Geometric structures on moment-angle manifolds

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Intersections of quadrics.

Given a set of $m$ vectors

$$\Gamma = \left\{ \gamma_k = (\gamma_{1,k}, \ldots, \gamma_{m-n,k})^t \in \mathbb{R}^{m-n}, \quad k = 1, \ldots, m \right\},$$

and a vector $c = (c_1, \ldots, c_{m-n})^t \in \mathbb{R}^{m-n}$, we consider the following intersections of $m - n$ real quadrics $\mathbb{R}^m$ and $\mathbb{C}^m$:

$$\mathcal{R}_\Gamma = \left\{ u = (u_1, \ldots, u_m) \in \mathbb{R}^m: \sum_{k=1}^{m} \gamma_{jk} u_k^2 = c_j, \quad \text{for } 1 \leq j \leq m - n \right\},$$

$$\mathcal{Z}_\Gamma = \left\{ z = (z_1, \ldots, z_m) \in \mathbb{C}^m: \sum_{k=1}^{m} \gamma_{jk} |z_k|^2 = c_j, \quad \text{for } 1 \leq j \leq m - n \right\}.$$
Prop 1. Intersections of quadrics $\mathcal{R}_\Gamma$ and $\mathcal{Z}_\Gamma$ are nonempty and non-degenerate if and only if the following two conditions are satisfied:

(a) $c \in \sigma\langle \gamma_1, \ldots, \gamma_m \rangle$;

(b) if $c \in \sigma\langle \gamma_{i_1}, \ldots, \gamma_{i_k} \rangle$, then $k \geq m - n$.

Under these conditions, $\mathcal{R}_\Gamma$ and $\mathcal{Z}_\Gamma$ are smooth submanifolds in $\mathbb{R}^m$ and $\mathbb{C}^m$ of dimension $n$ and $m + n$ respectively, and the vectors $\gamma_1, \ldots, \gamma_m$ span $\mathbb{R}^{m-n}$.

From now on we assume that the conditions of Proposition 1 are satisfied. Moreover, we assume that

(c) the vectors $\gamma_1, \ldots, \gamma_m$ generate a lattice $L$ in $\mathbb{R}^{m-n}$.
Let

\[ L^* = \{ \lambda^* \in \mathbb{R}^{m-n} : \langle \lambda^*, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in L \} \]

be the dual lattice.

The torus \( \mathbb{T}^m = \{ (e^{2\pi i \chi_1}, \ldots, e^{2\pi i \chi_m}) \in \mathbb{C}^m \} \), where \( (\chi_1, \ldots, \chi_m) \in \mathbb{R}^m \), acts on \( \mathcal{Z}_\Gamma \) coordinatewise. Similarly, the ‘real torus’ \( (\mathbb{Z}/2)^m \subset \mathbb{T}^m \) (corresponding to \( (\chi_1, \ldots, \chi_m) \in \frac{1}{2}\mathbb{Z}^m \)) acts on \( \mathcal{R}_\Gamma \).

The vectors \( \gamma_i \) define an \( (m-n) \)-dimensional torus subgroup in \( \mathbb{T}^m \) whose lattice of characters is \( L \):

\[ T_\Gamma = \{ (e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle}) \in \mathbb{T}^m \} \cong \mathbb{T}^{m-n}, \]

where \( \varphi \in \mathbb{R}^{m-n} \). We also define

\[ D_\Gamma = \frac{1}{2}L^*/L^* \cong (\mathbb{Z}/2)^{m-n}. \]

Note that \( D_\Gamma \) embeds canonically as a subgroup in \( T_\Gamma = \mathbb{R}^{m-n}/L^* \).
Given a subset $I \subset [m] = \{1, \ldots, m\}$, define the sublattice

$$L_I = \mathbb{Z}\langle \gamma_i : i \notin I \rangle \subset L.$$ 

Given $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, define its zero set as

$$I_u = \{i : u_i = 0\} \subset [m],$$

and define $I_z$ similarly for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$.

