

Complex-analytic structures on moment-angle manifolds

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1. Moment-angle complexes and manifolds.

\mathcal{K} an (abstract) **simplicial complex** on the set $[m] = \{1, \dots, m\}$.

$I = \{i_1, \dots, i_k\} \in \mathcal{K}$ a **simplex**. Always assume $\emptyset \in \mathcal{K}$.

Allow $\{i\} \notin \mathcal{K}$ for some i (**ghost vertices**).

Consider the unit polydisc in \mathbb{C}^m ,

$$\mathbb{D}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| \leq 1, \quad i = 1, \dots, m\}.$$

Given $I \subset [m]$, set

$$B_I := \{(z_1, \dots, z_m) \in \mathbb{D}^m : |z_j| = 1 \text{ for } j \notin I\}.$$

Following **[BP]** define the **moment-angle complex**

$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| = 1, \quad i = 1, \dots, m\}$$

on \mathbb{C}^m .

Constr 1 (\mathcal{K} -power). Let X be a space, and W a subspace of X . Given $I \subset [m]$, set

$$(X, W)^I = \left\{ (x_1, \dots, x_m) \in X^m : x_j \in W \text{ for } j \notin I \right\} \cong \prod_{i \in I} X \times \prod_{i \notin I} W,$$

and define the \mathcal{K} -power (also known as the **polyhedral product**) of (X, W) as

$$(X, W)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (X, W)^I \subset X^m.$$

Then $\mathcal{Z}_{\mathcal{K}} = (\mathbb{D}, \mathbb{T})^{\mathcal{K}}$, where \mathbb{T} is the unit circle.

Another important example is the complement of the **coordinate subspace arrangement** corresponding to \mathcal{K} :

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_k\} \notin \mathcal{K}} \{z \in \mathbb{C}^m : z_{i_1} = \dots = z_{i_k} = 0\},$$

namely,

$$U(\mathcal{K}) = (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}},$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Clearly, $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$. Moreover, $\mathcal{Z}_{\mathcal{K}}$ is a \mathbb{T}^m -equivariant deformation retract of $U(\mathcal{K})$ for every \mathcal{K} [**BP**, Th. 8.9].

Prop 1 ([**BP**]). Assume $|\mathcal{K}| \cong S^{n-1}$ (a sphere triangulation with m vertices). Then $\mathcal{Z}_{\mathcal{K}}$ is a closed manifold of dimension $m + n$.

We refer to such $\mathcal{Z}_{\mathcal{K}}$ as **moment-angle manifolds**.

If $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a **simple convex polytope** P , then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ embeds in \mathbb{C}^m as a nondegenerate (transverse) intersection of $m - n$ real quadratic hypersurfaces [**BM**], [**BP**]. Therefore, \mathcal{Z}_P can be smoothed canonically.

Now we shall look at a wider class of simplicial complexes \mathcal{K} : **starshaped spheres**, or underlying complexes of **complete simplicial fans**.

A set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{R}^n$ generates a convex polyhedral **cone**

$$\sigma = \{\mu_1 \mathbf{a}_1 + \dots + \mu_k \mathbf{a}_k : \mu_i \in \mathbb{R}, \mu_i \geq 0\}.$$

A cone is **rational** if its generators can be chosen from $\mathbb{Z}^n \subset \mathbb{R}^n$, and is **strongly convex** if it does not contain a line. A cone is **simplicial** (respectively, **regular**) if it is generated by a part of basis of \mathbb{R}^n (respectively, \mathbb{Z}^n).

A **fan** is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex cones in \mathbb{R}^n such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is **rational** (respectively, **simplicial**, **regular**) if every cone in Σ is rational (respectively, simplicial, regular). A fan $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ is **complete** if $\sigma_1 \cup \dots \cup \sigma_s = \mathbb{R}^n$.

