Geometric structures on moment-angle manifolds

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Intersections of quadrics.

Given a set of m vectors

$$\Gamma = \left\{\gamma_k = (\gamma_{1,k}, \ldots, \gamma_{m-n,k})^t \in \mathbb{R}^{m-n}, \quad k = 1, \ldots, m\right\},$$

and a vector $\boldsymbol{c} = (c_1, \ldots, c_{m-n})^t \in \mathbb{R}^{m-n}$, we consider the following intersections of $m-n$ real quadrics $\mathbb{R}^m$ and $\mathbb{C}^m$:

$$\mathcal{R}_\Gamma = \left\{\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m \colon \sum_{k=1}^{m} \gamma_{jk} u_k^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m-n\right\},$$

$$\mathcal{Z}_\Gamma = \left\{\boldsymbol{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m \colon \sum_{k=1}^{m} \gamma_{jk} |z_k|^2 = c_j, \quad \text{for } 1 \leqslant j \leqslant m-n\right\}.$$

Prop 1. *Intersections of quadrics $\mathcal{R}_\Gamma$ and $\mathcal{Z}_\Gamma$ are nonempty and non-degenerate if and only if the following two conditions are satisfied:*

(a) $\boldsymbol{c} \in \sigma\langle\gamma_1, \dots, \gamma_m\rangle$*;*

(b) if $\boldsymbol{c} \in \sigma\langle\gamma_{i_1}, \dots \gamma_{i_k}\rangle$*, then* $k \geqslant m - n$*.*

Under these conditions, $\mathcal{R}_\Gamma$ and $\mathcal{Z}_\Gamma$ are smooth submanifolds in $\mathbb{R}^m$ and $\mathbb{C}^m$ of dimension n and $m + n$ respectively, and the vectors $\gamma_1, \dots, \gamma_m$ span $\mathbb{R}^{m-n}$.

From now on we assume that the conditions of Proposition 1 are satisfied. Moreover, we assume that

(c) the vectors $\gamma_1, \dots, \gamma_m$ generate a lattice L in $\mathbb{R}^{m-n}$.

Let

$$L^* = \{\lambda^* \in \mathbb{R}^{m-n} \colon \langle \lambda^*, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in L\}$$

be the dual lattice.

The torus $\mathbb{T}^m = \left\{(e^{2\pi i \chi_1}, \ldots, e^{2\pi i \chi_m}) \in \mathbb{C}^m\right\}$, where $(\chi_1, \ldots, \chi_m) \in \mathbb{R}^m$, acts on $\mathcal{Z}_\Gamma$ coordinatewise. Similarly, the 'real torus' $(\mathbb{Z}/2)^m \subset \mathbb{T}^m$ (corresponding to $(\chi_1, \ldots, \chi_m) \in \frac{1}{2}\mathbb{Z}^m$) acts on $\mathcal{R}_\Gamma$.

The vectors γ_i define an $(m-n)$-dimensional torus subgroup in $\mathbb{T}^m$ whose lattice of characters is L:

$$T_\Gamma = \left\{\left(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle}\right) \in \mathbb{T}^m\right\} \cong \mathbb{T}^{m-n},$$

where $\varphi \in \mathbb{R}^{m-n}$. We also define

$$D_\Gamma = \frac{1}{2} L^* / L^* \cong (\mathbb{Z}/2)^{m-n}.$$

Note that D_Γ embeds canonically as a subgroup in $T_\Gamma = \mathbb{R}^{m-n}/L^*$.

Given a subset $I \subset [m] = \{1, \dots, m\}$, define the sublattice

$$L_I = \mathbb{Z}\langle \gamma_i \colon i \notin I \rangle \subset L.$$

Given $\boldsymbol{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$, define its *zero set* as

$$I_{\boldsymbol{u}} = \{i \colon u_i = 0\} \subset [m],$$

and define $I_{\boldsymbol{z}}$ similarly for $\boldsymbol{z} = (z_1, \dots, z_m) \in \mathbb{C}^m$.

A G-action is *almost free* if all isotropy subgroups are finite.

Prop 2. *The torus T_Γ acts on $\mathcal{Z}_\Gamma$ almost freely. The isotropy subgroup of $\boldsymbol{z} \in \mathcal{Z}_\Gamma$ is given by $L^*_{I_{\boldsymbol{z}}}/L^*$, where $L_{I_{\boldsymbol{z}}} = \mathbb{Z}\langle \gamma_k \colon k \notin I_{\boldsymbol{z}} \rangle \subset L$.*

Proof. An element $(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \dots, e^{2\pi i \langle \gamma_m, \varphi \rangle}) \in T_\Gamma$ fixes the given $\boldsymbol{z} \in \mathcal{Z}_\Gamma$ whenever $e^{2\pi i \langle \gamma_k, \varphi \rangle} = 1$ for every $k \notin I_{\boldsymbol{z}}$. The latter condition is equivalent to $\langle \gamma_k, \varphi \rangle \in \mathbb{Z}$, that is, $\varphi \in L^*_{I_{\boldsymbol{z}}}$. Since $\varphi \in L^*$ maps to $1 \in T_\Gamma$, the isotropy subgroup of $\boldsymbol{z}$ is indeed $L^*_{I_{\boldsymbol{z}}}/L^*$. □

Lagrangian immersions.