A $G$-action is *almost free* if all isotropy subgroups are finite.

**Prop 2.** The torus $T_\Gamma$ acts on $\mathbb{Z}_\Gamma$ almost freely. The isotropy subgroup of $z \in \mathbb{Z}_\Gamma$ is given by $L_{I_z}^*/L^*$, where $L_{I_z} = \mathbb{Z}\langle \gamma_k : k \notin I_z \rangle \subset L$.

**Proof.** An element $(e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle}) \in T_\Gamma$ fixes the given $z \in \mathbb{Z}_\Gamma$ whenever $e^{2\pi i \langle \gamma_k, \varphi \rangle} = 1$ for every $k \notin I_z$. The latter condition is equivalent to $\langle \gamma_k, \varphi \rangle \in \mathbb{Z}$, that is, $\varphi \in L_{I_z}^*$. Since $\varphi \in L^*$ maps to $1 \in T_\Gamma$, the isotropy subgroup of $z$ is indeed $L_{I_z}^*/L^*$. \qed
Lagrangian immersions.

Let \((M, \omega)\) be a symplectic \(2n\)-manifold. An immersion \(i: N \hookrightarrow M\) of an \(n\)-manifold \(N\) is Lagrangian if \(i^*(\omega) = 0\). If \(i\) is an embedding, then \(i(N)\) is a Lagrangian submanifold of \(M\). A vector field \(\xi\) on \(M\) is Hamiltonian if the 1-form \(\omega(\cdot, \xi)\) is exact.

Assume that a compatible Riemannian metric is chosen on \(M\). A Lagrangian immersion \(i: N \hookrightarrow M\) is Hamiltonian minimal (\(H\)-minimal) if the variations of the volume of \(i(N)\) along all Hamiltonian vector fields with compact support are zero, that is,

\[
\frac{d}{dt} \left. \text{vol}(i_t(N)) \right|_{t=0} = 0,
\]

where \(i_0(N) = i(N)\), \(i_t(N)\) is a deformation of \(i(N)\) along a Hamiltonian vector field, and \(\text{vol}(i_t(N))\) is the volume of the deformed part of \(i_t(N)\). An immersion is minimal if the variations of the volume of \(i(N)\) along all vector fields are zero.
Consider the map

\[ j: \mathcal{R}_\Gamma \times T_\Gamma \rightarrow \mathbb{C}^m, \]

\[ (u, \varphi) \mapsto u \cdot \varphi = (u_1 e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, u_m e^{2\pi i \langle \gamma_m, \varphi \rangle}). \]

Note that \( j(\mathcal{R}_\Gamma \times T_\Gamma) \subset \mathcal{Z}_\Gamma \). The quotient

\[ N_\Gamma = \mathcal{R}_\Gamma \times_{D_\Gamma} T_\Gamma \]

is an \( m \)-dimensional manifold.

**Lemma 1.** The map \( j: \mathcal{R}_\Gamma \times T_\Gamma \rightarrow \mathbb{C}^m \) induces an immersion \( i_\Gamma: N_\Gamma \hookrightarrow \mathbb{C}^m \).

**Thm 1 (Mironov).** The immersion \( i_\Gamma: N_\Gamma \hookrightarrow \mathbb{C}^m \) is H-minimal Lagrangian. Moreover, if \( \sum_{k=1}^m \gamma_k = 0 \), then \( i_\Gamma \) is a minimal Lagrangian immersion.
Lagrangian embeddings and moment-angle manifolds.

**Thm 2.** The following conditions are equivalent:

1. $i_\Gamma: N_\Gamma \to \mathbb{C}^m$ is an embedding of an $H$-minimal Lagrangian submanifold;
2. $L_{Iu} = L$ for every $u \in \mathcal{R}_\Gamma$;
3. $T_\Gamma$ acts on $Z_\Gamma$ freely.