Let Σ be a simplicial fan in \mathbb{R}^n with m one-dimensional cones generated by $\mathbf{a}_1, \dots, \mathbf{a}_m$. Its **underlying simplicial complex** is

$$\mathcal{K}_\Sigma = \left\{ I \subset [m] : \{\mathbf{a}_i : i \in I\} \text{ spans a cone of } \Sigma \right\}$$

Note: Σ is complete iff $|\mathcal{K}_\Sigma|$ is a triangulation of S^{n-1} .

Now consider the linear map

$$\Lambda_{\mathbb{R}}: \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad \mathbf{e}_i \mapsto \mathbf{a}_i,$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ is the standard basis of \mathbb{R}^m . Set

$$\mathbb{R}_{>}^m = \{(y_1, \dots, y_m) \in \mathbb{R}^m : y_i > 0\},$$

and define

$$R_{\Sigma} := \exp(\text{Ker } \Lambda_{\mathbb{R}}) = \left\{ (y_1, \dots, y_m) \in \mathbb{R}_{>}^m : \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u} \rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n \right\},$$

Note: $R_{\Sigma} \cong \mathbb{R}_{>}^{m-n}$ if Σ is complete (or contains an n -dimensional cone).

Both $\mathbb{R}_{>}^m$ and its subgroup R_{Σ} act on the complement $U(\mathcal{K}_{\Sigma}) \subset \mathbb{C}^m$ by coordinatewise multiplications.

Thm 1. *Let Σ be a complete simplicial fan in \mathbb{R}^n with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_{\Sigma}$ be its underlying simplicial complex. Then*

- (a) *the group R_{Σ} acts on $U(\mathcal{K})$ freely and properly, and the quotient $U(\mathcal{K})/R_{\Sigma}$ has a canonical structure of a smooth $(m+n)$ -dimensional manifold;*
- (b) *$U(\mathcal{K})/R_{\Sigma}$ is \mathbb{T}^m -equivariantly homeomorphic to $\mathcal{Z}_{\mathcal{K}}$.*

Therefore, $\mathcal{Z}_{\mathcal{K}}$ can be smoothed canonically.

Rem 1. The construction of the smooth structure on $\mathcal{Z}_{\mathcal{K}_\Sigma}$ from Thm 1 *does* depend on the geometry of the fan Σ .

However, we expect that the smooth structures coming from fans Σ and Σ' are equivalent whenever the underlying simplicial complexes \mathcal{K}_Σ and $\mathcal{K}_{\Sigma'}$ are the same. This question is equivalent to that the quotients $\mathcal{Z}_{\mathcal{K}_\Sigma}/\mathbb{T}^m$ and $\mathcal{Z}_{\mathcal{K}_{\Sigma'}}/\mathbb{T}^m$ are diffeomorphic as **manifolds with corners** whenever $\mathcal{K}_\Sigma = \mathcal{K}_{\Sigma'}$. It is true in the polytopal case, and also for those fans Σ which are **shellable**.

Question 1. *Describe the class of sphere triangulations \mathcal{K} for which the moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$ admits a smooth structure.*

2. Complex-analytic structures.

We shall show that the even-dimensional moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$ corresponding to a complete simplicial fan admits a structure of a complex manifold. The idea is to replace the action of $\mathbb{R}_{>}^{m-n}$ on $U(\mathcal{K})$ (whose quotient is $\mathcal{Z}_{\mathcal{K}}$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

Assume $m - n$ is even from now on. We can always achieve this by formally adding an ‘empty’ one-dimensional cone to Σ ; this corresponds to adding a ghost vertex to \mathcal{K} , or multiplying $\mathcal{Z}_{\mathcal{K}}$ by a circle. The column of matrix $\Lambda_{\mathbb{R}}$ corresponding to the ‘empty’ 1-cone is set to be zero.

Set $\ell = \frac{m-n}{2}$.

Constr 2. Choose a linear map $\Psi: \mathbb{C}^\ell \rightarrow \mathbb{C}^m$ satisfying two conditions:

(a) $\text{Re} \circ \Psi: \mathbb{C}^\ell \rightarrow \mathbb{R}^m$ is a monomorphism.

(b) $\Lambda_{\mathbb{R}} \circ \text{Re} \circ \Psi = 0$.