Let (M,ω) be a symplectic $2n$-manifold. An immersion $i\colon N \looparrowright M$ of an n-manifold N is *Lagrangian* if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a *Lagrangian submanifold* of M. A vector field ξ on M is *Hamiltonian* if the 1-form $\omega(\,\cdot\,,\xi)$ is exact.

Assume that a compatible Riemannian metric is chosen on M. A Lagrangian immersion $i\colon N \looparrowright M$ is *Hamiltonian minimal* (*H-minimal*) if the variations of the volume of $i(N)$ along all Hamiltonian vector fields with compact support are zero, that is,

$$\frac{d}{dt}\mathrm{vol}(i_t(N))\Big|_{t=0} = 0,$$

where $i_0(N) = i(N)$, $i_t(N)$ is a deformation of $i(N)$ along a Hamiltonian vector field, and $\mathrm{vol}(i_t(N))$ is the volume of the deformed part of $i_t(N)$. An immersion is *minimal* if the variations of the volume of $i(N)$ along *all* vector fields are zero.

Consider the map

$$j\colon \mathcal{R}_\Gamma \times T_\Gamma \longrightarrow \mathbb{C}^m,$$
$$(\boldsymbol{u}, \varphi) \mapsto \boldsymbol{u} \cdot \varphi = (u_1 e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, u_m e^{2\pi i \langle \gamma_m, \varphi \rangle}).$$

Note that $j(\mathcal{R}_\Gamma \times T_\Gamma) \subset \mathcal{Z}_\Gamma$. The quotient

$$N_\Gamma = \mathcal{R}_\Gamma \times_{D_\Gamma} T_\Gamma$$

is an m-dimensional manifold.

Lemma 1. *The map $j\colon \mathcal{R}_\Gamma \times T_\Gamma \to \mathbb{C}^m$ induces an immersion $i_\Gamma\colon N_\Gamma \looparrowright \mathbb{C}^m$.*

Thm 1 (Mironov)**.** *The immersion $i_\Gamma\colon N_\Gamma \looparrowright \mathbb{C}^m$ is H-minimal Lagrangian. Moreover, if $\sum_{k=1}^m \gamma_k = 0$, then i_Γ is a minimal Lagrangian immersion.*

Lagrangian embeddings and moment-angle manifolds.

Thm 2. *The following conditions are equivalent:*

(1) $i_\Gamma \colon N_\Gamma \to \mathbb{C}^m$ *is an embedding of an H-minimal Lagrangian submanifold;*

(2) $L_{I_{\boldsymbol{u}}} = L$ *for every* $\boldsymbol{u} \in \mathcal{R}_\Gamma$*;*

(3) T_Γ *acts on* $\mathcal{Z}_\Gamma$ *freely.*

This result opens a way to construct explicitly new families of H-minimal Lagrangian submanifolds, once we have an effective method to produce nondegenerate intersections of quadrics $\mathcal{R}_\Gamma$ satisfying conditions (2) or (3) of Theorem 2. Toric topology provides such a method.

The quotient of $\mathcal{R}_\Gamma$ by the action of $(\mathbb{Z}/2)^m$ (or the quotient of $\mathcal{Z}_\Gamma$ by the action of $\mathbb{T}^m$) is identified with the set of nonnegative solutions of the following system of $m-n$ linear equations:

$$\sum_{k=1}^{m} \gamma_k y_k = \boldsymbol{c}.$$

This set may be described as a convex polyhedron obtained by intersecting m halfspaces in $\mathbb{R}^n$:

$$P = \Big\{ \boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geqslant 0 \quad \text{for } i = 1, \dots, m \Big\}, \tag{1}$$

Note that P may be unbounded; in fact P is bounded if and only if $\mathcal{R}_\Gamma$ is bounded (compact). Bounded polyhedra are known as *polytopes*.

We refer to (1) as a *presentation* of P by inequalities. A presentation is *generic* if P is n-dimensional, has at least one vertex, and the hyperplanes defined by the equations $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i = 0$ are in general position at every point of P. If P is a polytope, then the existence of a generic presentation implies that P is *simple*.

Given a generic presentation of a polyhedron P, we may reconstruct the intersections of quadrics $\mathcal{R}_\Gamma$ and $\mathcal{Z}_\Gamma$ as follows.

