# Non-Kähler complex structures on moment-angle manifolds and other toric spaces

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## 1. Moment-angle complexes and manifolds.

$\mathcal{K}$ an (abstract) simplicial complex on the set $[m] = \{1, \ldots, m\}$.

$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex. Always assume $\varnothing \in \mathcal{K}$.
Allow $\{i\} \notin \mathcal{K}$ for some $i$ (ghost vertices).

Consider the unit polydisc in $\mathbb{C}^m$,

$$\mathbb{D}^m = \Bigl\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1, \quad i = 1, \ldots, m\Bigr\}.$$

Given $I \subset [m]$, set

$$B_I := \Bigl\{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon |z_j| = 1 \text{ for } j \notin I\Bigr\}.$$

Define the moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the standard torus

$$\mathbb{T}^m = \Bigl\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| = 1, \quad i = 1, \ldots, m\Bigr\}$$

on $\mathbb{C}^m$.

**Constr 1** (polyhedral product)**.** Given spaces $W \subset X$ and $I \subset [m]$, set

$$(X,W)^I = \bigl\{(x_1,\ldots,x_m) \in X^m \colon x_j \in W \text{ for } j \notin I\bigr\} \cong \prod_{i\in I} X \times \prod_{i\notin I} W,$$

and define the polyhedral product of $(X,W)$ as

$$(X,W)^{\mathcal K} = \bigcup_{I\in\mathcal K} (X,W)^I \subset X^m.$$

Then $\mathcal Z_{\mathcal K} = (\mathbb D, \mathbb T)^{\mathcal K}$, where $\mathbb T$ is the unit circle.

Another example is the complement of a coordinate subspace arrangement:

$$U(\mathcal K) = \mathbb C^m \setminus \bigcup_{\{i_1,\ldots,i_k\}\notin\mathcal K} \{z \in \mathbb C^m \colon z_{i_1} = \ldots = z_{i_k} = 0\},$$

namely,

$$U(\mathcal K) = (\mathbb C, \mathbb C^\times)^{\mathcal K} = \bigcup_{I\in\mathcal K} \Bigl(\prod_{i\in I} \mathbb C \times \prod_{i\notin I} \mathbb C^\times\Bigr),$$

where $\mathbb C^\times = \mathbb C \setminus \{0\}$.

Clearly, $\mathcal Z_{\mathcal K} \subset U(\mathcal K)$. Moreover, $\mathcal Z_{\mathcal K}$ is a $\mathbb T^m$-equivariant deformation retract of $U(\mathcal K)$ for every $\mathcal K$ [**Buchstaber-P**].

**Prop 1** (**[Buchstaber-P]**). *Assume $|\mathcal{K}| \cong S^{n-1}$ (a sphere triangulation with $m$ vertices). Then $\mathcal{Z}_K$ is a closed manifold of dimension $m+n$.*

We refer to such $\mathcal{Z}_{\mathcal{K}}$ as moment-angle manifolds.

If $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope $P$, then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ embeds in $\mathbb{C}^m$ as a nondegenerate (transverse) intersection of $m-n$ real quadratic hypersurfaces. Therefore, $\mathcal{Z}_P$ can be smoothed canonically.

Now assume $\mathcal{K}$ is the underlying complex of a complete simplicial fan $\Sigma$ (a starshaped sphere).

A fan is a finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of strongly convex cones in $\mathbb{R}^n$ such that every face of a cone in $\Sigma$ belongs to $\Sigma$ and the intersection of any two cones in $\Sigma$ is a face of each.

A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is complete if $\sigma_1 \cup \ldots \cup \sigma_s = \mathbb{R}^n$.

Let $\Sigma$ be a simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones generated by $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Its underlying simplicial complex is

$$\mathcal{K}_\Sigma = \Bigl\{ I \subset [m] \colon \{\mathbf{a}_i \colon i \in I\} \text{ spans a cone of } \Sigma \Bigr\}$$

Note: $\Sigma$ is complete iff $|\mathcal{K}_\Sigma|$ is a triangulation of $S^{n-1}$.