This corresponds to choosing a complex structure and specifying a complex basis in the real vector space $\text{Ker } \Lambda_{\mathbb{R}} \cong \mathbb{R}^{2\ell}$. We also obtain that the composite map of the top line in the following diagram is zero:

$$\begin{array}{ccccccc}
 \mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\text{Re}} & \mathbb{R}^m & \xrightarrow{\Lambda_{\mathbb{R}}} & \mathbb{R}^n \\
 & & \downarrow \text{exp} & & \downarrow \text{exp} & & \downarrow \text{exp} \\
 & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}_{>}^m & \xrightarrow{\text{exp } \Lambda_{\mathbb{R}}} & \mathbb{R}_{>}^n
 \end{array}$$

where $|\cdot|$ denotes the map $(z_1, \dots, z_m) \mapsto (|z_1|, \dots, |z_m|)$. Now set

$$C_{\Psi, \Sigma} = \text{exp } \Psi(\mathbb{C}^\ell) = \left\{ \left(e^{\langle \psi_1, \mathbf{w} \rangle}, \dots, e^{\langle \psi_m, \mathbf{w} \rangle} \right) \in (\mathbb{C}^\times)^m \right\}$$

where $\mathbf{w} = (w_1, \dots, w_\ell) \in \mathbb{C}^\ell$, ψ_i denotes the i th row of the $m \times \ell$ -matrix $\Psi = (\psi_{ij})$.

Then $C_{\Psi, \Sigma} \cong \mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

Ex 1. Let \mathcal{K} be empty on 2 elements (that is, \mathcal{K} has two ghost vertices). We therefore have $n = 0$, $m = 2$, $\ell = 1$, and $\Lambda_{\mathbb{R}}: \mathbb{R}^2 \rightarrow 0$ is a zero map. Let $\Psi: \mathbb{C} \rightarrow \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$C = C_{\Psi, \Sigma} = \{(e^z, e^{\alpha z})\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) of Constr 2 is void, while (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp \Psi: \mathbb{C} \rightarrow (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2 / C$ with the natural complex structure is a complex torus $T_{\mathbb{C}}^2$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2 / C \cong \mathbb{C} / (\mathbb{Z} \oplus \alpha \mathbb{Z}) = T_{\mathbb{C}}^2(\alpha).$$

Similarly, if \mathcal{K} is empty on 2ℓ elements (so that $n = 0$, $m = 2\ell$), we may obtain any complex torus $T_{\mathbb{C}}^{2\ell}$ as the quotient $(\mathbb{C}^\times)^{2\ell} / C_{\Psi, \Sigma}$ [Meersseman].

Thm 2. *Let Σ be a complete simplicial fan in \mathbb{R}^n with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_{\Sigma}$ be its underlying simplicial complex. Assume that $m - n = 2\ell$. Then*

- (a) *the holomorphic action of the group $C_{\Psi, \Sigma}$ on $U(\mathcal{K})$ is free and proper, and the quotient $U(\mathcal{K}) / C_{\Psi, \Sigma}$ has a canonical structure of a compact complex manifold of complex dimension $m - \ell$;*
- (b) *there is a \mathbb{T}^m -equivariant diffeomorphism $U(\mathcal{K}) / C_{\Psi, \Sigma} \cong \mathcal{Z}_{\mathcal{K}}$ defining a complex structure on $\mathcal{Z}_{\mathcal{K}}$ in which \mathbb{T}^m acts holomorphically.*

Rem 2. Unlike the smooth structure, the complex structure on $\mathcal{Z}_{\mathcal{K}}$ depends on both the geometry of Σ and the choice of Ψ . (The latter is already clear from the torus example (Ex 1).

Ex 2 (Hopf manifold). Let Σ be the complete fan in \mathbb{R}^n whose cones are generated by all proper subsets of $n + 1$ vectors $\mathbf{e}_1, \dots, \mathbf{e}_n, -\mathbf{e}_1 - \dots - \mathbf{e}_n$.