This result opens a way to construct explicitly new families of $H$-minimal Lagrangian submanifolds, once we have an effective method to produce nondegenerate intersections of quadrics $\mathcal{R}_\Gamma$ satisfying conditions (2) or (3) of Theorem 2. Toric topology provides such a method.
The quotient of $\mathcal{R}_\Gamma$ by the action of $(\mathbb{Z}/2)^m$ (or the quotient of $\mathbb{Z}_\Gamma$ by the action of $\mathbb{T}^m$) is identified with the set of nonnegative solutions of the following system of $m - n$ linear equations:

$$\sum_{k=1}^{m} \gamma_k y_k = c.$$ 

This set may be described as a convex polyhedron obtained by intersecting $m$ halfspaces in $\mathbb{R}^n$:

$$P = \left\{ x \in \mathbb{R}^n : \langle a_i, x \rangle + b_i \geq 0 \quad \text{for } i = 1, \ldots, m \right\}, \quad (1)$$

Note that $P$ may be unbounded; in fact $P$ is bounded if and only if $\mathcal{R}_\Gamma$ is bounded (compact). Bounded polyhedra are known as polytopes.

We refer to (1) as a presentation of $P$ by inequalities. A presentation is generic if $P$ is $n$-dimensional, has at least one vertex, and the hyperplanes defined by the equations $\langle a_i, x \rangle + b_i = 0$ are in general position at every point of $P$. If $P$ is a polytope, then the existence of a generic presentation implies that $P$ is simple.
Given a generic presentation of a polyhedron $P$, we may reconstruct the intersections of quadrics $\mathcal{R}_\Gamma$ and $\mathcal{Z}_\Gamma$ as follows.

Consider the affine map

$$i_P : \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(x) = \left(\langle a_1, x \rangle + b_1, \ldots, \langle a_m, x \rangle + b_m \right).$$

It is monomorphic onto a certain $n$-dimensional plane in $\mathbb{R}^m$ (because $P$ has a vertex), and $i_P(P)$ is the intersection of this plane with $\mathbb{R}^m$.

We define the space $\mathcal{Z}_P$ from the commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\
\downarrow & & \downarrow \mu \\
P & \xrightarrow{i_P} & \mathbb{R}^m \\
\end{array}$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. Note that $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient $P$, and $i_Z$ is a $\mathbb{T}^m$-equivariant embedding.
If the presentation of $P$ is generic, then $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$, known as the (polytopal) moment-angle manifold corresponding to $P$.

Now we can write the $n$-dimensional plane $i_P(\mathbb{R}^n)$ by $m - n$ linear equations in $\mathbb{R}^m$. Replacing each $y_k$ by $|z_k|^2$ we obtain a presentation of the moment-angle manifold $\mathcal{Z}_P$ as an intersection of quadrics.

By replacing $\mathbb{C}^m$ by $\mathbb{R}^m$ we obtain the real moment-angle manifold $\mathcal{R}_P$. 
Clearly, $\gamma_1, \ldots, \gamma_m$ generate a lattice $L$ in $\mathbb{R}^{m-n}$ if and only if $a_1, \ldots, a_m$ generate a lattice $\Lambda$ in $\mathbb{R}^n$. The corresponding $P$ are called rational.

If $P$ is rational, then we have a map of lattices

$$A_P: \Lambda^* \to \mathbb{Z}^m, \quad \mathbf{x} \mapsto (\langle a_1, \mathbf{x} \rangle, \ldots, \langle a_m, \mathbf{x} \rangle).$$

Its conjugate gives rise to a map of tori $\mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^n/\Lambda$, whose kernel we denote by $T_P$. It becomes $T_{\Gamma}$ under the identification of $\mathbb{Z}_P$ with $\mathbb{Z}_{\Gamma}$. We also have $D_P \cong (\mathbb{Z}/2)^{m-n}$ and $N_P = \mathcal{R}_P \times_{D_P} T_P$.