Given $\Sigma$ with 1-cones generated by $\mathbf{a}_1, \ldots, \mathbf{a}_m$, define a map

$$\Lambda_{\mathbb{R}} \colon \mathbb{R}^m \to \mathbb{R}^n, \quad \mathbf{e}_i \mapsto \mathbf{a}_i,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the standard basis of $\mathbb{R}^m$. Set

$$\mathbb{R}^m_> = \{(y_1, \ldots, y_m) \in \mathbb{R}^m \colon y_i > 0\},$$

and define

$$R_\Sigma := \exp(\operatorname{Ker} \Lambda_{\mathbb{R}}) = \Big\{(y_1, \ldots, y_m) \in \mathbb{R}^m_> \colon \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u} \rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n\Big\},$$

$R_\Sigma \subset \mathbb{R}^m_>$ acts on $U(\mathcal{K}_\Sigma) \subset \mathbb{C}^m$ by coordinatewise multiplications.

**Thm 1.** *Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Then*

*(a) the group $R_\Sigma \cong \mathbb{R}^{m-n}$ acts on $U(\mathcal{K})$ freely and properly, so the quotient $U(\mathcal{K})/R_\Sigma$ is a smooth $(m+n)$-dimensional manifold;*

*(b) $U(\mathcal{K})/R_\Sigma$ is $\mathbb{T}^m$-equivariantly homeomorphic to $\mathcal{Z}_\mathcal{K}$.*

*Therefore, $\mathcal{Z}_\mathcal{K}$ can be smoothed canonically.*

## 2. Complex-analytic structures.

We shall show that the even-dimensional moment-angle manifold $\mathcal{Z}_\mathcal{K}$ corresponding to a complete simplicial fan admits a structure of a complex manifold. The idea is to replace the action of $\mathbb{R}^{m-n}_>$ on $U(\mathcal{K})$ (whose quotient is $\mathcal{Z}_\mathcal{K}$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

**Rem 1.** Complex structures on *polytopal* moment-angle manifolds $\mathcal{Z}_P$ were described by Bosio and Meersseman. Existence of complex structure on moment-angle manifolds corresponding to complete simplicial fans has been also recently and independently established by Tambour.

Assume $m-n$ is even from now on. We can always achieve this by formally adding an 'empty' one-dimensional cone to $\Sigma$; this corresponds to adding a ghost vertex to $\mathcal{K}$, or multiplying $\mathcal{Z}_\mathcal{K}$ by a circle.

Set $\ell = \frac{m-n}{2}$.

**Constr 2.** Choose a linear map $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying the two conditions:

(a) $\mathrm{Re}\circ\Psi\colon \mathbb{C}^\ell \to \mathbb{R}^m$ is a monomorphism.

(b) $\Lambda_\mathbb{R} \circ \mathrm{Re}\circ\Psi = 0$.

The composite map of the top line in the following diagram is zero:

$$\begin{array}{ccccccc}
\mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\mathrm{Re}} & \mathbb{R}^m & \xrightarrow{\Lambda_\mathbb{R}} & \mathbb{R}^n \\
 & & \Big\downarrow{\scriptstyle\exp} & & \Big\downarrow{\scriptstyle\exp} & & \Big\downarrow{\scriptstyle\exp} \\
 & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}^m_> & \xrightarrow{\exp\Lambda_\mathbb{R}} & \mathbb{R}^n_>
\end{array}$$

where $|\cdot|$ denotes the map $(z_1,\ldots,z_m)\mapsto(|z_1|,\ldots,|z_m|)$. Now set

$$C_{\Psi,\Sigma} = \exp\Psi(\mathbb{C}^\ell) = \left\{\left(e^{\langle\psi_1,\mathbf{w}\rangle},\ldots,e^{\langle\psi_m,\mathbf{w}\rangle}\right)\in(\mathbb{C}^\times)^m\right\}$$

where $\mathbf{w}=(w_1,\ldots,w_\ell)\in\mathbb{C}^\ell$, $\psi_i$ denotes the $i$th row of the $m\times\ell$-matrix $\Psi=(\psi_{ij})$.

