathbb{Z}^{10}, \quad H^6 = \mathbb{Z}^{15}, \quad H^7 = \mathbb{Z}^6$$
$$H^9 = \mathbb{Z}/2.$$

All Massey products vanish for dimensional reasons, so $\mathcal{K}$ is Golod (over any field).

Nevertheless, $\mathcal{Z}_\mathcal{K}$ is not homotopy equivalent to a wedge of spheres because of the torsion. In fact,

$$\mathcal{Z}_\mathcal{K} \simeq (S^5)^{\vee 10} \vee (S^6)^{\vee 15} \vee (S^7)^{\vee 6} \vee \Sigma^7 \mathbb{R}P^2.$$

Question

Assume that $H^*(\mathcal{Z}_\mathcal{K})$ has trivial multiplication, so that $\mathcal{K}$ is Golod, over any field. Is it true that $\mathcal{Z}_\mathcal{K}$ is a co-H-space, or even a suspension, as in the previous example?

How many spheres in the wedge? What if $\mathcal{Z}_\mathcal{K}$ is not a wedge of spheres?

Theorem

For any flag complex $\mathcal{K}$, there is an isomorphism

$$H_*\big(\Omega DJ(\mathcal{K})\big) \cong T\langle u_1,\ldots,u_m\rangle\big/\big(u_i^2=0,\ u_iu_j+u_ju_i=0 \text{ for } \{i,j\}\in\mathcal{K}\big)$$

where $T\langle u_1,\ldots,u_m\rangle$ is the free algebra on m generators of degree 1.

Remember the exact sequence of non-commutative algebras

$$1 \longrightarrow H_*(\Omega\mathcal{Z}_\mathcal{K}) \longrightarrow H_*(\Omega DJ(\mathcal{K})) \xrightarrow{\mathrm{Ab}} \Lambda[u_1,\ldots,u_m] \longrightarrow 1$$

Proposition

For any flag complex $\mathcal{K}$, the Poincaré series of $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is given by

$$P\big(H_*(\Omega\mathcal{Z}_\mathcal{K});t\big) \;=\; \frac{1}{(1+t)^{m-n}(1-h_1t+\cdots+(-1)^nh_nt^n)}\,,$$

where $\boldsymbol{h}(\mathcal{K})=(h_0,h_1\ldots,h_n)$ is the h-vector of $\mathcal{K}$.

Theorem

Assume that $\mathcal{K}$ *is flag. The algebra* $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$*, viewed as the commutator subalgebra of* $H_*(\Omega DJ(\mathcal{K}))$*, is multiplicatively generated by* $\sum_{I\subset[m]} \dim \widetilde{H}^0(\mathcal{K}_I)$ *iterated commutators of the form*

$$[u_j, u_i], \quad [u_{k_1}, [u_j, u_i]], \quad \ldots, \quad [u_{k_1}, [u_{k_2}, \cdots [u_{k_{m-2}}, [u_j, u_i]] \cdots]]$$

where $k_1 < k_2 < \cdots < k_p < j > i$*,* $k_s \neq i$ *for any* s*, and* i *is the smallest vertex in a connected component not containing* j *of the subcomplex* $\mathcal{K}_{\{k_1,\ldots,k_p,j,i\}}$*.*

Furthermore, this multiplicative generating set is minimal.

Here is an important particular case (corresponding to $\mathcal{K} = m$ points).
It is an analogue of the description of a basis in the commutator subalgebra of a free algebra, given by Cohen and Neisendorfer:

Lemma

Let A be the commutator subalgebra of $T\langle u_1, \ldots, u_m\rangle/(u_i^2 = 0)$:

$$1 \longrightarrow A \longrightarrow T\langle u_1, \ldots, u_m\rangle/(u_i^2 = 0) \longrightarrow \Lambda[u_1, \ldots, u_m] \longrightarrow 1$$

where $\deg u_i = 1$.

Then A is a free associative algebra.
It is minimally generated by the iterated commutators of the form

$$[u_j, u_i], \quad [u_{k_1}, [u_j, u_i]], \quad \ldots, \quad [u_{k_1}, [u_{k_2}, \cdots [u_{k_{m-2}}, [u_j, u_i]] \cdots]]$$

where $k_1 < k_2 < \cdots < k_p < j > i$ and $k_s \neq i$ for any s.
The number of commutators of length ℓ is equal to $(\ell - 1)\binom{m}{\ell}$.

Corollary

Assume that $\mathcal{K}$ *is flag and* $\mathcal{Z}_{\mathcal{K}}$ *has homotopy type of a wedge of spheres.*

Then the number of spheres of dimension $\ell+1$ *in the wedge is given by* $\sum_{|I|=\ell}\dim \widetilde{H}^0(\mathcal{K}_I)$, *for* $2 \leqslant \ell \leqslant m$.

In particular, $H^i(\mathcal{K}_I)=0$ *for* $i>0$ *and all* I.

Reference

Jelena Grbic, Taras Panov, Stephen Theriault and Jie Wu.
Homotopy types of moment-angle complexes.
Preprint (2012); arXiv:1211.0873.