To make $m - n$ even we add one ‘empty’ 1-cone. We have $m = n + 2$, $\ell = 1$. Then $\Lambda_{\mathbb{R}}: \mathbb{R}^{n+2} \rightarrow \mathbb{R}^n$ is given by the matrix $(\mathbf{0} \ I \ -\mathbf{1})$, where I is the unit $n \times n$ matrix, and $\mathbf{0}$, $\mathbf{1}$ are the n -columns of zeros and units respectively.

We have that \mathcal{K} is the boundary of an n -dim simplex with $n + 1$ vertices and 1 ghost vertex, $\mathcal{Z}_{\mathcal{K}} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^{\times} \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\Psi: \mathbb{C} \rightarrow \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \dots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{R}$. Then

$$C = C_{\Psi, \Sigma} = \{(e^z, e^{\alpha z}, \dots, e^{\alpha z}) : z \in \mathbb{C}\} \subset (\mathbb{C}^{\times})^{n+2},$$

and $\mathcal{Z}_{\mathcal{K}}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^{\times} \times (\mathbb{C}^{n+1} \setminus \{0\}) / \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z} \mathbf{w})\} \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \{\mathbf{w} \sim e^{2\pi i \alpha} \mathbf{w}\},$$

where $t \in \mathbb{C}^{\times}$, $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. The latter quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ is known as the **Hopf manifold**.

3. Holomorphic bundles over toric varieties and Hodge numbers.

Manifolds $\mathcal{Z}_{\mathcal{K}}$ corresponding to complete *regular* simplicial fans are total spaces of **holomorphic principal bundles** over **toric varieties** with fibre a complex torus, by a generalisation of the construction of **Meersseman and Verjovsky**. This allows us to calculate invariants of complex structures on $\mathcal{Z}_{\mathcal{K}}$.

A **toric variety** is a normal algebraic variety X on which an algebraic torus $(\mathbb{C}^{\times})^n$ acts with a dense orbit.

Toric varieties are classified by rational fans. Under this correspondence,

complete fans	\longleftrightarrow	compact varieties
normal fans of polytopes	\longleftrightarrow	projective varieties
regular fans	\longleftrightarrow	nonsingular varieties
simplicial fans	\longleftrightarrow	orbifolds

Σ complete, simplicial, rational;

$\mathbf{a}_1, \dots, \mathbf{a}_m$ primitive integral generators of 1-cones.

Constr 3 ('Cox construction'). Let $\Lambda_{\mathbb{C}}: \mathbb{C}^m \rightarrow \mathbb{C}^n$, $\mathbf{e}_i \mapsto \mathbf{a}_i$,

$$\exp \Lambda_{\mathbb{C}}: (\mathbb{C}^{\times})^m \rightarrow (\mathbb{C}^{\times})^n,$$

$$(z_1, \dots, z_m) \mapsto \left(\prod_{i=1}^m z_i^{a_{i1}}, \dots, \prod_{i=1}^m z_i^{a_{in}} \right)$$

Set $G_{\Sigma} = \text{Ker } \exp \Lambda_{\mathbb{C}}$.

This is an $(m - n)$ -dimensional algebraic subgroup in $(\mathbb{C}^{\times})^m$.

It acts almost freely (with finite isotropy subgroups) on $U(\mathcal{K}_{\Sigma})$.

If Σ is regular, then $G_{\Sigma} \cong (\mathbb{C}^{\times})^{m-n}$ and the action is free.

$X_{\Sigma} = U(\mathcal{K}_{\Sigma})/G_{\Sigma}$ the **toric variety** associated to Σ .

The quotient torus $(\mathbb{C}^{\times})^m/G_{\Sigma} \cong (\mathbb{C}^{\times})^n$ acts on X_{Σ} with a dense orbit.

Observe that $C_{\psi, \Sigma} \subset G_{\Sigma}$ as a complex ℓ -dimensional subgroup.

Prop 2.