Then $C_{\Psi,\Sigma}\cong\mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

**Ex 1.** Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $\Lambda_{\mathbb{R}} \colon \mathbb{R}^2 \to 0$ is a zero map. Let $\Psi \colon \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$C = C_{\Psi,\Sigma} = \left\{(e^z, e^{\alpha z})\right\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) of Constr 2 is void, while (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp \Psi \colon \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ with the natural complex structure is a complex torus $T^2_{\mathbb{C}}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T^2_{\mathbb{C}}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on $2\ell$ elements (so that $n = 0$, $m = 2\ell$), we may obtain any complex torus $T^{2\ell}_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C_{\Psi,\Sigma}$.

**Thm 2.** *Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^n$ with $m$ one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then*

*(a) the holomorphic action of the group $C_{\Psi,\Sigma} \cong \mathbb{C}^\ell$ on $U(\mathcal{K})$ is free and proper, so the quotient $U(\mathcal{K})/C_{\Psi,\Sigma}$ is a compact complex $(m-\ell)$-manifold;*

*(b) there is a $\mathbb{T}^m$-equivariant diffeomorphism $U(\mathcal{K})/C_{\Psi,\Sigma} \cong \mathcal{Z}_\mathcal{K}$ defining a complex structure on $\mathcal{Z}_\mathcal{K}$ in which $\mathbb{T}^m$ acts holomorphically.*

**Ex 2** (Hopf manifold). Let $\Sigma$ be the complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n+1$ vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_1 - \ldots - \mathbf{e}_n$.

To make $m-n$ even we add one 'empty' 1-cone. We have $m = n+2$, $\ell = 1$. Then $\Lambda_{\mathbb{R}} \colon \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given by the matrix $(\mathbf{0}\ I\ -\mathbf{1})$, where $I$ is the unit $n \times n$ matrix, and $\mathbf{0}$, $\mathbf{1}$ are the $n$-columns of zeros and units respectively.

We have that $\mathcal{K}$ is the boundary of an $n$-dim simplex with $n+1$ vertices and 1 ghost vertex, $\mathcal{Z}_{\mathcal{K}} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\Psi \colon \mathbb{C} \to \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \ldots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{R}$. Then

$$C = C_{\Psi,\Sigma} = \left\{(e^z, e^{\alpha z}, \ldots, e^{\alpha z}) \colon z \in \mathbb{C}\right\} \subset (\mathbb{C}^\times)^{n+2},$$

and $\mathcal{Z}_{\mathcal{K}}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^\times \times \left(\mathbb{C}^{n+1} \setminus \{0\}\right) \big/ \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z}\mathbf{w})\} \cong \left(\mathbb{C}^{n+1} \setminus \{0\}\right) \big/ \{\mathbf{w} \sim e^{2\pi i \alpha}\mathbf{w}\},$$

where $t \in \mathbb{C}^\times$, $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. The latter quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ is known as the Hopf manifold.

## 3. Holomorphic bundles over toric varieties, and Hodge numbers.

Manifolds $\mathcal{Z}_{\mathcal{K}}$ corresponding to complete *regular* simplicial fans are total spaces of holomorphic principal bundles over toric varieties with fibre a complex torus. This allows us to calculate invariants of the complex structures on $\mathcal{Z}_{\mathcal{K}}$.

A toric variety is a normal algebraic variety $X$ on which an algebraic torus $(\mathbb{C}^{\times})^n$ acts with a dense (Zariski open) orbit.