Consider the affine map

$$i_P\colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = \bigl(\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x}\rangle + b_m\bigr).$$

It is monomorphic onto a certain n-dimensional plane in $\mathbb{R}^m$ (because P has a vertex), and $i_P(P)$ is the intersection of this plane with $\mathbb{R}^m_\geqslant$.

We define the space $\mathcal{Z}_P$ from the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \stackrel{i_Z}{\longrightarrow} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \mu \\ P & \stackrel{i_P}{\longrightarrow} & \mathbb{R}^m_\geqslant \end{array}$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. Note that $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient P, and i_Z is a $\mathbb{T}^m$-equivariant embedding.

If the presentation of P is generic, then $\mathcal{Z}_P$ is a smooth manifold of dimension $m+n$, known as the *(polytopal) moment-angle manifold* corresponding to P.

Now we can write the n-dimensional plane $i_P(\mathbb{R}^n)$ by $m-n$ linear equations in $\mathbb{R}^m$. Replacing each y_k by $|z_k|^2$ we obtain a presentation of the moment-angle manifold $\mathcal{Z}_P$ as an intersection of quadrics.

By replacing $\mathbb{C}^m$ by $\mathbb{R}^m$ we obtain the *real moment-angle manifold* $\mathcal{R}_P$.

Clearly, $\gamma_1, \ldots, \gamma_m$ generate a lattice L in $\mathbb{R}^{m-n}$ if and only if $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ generate a lattice Λ in $\mathbb{R}^n$. The corresponding P are called *rational*.

If P is rational, then we have a map of lattices

$$A_P \colon \Lambda^* \to \mathbb{Z}^m, \quad \boldsymbol{x} \mapsto (\langle \boldsymbol{a}_1, \boldsymbol{x} \rangle, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{x} \rangle).$$

Its conjugate gives rise to a map of tori $\mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^n/\Lambda$, whose kernel we denote by T_P. It becomes T_Γ under the identification of $\mathcal{Z}_P$ with $\mathcal{Z}_\Gamma$. We also have $D_P \cong (\mathbb{Z}/2)^{m-n}$ and $N_P = \mathcal{R}_P \times_{D_P} T_P$.

The manifolds $\mathcal{R}_P, \mathcal{Z}_P, N_P$ represent the same geometric objects as $\mathcal{R}_\Gamma, \mathcal{Z}_\Gamma, N_\Gamma$, although a different initial data is used in their definition.

P is *Delzant* if it is rational and for every vertex $\boldsymbol{x} \in P$ the vectors $\boldsymbol{a}_{j_1}, \ldots, \boldsymbol{a}_{j_n}$ normal to the facets meeting at $\boldsymbol{x}$ constitute a basis of $\Lambda = \mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$.

Thm 3. *The map $N_P = \mathcal{R}_P \times_{D_P} T_P \to \mathbb{C}^m$ is an embedding if and only if P is a Delzant polyhedron.*

Topology of Lagrangian submanifolds N.

Toric topology provides large families of explicitly constructed Delzant polytopes:

- simplices and cubes in all dimensions;
- products and face cuts;
- *associahedra* (Stasheff polytopes), *permutahedra*, and general *nestohedra*.

Nevertheless, the topology of $\mathcal{Z}_P$ (and therefore of N_P) is very complicated in general. Cohomology rings of $\mathcal{Z}_P$ are described by [Buchstaber-P.], and explicit homotopy and diffeomorphism type for some particular families of P are given by [Bahri–Bendersky–Cohen–Gitler], [Gitler–Lopez de Medrano], [Grbić–Theriault], and others.

Prop 3. *(a) The immersion of N in $\mathbb{C}^m$ factors as $N \looparrowright \mathcal{Z} \hookrightarrow \mathbb{C}^m$;*

(b) N is the total space of a bundle over a torus T^{m-n} with fibre $\mathcal{R}$;

(c) if $N \to \mathbb{C}^m$ is an embedding, then N is the total space of a principal T^{m-n}-bundle over the n-dimensional manifold $\mathcal{R}/D_P$.

Proof. Statement (a) is clear. Since D_P acts freely on T_P, the projection $N = \mathcal{R} \times_{D_P} T_P \to T_P/D_P$ onto the second factor is a fibre bundle with fibre $\mathcal{R}$. Then (b) follows from the fact that $T_P/D_P \cong T^{m-n}$.