- (a) *The toric variety X_{Σ} is homeomorphic to the quotient of $\mathcal{Z}_{\mathcal{K}_{\Sigma}}$ by the holomorphic action of $G_{\Sigma}/C_{\psi, \Sigma}$.*
- (b) *If Σ is regular, then there is a holomorphic principal bundle $\mathcal{Z}_{\mathcal{K}_{\Sigma}} \rightarrow X_{\Sigma}$ with fibre the compact complex torus $G_{\Sigma}/C_{\psi, \Sigma}$ of dimension ℓ .*

Rem 3. For singular varieties X_{Σ} the quotient projection $\mathcal{Z}_{\mathcal{K}_{\Sigma}} \rightarrow X_{\Sigma}$ is a holomorphic principal **Seifert bundle** for an appropriate orbifold structure on X_{Σ} (same as in the projective case of [[Meersseman–Verjovsky](#)]).

Given a complex n -dimensional manifold M , there is a decomposition $\Omega_{\mathbb{C}}^*(M) = \bigoplus \Omega^{p,q}(M)$ of the space of differential \mathbb{C} -forms on M into a direct sum of the subspaces of (p, q) -forms for $1 \leq p, q \leq n$, and the **Dolbeault differential** $\bar{\partial}: \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M)$.

$h^{p,q}(M) = \dim H_{\bar{\partial}}^{p,q}(M)$: the **Hodge numbers** of M .

The Dolbeault cohomology of a complex torus is given by

$$H_{\bar{\partial}}^{*,*}(T_{\mathbb{C}}^{2\ell}) \cong \wedge[\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell],$$

where $\xi_1, \dots, \xi_\ell \in H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell})$, $\eta_1, \dots, \eta_\ell \in H_{\bar{\partial}}^{0,1}(T_{\mathbb{C}}^{2\ell})$. Hence, $h^{p,q}(T_{\mathbb{C}}^{2\ell}) = \binom{\ell}{p} \binom{\ell}{q}$.

The Dolbeault cohomology of a complete nonsingular toric variety X_{Σ} is given by [**Danilov–Jurkiewicz**]:

$$H_{\bar{\partial}}^{*,*}(X_{\Sigma}) \cong \mathbb{C}[v_1, \dots, v_m] / (\mathcal{I}_{\mathcal{K}_{\Sigma}} + \mathcal{J}_{\Sigma}),$$

where $v_i \in H_{\bar{\partial}}^{1,1}(X_{\Sigma})$,

$\mathcal{I}_{\mathcal{K}_{\Sigma}} = (v_{i_1} \cdots v_{i_k} : \{i_1, \dots, i_k\} \notin \mathcal{K}_{\Sigma})$ (the **Stanley–Reisner ideal**),

$\mathcal{J}_{\Sigma} = (\sum_{k=1}^m a_{kj} v_k, \quad 1 \leq j \leq n)$.

We have $h^{p,p}(X_{\Sigma}) = h_p$, where (h_0, h_1, \dots, h_n) is the **h -vector** of \mathcal{K}_{Σ} , and $h^{p,q}(X_{\Sigma}) = 0$ for $p \neq q$.

By an application of the **Borel spectral sequence** to the holomorphic bundle $\mathcal{Z}_{\mathcal{K}} \rightarrow X_{\Sigma}$ we obtain the following description of the Dolbeault cohomology.

Thm 3. *Let Σ be a complete rational nonsingular fan. Then the Dolbeault cohomology group $H_{\bar{\partial}}^{p,q}(\mathcal{Z}_{\mathcal{K}})$ is isomorphic to the (p, q) -th cohomology group of the differential bigraded algebra*

$$\left[\wedge[\xi_1, \dots, \xi_{\ell}, \eta_1, \dots, \eta_{\ell}] \otimes H_{\bar{\partial}}^{*,*}(X_{\Sigma}), d \right]$$

whose differential d of bidegree $(0, 1)$ is defined on the generators as

$$dv_i = d\eta_j = 0, \quad d\xi_j = c(\xi_j), \quad 1 \leq i \leq m, \quad 1 \leq j \leq \ell,$$

where $c: H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{2\ell}) \rightarrow H^2(X_{\Sigma}, \mathbb{C}) = H_{\bar{\partial}}^{1,1}(X_{\Sigma})$ is the first Chern class map of the torus principal bundle $\mathcal{Z}_{\mathcal{K}} \rightarrow X_{\Sigma}$.