Toric varieties are classified by rational fans. Under this correspondence,

$$\begin{aligned} \text{complete fans} &\longleftrightarrow \text{compact varieties} \\ \text{normal fans of polytopes} &\longleftrightarrow \text{projective varieties} \\ \text{regular fans} &\longleftrightarrow \text{nonsingular varieties} \\ \text{simplicial fans} &\longleftrightarrow \text{orbifolds} \end{aligned}$$

$\Sigma$ complete, simplicial, rational;
$\mathbf{a}_1, \ldots, \mathbf{a}_m$ primitive integral generators of 1-cones.

**Constr 3** ('Cox construction'). Let $\Lambda_{\mathbb{C}} \colon \mathbb{C}^m \to \mathbb{C}^n$, $\mathbf{e}_i \mapsto \mathbf{a}_i$,

$$\exp \Lambda_{\mathbb{C}} \colon (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n,$$
$$(z_1, \ldots, z_m) \mapsto \Bigl( \prod_{i=1}^m z_i^{a_{i1}}, \ldots, \prod_{i=1}^m z_i^{a_{in}} \Bigr)$$

Set $G_\Sigma = \operatorname{Ker} \exp \Lambda_{\mathbb{C}}$.
This is an $(m-n)$-dimensional algebraic subgroup in $(\mathbb{C}^\times)^m$.
It acts almost freely (with finite isotropy subgroups) on $U(\mathcal{K}_\Sigma)$.
If $\Sigma$ is regular, then $G_\Sigma \cong (\mathbb{C}^\times)^{m-n}$ and the action is free.

$X_\Sigma = U(\mathcal{K}_\Sigma)/G_\Sigma$ the toric variety associated to $\Sigma$.
The quotient torus $(\mathbb{C}^\times)^m / G_\Sigma \cong (\mathbb{C}^\times)^n$ acts on $X_\Sigma$ with a dense orbit.

Observe that $\mathbb{C}^\ell \cong C_{\Psi,\Sigma} \subset G_\Sigma \cong (\mathbb{C}^\times)^m$ as a complex subgroup.

**Prop 2.**

*(a) The toric variety $X_\Sigma$ is homeomorphic to the quotient of $\mathcal{Z}_{\mathcal{K}_\Sigma}$ by the holomorphic action of $G_\Sigma/C_{\Psi,\Sigma}$.*

*(b) If $\Sigma$ is regular, then there is a holomorphic principal bundle $\mathcal{Z}_{\mathcal{K}_\Sigma} \to X_\Sigma$ with fibre the compact complex torus $G_\Sigma/C_{\Psi,\Sigma}$ of dimension $\ell$.*

**Rem 2.** For singular varieties $X_\Sigma$ the quotient projection $\mathcal{Z}_{\mathcal{K}_\Sigma} \to X_\Sigma$ is a holomorphic principal Seifert bundle for an appropriate orbifold structure on $X_\Sigma$.

$h^{p,q}(M) = \dim H^{p,q}_{\bar\partial}(M)$: the Hodge numbers of a complex manifold $M$.

The Dolbeault cohomology of a complex torus is given by

$$H^{*,*}_{\bar\partial}(T^{2\ell}_{\mathbb{C}}) \cong \Lambda[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell],$$

where $\xi_1, \dots, \xi_\ell \in H^{1,0}_{\bar\partial}(T^{2\ell}_{\mathbb{C}})$, $\eta_1, \dots, \eta_\ell \in H^{0,1}_{\bar\partial}(T^{2\ell}_{\mathbb{C}})$. Hence, $h^{p,q}(T^{2\ell}_{\mathbb{C}}) = \binom{\ell}{p}\binom{\ell}{q}$.