The manifolds $\mathcal{R}_P, \mathbb{Z}_P, N_P$ represent the same geometric objects as $\mathcal{R}_{\Gamma}, \mathbb{Z}_{\Gamma}, N_{\Gamma}$, although a different initial data is used in their definition.

$P$ is Delzant if it is rational and for every vertex $\mathbf{x} \in P$ the vectors $a_{j_1}, \ldots, a_{j_n}$ normal to the facets meeting at $\mathbf{x}$ constitute a basis of $\Lambda = \mathbb{Z}\langle a_1, \ldots, a_m \rangle$.

**Thm 3.** The map $N_P = \mathcal{R}_P \times_{D_P} T_P \to \mathbb{C}^m$ is an embedding if and only if $P$ is a Delzant polyhedron.
Topology of Lagrangian submanifolds $N$.

Toric topology provides large families of explicitly constructed Delzant polytopes:

- simplices and cubes in all dimensions;
- products and face cuts;
- *associahedra* (Stasheff polytopes), *permutahedra*, and general *nestohedra*.

Nevertheless, the topology of $\mathcal{Z}_P$ (and therefore of $N_P$) is very complicated in general. Cohomology rings of $\mathcal{Z}_P$ are described by [Buchstaber-P.], and explicit homotopy and diffeomorphism type for some particular families of $P$ are given by [Bahri–Bendersky–Cohen–Gitler], [Gitler–Lopez de Medrano], [Grbić–Theriault], and others.
Prop 3. (a) The immersion of $N$ in $\mathbb{C}^m$ factors as $N \hookrightarrow \mathcal{Z} \hookrightarrow \mathbb{C}^m$;

(b) $N$ is the total space of a bundle over a torus $T^{m-n}$ with fibre $\mathcal{R}$;

(c) if $N \to \mathbb{C}^m$ is an embedding, then $N$ is the total space of a principal $T^{m-n}$-bundle over the $n$-dimensional manifold $\mathcal{R}/D_P$.

Proof. Statement (a) is clear. Since $D_P$ acts freely on $T_P$, the projection $N = \mathcal{R} \times_{D_P} T_P \to T_P/D_P$ onto the second factor is a fibre bundle with fibre $\mathcal{R}$. Then (b) follows from the fact that $T_P/D_P \cong T^{m-n}$.

If $N \to \mathbb{C}^m$ is an embedding, then $T_P$ acts freely on $\mathcal{Z}$. The action of $D_P$ on $\mathcal{R}$ is also free. Therefore, the projection $N = \mathcal{R} \times_{D_P} T_P \to \mathcal{R}/D_P$ onto the first factor is a principal $T_P$-bundle, which proves (c). $\square$
\textbf{Ex 1} (one quadric). Let \( m - n = 1 \), that is, \( \mathcal{R} \) is given by
\[
\gamma_1 u_1^2 + \ldots + \gamma_m u_m^2 = c. \tag{2}
\]
If \( \mathcal{R} \) is compact, then \( \mathcal{R} \cong S^{m-1} \), and
\[
N \cong S^{m-1} \times \mathbb{Z}/2 S^1 \cong \begin{cases} 
S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orientation of } S^{m-1}, \\
\mathcal{K}^m & \text{if } \tau \text{ reverses the orientation of } S^{m-1},
\end{cases}
\]
where \( \mathcal{K}^m \) is an \( m \)-dimensional Klein bottle.

\textbf{Prop 4.} We obtain an H-minimal Lagrangian embedding of \( N \cong S^{m-1} \times \mathbb{Z}/2 S^1 \) in \( \mathbb{C}^m \) if and only if \( \gamma_1 = \ldots = \gamma_m \) in (2). The topological type of \( N = N(m) \) depends only on the parity of \( m \), and is given by
\[
N(m) \cong S^{m-1} \times S^1 \quad \text{if } m \text{ is even},
N(m) \cong \mathcal{K}^m \quad \text{if } m \text{ is odd}.
\]

The Klein bottle \( \mathcal{K}^m \) with even \( m \) does \textit{not} admit Lagrangian embeddings in \( \mathbb{C}^m \) [Nemirovsky, Shevchishin].
Ex 2 (two quadrics).