If $N \to \mathbb{C}^m$ is an embedding, then T_P acts freely on $\mathcal{Z}$. The action of D_P on $\mathcal{R}$ is also free. Therefore, the projection $N = \mathcal{R} \times_{D_P} T_P \to \mathcal{R}/D_P$ onto the first factor is a principal T_P-bundle, which proves (c). $\square$

Ex 1 (one quadric)**.** Let $m-n=1$, that is, $\mathcal{R}$ is given by

$$\gamma_1 u_1^2 + \ldots + \gamma_m u_m^2 = c. \qquad (2)$$

If $\mathcal{R}$ is compact, then $\mathcal{R} \cong S^{m-1}$, and

$$N \cong S^{m-1} \times_{\mathbb{Z}/2} S^1 \cong \begin{cases} S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orientation of } S^{m-1}, \\ \mathcal{K}^m & \text{if } \tau \text{ reverses the orientation of } S^{m-1}, \end{cases}$$

where $\mathcal{K}^m$ is an m-*dimensional Klein bottle*.

Prop 4. *We obtain an H-minimal Lagrangian embedding of* $N \cong S^{n-1} \times_{\mathbb{Z}/2} S^1$ *in* $\mathbb{C}^m$ *if and only if* $\gamma_1 = \ldots = \gamma_m$ *in* (2)*. The topological type of* $N = N(m)$ *depends only on the parity of* m*, and is given by*

$$N(m) \cong S^{m-1} \times S^1 \qquad \text{if } m \text{ is even},$$
$$N(m) \cong \mathcal{K}^m \qquad \text{if } m \text{ is odd}.$$

The Klein bottle $\mathcal{K}^m$ with even m does *not* admit Lagrangian embeddings in $\mathbb{C}^m$ [Nemirovsky, Shevchishin].

Ex 2 (two quadrics).

Thm 4. *Let* $m - n = 2$.

(a) $\mathcal{R}_\Gamma$ *is diffeormorphic to* $\mathcal{R}(p,q) \cong S^{p-1} \times S^{q-1}$ *given by*

$$\begin{aligned} &u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \ldots + u_p^2 && = 1, \\ &u_1^2 + \ldots + u_k^2 && +u_{p+1}^2 + \ldots + u_m^2 = 2, \end{aligned} \tag{3}$$

where $p + q = m$, $0 < p < m$ *and* $0 \leqslant k \leqslant p$.

(b) If $N_\Gamma \to \mathbb{C}^m$ *is an embedding, then* N_Γ *is diffeomorphic to*

$$N_k(p,q) = \mathcal{R}(p,q) \times_{\mathbb{Z}/2 \times \mathbb{Z}/2} (S^1 \times S^1), \tag{4}$$

where $R(p,q)$ *is given by* (3) *and the two involutions act on it by*

$$\begin{aligned} \psi_1 \colon (u_1, \ldots, u_m) &\mapsto (-u_1, \ldots, -u_k, -u_{k+1}, \ldots, -u_p, u_{p+1}, \ldots, u_m), \\ \psi_2 \colon (u_1, \ldots, u_m) &\mapsto (-u_1, \ldots, -u_k, u_{k+1}, \ldots, u_p, -u_{p+1}, \ldots, -u_m). \end{aligned} \tag{5}$$

There is a fibration $N_k(p,q) \to S^{q-1} \times_{\mathbb{Z}/2} S^1 = N(q)$ with fibre $N(p)$ (the manifold from the previous Example), which is trivial for $k = 0$.

Ex 3 (three quadrics). In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}$ and $\mathcal{Z}$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest polytope P with $m - n = 3$ is a pentagon. It has many Delzant realisations, for instance,

$$P = \Big\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -x_1+2 \geqslant 0,\ -x_2+2 \geqslant 0,\ -x_1-x_2+3 \geqslant 0\Big\}.$$

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3 \times S^4$.

We therefore obtain an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^5$ which is the total space of a bundle over T^3 with fibre a surface of genus 5.

Manifolds $\mathcal{R}_P$ corresponding to polygons are described as follows.

Prop 5. *Assume that $n = 2$ the 2-dimensional polytope P corresponding to $\mathcal{R}$ is an m-gon. Then $\mathcal{R}$ is an orientable surface S_g of genus $g = 1 + 2^{m-3}(m-4)$.*

The H-minimal Lagrangian submanifold $N \subset \mathbb{C}^m$ corresponding to $\mathcal{R}$ from Proposition 5 is a total space of a bundle over T^{m-2} with fibre S_g. It is an aspherical manifold (for $m \geqslant 4$) whose fundamental group enters the short exact sequence

$$1 \longrightarrow \pi_1(S_g) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z}^{m-2} \longrightarrow 1.$$

For $n > 2$ and $m - n > 3$ the topology of $\mathcal{R}$ and $\mathcal{Z}$ is even more complicated.

Other geometric structures on moment-angle manifolds $\mathcal{Z}_P$ include

- non-Kähler complex-analytic structures [Bosio–Meersseman, P.-Ustinovsky, Tambour]

- T^m-invariant metrics of positive Ricci curvature [Bazaikin]

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