The Dolbeault cohomology of a complete nonsingular toric variety $X_\Sigma$ is given by [Danilov–Jurkiewicz]:

$$H^{*,*}_{\bar\partial}(X_\Sigma) \cong \mathbb{C}[v_1, \dots, v_m]/(\mathcal{I}_{\mathcal{K}_\Sigma} + \mathcal{J}_\Sigma),$$

where $v_i \in H^{1,1}_{\bar\partial}(X_\Sigma)$,
$\mathcal{I}_{\mathcal{K}_\Sigma} = \Big(v_{i_1} \cdots v_{i_k} \colon \{i_1, \dots, i_k\} \notin \mathcal{K}_\Sigma\Big)$ (the Stanley–Reisner ideal),
$\mathcal{J}_\Sigma = (\sum_{k=1}^m a_{kj} v_k, \quad 1 \leqslant j \leqslant n)$.
We have $h^{p,p}(X_\Sigma) = h_p$, where $(h_0, h_1, \dots, h_n)$ is the $h$-vector of $\mathcal{K}_\Sigma$, and $h^{p,q}(X_\Sigma) = 0$ for $p \neq q$.

By an application of the Borel spectral sequence to the holomorphic bundle $\mathcal{Z}_\mathcal{K} \to X_\Sigma$ we obtain the following description of the Dolbeault cohomology.

**Thm 3.** *Let $\Sigma$ be a complete rational nonsingular fan. Then the Dolbeault cohomology group $H^{p,q}_{\bar\partial}(\mathcal{Z}_\mathcal{K})$ is isomorphic to the $(p,q)$-th cohomology group of the differential bigraded algebra*

$$\Big[\Lambda[\xi_1,\ldots,\xi_\ell,\eta_1,\ldots,\eta_\ell] \otimes H^{*,*}_{\bar\partial}(X_\Sigma), d\Big]$$

*whose differential $d$ of bidegree $(0,1)$ is defined on the generators as*

$$dv_i = d\eta_j = 0, \quad d\xi_j = c(\xi_j), \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant \ell,$$

*where $c\colon H^{1,0}_{\bar\partial}(T^{2\ell}_\mathbb{C}) \to H^2(X_\Sigma,\mathbb{C}) = H^{1,1}_{\bar\partial}(X_\Sigma)$ is the first Chern class map of the torus principal bundle $\mathcal{Z}_\mathcal{K} \to X_\Sigma$.*

This result may be compared to the analogous description of the ordinary cohomology of $\mathcal{Z}_\mathcal{K}$ from [Buchstaber-P]:

**Thm 4.** *$H^*(\mathcal{Z}_\mathcal{K})$ is isomorphic to the cohomology of the dga*

$$\Big[\Lambda[u_1,\ldots,u_{m-n}] \otimes H^*(X_\Sigma), d\Big],$$

*with* $\deg u_j = 1$, $\deg v_i = 2$, *and differential $d$ defined on the generators as*

$$dv_i = 0, \quad du_j = \gamma_{j1}v_1 + \ldots + \gamma_{jm}v_m, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m-n.$$

**Thm 5.** *Let $\mathcal{Z}_\mathcal{K}$ be as in above, and let $k$ be the number of ghost vertices in $\mathcal{K}$. Then the Hodge numbers $h^{p,q} = h^{p,q}(\mathcal{Z}_\mathcal{K})$ satisfy*

*(a)* $\binom{k-\ell}{p} \leqslant h^{p,0} \leqslant \binom{[k/2]}{p}$ *for* $p \geqslant 0$*;*

*(b)* $h^{0,q} = \binom{\ell}{q}$ *for* $q \geqslant 0$*;*

*(c)* $h^{1,q} = (\ell - k)\binom{\ell}{q-1} + h^{1,0}\binom{\ell+1}{q}$ *for* $q \geqslant 1$*;*

*(d)* $\frac{\ell(3\ell+1)}{2} - h_2(\mathcal{K}) - \ell k + (\ell+1)h^{2,0} \leqslant h^{2,1} \leqslant \frac{\ell(3\ell+1)}{2} - \ell k + (\ell+1)h^{2,0}$.