Thm 4. Let $m - n = 2$.

(a) $\mathcal{R}_\Gamma$ is diffeomorphic to $\mathcal{R}(p, q) \cong S^{p-1} \times S^{q-1}$ given by

$$u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \ldots + u_p^2 = 1,$$

$$u_1^2 + \ldots + u_k^2 + u_{p+1}^2 + \ldots + u_m^2 = 2,$$

where $p + q = m$, $0 < p < m$ and $0 \leq k \leq p$.

(b) If $N_\Gamma \to \mathbb{C}^m$ is an embedding, then $N_\Gamma$ is diffeomorphic to

$$N_k(p, q) = \mathcal{R}(p, q) \times \mathbb{Z}/2 \times \mathbb{Z}/2 (S^1 \times S^1),$$

where $R(p, q)$ is given by (3) and the two involutions act on it by

$$\psi_1: (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, -u_{k+1}, \ldots, -u_p, u_{p+1}, \ldots, u_m),$$

$$\psi_2: (u_1, \ldots, u_m) \mapsto (-u_1, \ldots, -u_k, u_{k+1}, \ldots, u_p, -u_{p+1}, \ldots, -u_m).$$

There is a fibration $N_k(p, q) \to S^{q-1} \times \mathbb{Z}/2 S^1 = N(q)$ with fibre $N(p)$ (the manifold from the previous Example), which is trivial for $k = 0$.  

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**Ex 3** (three quadrics). In the case $m - n = 3$ the topology of compact manifolds $\mathcal{R}$ and $\mathcal{Z}$ was fully described by [Lopez de Medrano]. Each manifold is diffeomorphic to a product of three spheres, or to a connected sum of products of spheres, with two spheres in each product.

The simplest polytope $P$ with $m - n = 3$ is a pentagon. It has many Delzant realisations, for instance,

$$P = \{(x_1, x_2) \in \mathbb{R}^2: x_1 \geq 0, x_2 \geq 0, -x_1 + 2 \geq 0, -x_2 + 2 \geq 0, -x_1 - x_2 + 3 \geq 0\}$$

In this case $\mathcal{R}_P$ is an oriented surface of genus 5, and $\mathcal{Z}_P$ is diffeomorphic to a connected sum of 5 copies of $S^3 \times S^4$.

We therefore obtain an H-minimal Lagrangian submanifold $N_P \subset \mathbb{C}^5$ which is the total space of a bundle over $T^3$ with fibre a surface of genus 5.
Manifolds $\mathcal{R}_P$ corresponding to polygons are described as follows.

**Prop 5.** Assume that $n = 2$ the 2-dimensional polytope $P$ corresponding to $\mathcal{R}$ is an $m$-gon. Then $\mathcal{R}$ is an orientable surface $S_g$ of genus $g = 1 + 2^{m-3}(m - 4)$.

The H-minimal Lagrangian submanifold $N \subset \mathbb{C}^m$ corresponding to $\mathcal{R}$ from Proposition 5 is a total space of a bundle over $T^{m-2}$ with fibre $S_g$. It is an aspherical manifold (for $m \geq 4$) whose fundamental group enters the short exact sequence

$$1 \to \pi_1(S_g) \to \pi_1(N) \to \mathbb{Z}^{m-2} \to 1.$$ 

For $n > 2$ and $m - n > 3$ the topology of $\mathcal{R}$ and $\mathcal{Z}$ is even more complicated.
Other geometric structures on moment-angle manifolds $\mathcal{Z}_P$ include

- non-Kähler complex-analytic structures [Bosio–Meersseman, P.-Ustinovsky, Tambour]

- $T^m$-invariant metrics of positive Ricci curvature [Bazaikin]