**Rem 3.** At most one ghost vertex is required to make $\dim \mathcal{Z}_\mathcal{K} = m + n$ even. Note that $k \leqslant 1$ implies $h^{p,0}(\mathcal{Z}_\mathcal{K}) = 0$, so that $\mathcal{Z}_\mathcal{K}$ does not have holomorphic forms of any degree in this case.

If $\mathcal{Z}_\mathcal{K}$ is a torus, then $m = k = 2\ell$, and $h^{1,0}(\mathcal{Z}_\mathcal{K}) = h^{0,1}(\mathcal{Z}_\mathcal{K}) = \ell$. Otherwise Thm 5 implies that $h^{1,0}(\mathcal{Z}_\mathcal{K}) < h^{0,1}(\mathcal{Z}_\mathcal{K})$, and therefore $\mathcal{Z}_\mathcal{K}$ is not Kähler.

**Ex 3** (Calabi–Eckmann manifold). Let $\mathcal{K}_\Sigma = \partial\Delta^p \times \partial\Delta^q$ with $p \leqslant q$, so $n = p+q$, $m = n+2$ and $\ell = 1$.

Then $U(\mathcal{K}) = (\mathbb{C}^{p+1} \setminus \{0\}) \times (\mathbb{C}^{q+1} \setminus \{0\})$. Choose $\Psi = (1,\ldots,1,\alpha,\ldots,\alpha)^t$ where the number of units is $p+1$ and $\alpha \notin \mathbb{R}$. Have $\exp\Psi\colon \mathbb{C} \to (\mathbb{C}^\times)^m$.

This gives $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/\mathbb{C} \cong S^{2p+1} \times S^{2q+1}$ a structure of a complex manifold. It is the total space of a holomorphic principal bundle over $X_\Sigma = \mathbb{C}P^p \times \mathbb{C}P^q$ with fibre a complex torus $\mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z})$, a Calabi–Eckmann manifold $CE(p,q)$.

By Thm 3, $\quad H^{*,*}_{\bar\partial}\bigl(CE(p,q)\bigr) \cong H\Bigl[\Lambda[\xi,\eta] \otimes \mathbb{C}[x,y]/(x^{p+1},y^{q+1}), d\Bigr]$,
where $dx = dy = d\eta = 0$ and $d\xi = x - y$ for an appropriate choice of $x,y$.
We therefore obtain

$$H^{*,*}_{\bar\partial}\bigl(CE(p,q)\bigr) \cong \Lambda[\omega,\eta] \otimes \mathbb{C}[x]/(x^{p+1}),$$

where $\omega \in H^{q+1,q}_{\bar\partial}\bigl(CE(p,q)\bigr)$ is the cohomology class of the cocycle $\xi\frac{x^{q+1}-y^{q+1}}{x-y}$.
This calculation is originally due to Borel.

**Ex 4.** The product $S^3 \times S^3 \times S^5 \times S^5$ has has two complex structures as a product of Calabi–Eckmann manifolds, namely, $CE(1,1) \times CE(2,2)$ and $CE(1,2) \times CE(1,2)$.

In the first case $h^{2,1} = 1$, and $h^{2,1} = 0$ in the second.

[BM] Frédéric Bosio and Laurent Meersseman. *Real quadrics in $\mathbb{C}^n$, complex manifolds and convex polytopes.* Acta Math. **197** (2006), no. 1, 53–127.

[BP] Victor Buchstaber and Taras Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* University Lecture Series, vol. **24**, Amer. Math. Soc., Providence, R.I., 2002.

[LV] Santiago López de Medrano and Alberto Verjovsky. *A new family of complex, compact, non-symplectic manifolds*. Bol. Soc. Mat. Brasil. **28** (1997), 253–269.

[PU] Taras Panov and Yuri Ustinovsky. *Complex-analytic structures on moment-angle manifolds*. Moscow Math. J. **12** (2012), no. 1; arXiv:1008.476

[Ta] Jérôme Tambour. *LVMB manifolds and simplicial spheres.* Preprint (2010); arXiv:1006.1797.