Thm 4. Let $\mathcal{Z}_{\mathcal{K}}$ be as in Thm 3, and let k be the number of ghost vertices in \mathcal{K} . Then the Hodge numbers $h^{p,q} = h^{p,q}(\mathcal{Z}_{\mathcal{K}})$ satisfy

$$(a) \binom{k-\ell}{p} \leq h^{p,0} \leq \binom{\lfloor k/2 \rfloor}{p} \text{ for } p \geq 0;$$

$$(b) h^{0,q} = \binom{\ell}{q} \text{ for } q \geq 0;$$

$$(c) h^{1,q} = (\ell - k) \binom{\ell}{q-1} + h^{1,0} \binom{\ell+1}{q} \text{ for } q \geq 1;$$

$$(d) \frac{\ell(3\ell+1)}{2} - h_2(\mathcal{K}) - \ell k + (\ell + 1)h^{2,0} \leq h^{2,1} \leq \frac{\ell(3\ell+1)}{2} - \ell k + (\ell + 1)h^{2,0}.$$

Rem 4. At most one ghost vertex is required to make $\dim \mathcal{Z}_{\mathcal{K}} = m + n$ even. Note that $k \leq 1$ implies $h^{p,0}(\mathcal{Z}_{\mathcal{K}}) = 0$, so that $\mathcal{Z}_{\mathcal{K}}$ does not have holomorphic forms of any degree in this case.

If $\mathcal{Z}_{\mathcal{K}}$ is a torus, then $m = k = 2\ell$, and $h^{1,0}(\mathcal{Z}_{\mathcal{K}}) = h^{0,1}(\mathcal{Z}_{\mathcal{K}}) = \ell$. Otherwise Thm 4 implies that $h^{1,0}(\mathcal{Z}_{\mathcal{K}}) < h^{0,1}(\mathcal{Z}_{\mathcal{K}})$, and therefore $\mathcal{Z}_{\mathcal{K}}$ is not Kähler (this was observed by [Meersseman] in the polytopal case).

Ex 3 (Calabi–Eckmann manifold). Let $X_\Sigma = \mathbb{C}P^p \times \mathbb{C}P^q$ with $p \leq q$, so $n = p+q$, $m = n + 2$ and $\ell = 1$. The cohomology ring is $\mathbb{C}[x, y]/(x^{p+1}, y^{q+1})$.

Choose $\Psi = (1, \dots, 1, \alpha, \dots, \alpha)^t$ where the number of units is $p + 1$ and $\alpha \notin \mathbb{R}$.

This provides $Z_K \cong S^{2p+1} \times S^{2q+1}$ with a structure of a complex manifold.

It is the total space of a holomorphic principal bundle over $\mathbb{C}P^p \times \mathbb{C}P^q$ with fibre a complex torus $\mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z})$, a **Calabi–Eckmann manifold** $CE(p, q)$.

By Thm 3, $H_{\bar{\partial}}^{*,*}(CE(p, q)) \cong H[\Lambda[\xi, \eta] \otimes \mathbb{C}[x, y]/(x^{p+1}, y^{q+1}), d]$,

where $dx = dy = d\eta = 0$ and $d\xi = x - y$ for an appropriate choice of x, y .

We therefore obtain

$$H_{\bar{\partial}}^{*,*}(CE(p, q)) \cong \Lambda[\omega, \eta] \otimes \mathbb{C}[x]/(x^{p+1}),$$

where $\omega \in H_{\bar{\partial}}^{q+1, q}(CE(p, q))$ is the cohomology class of the cocycle $\xi \frac{x^{q+1} - y^{q+1}}{x - y}$.

This calculation is originally due to **Borel**.

Ex 4. The product $S^3 \times S^3 \times S^5 \times S^5$ has two complex structures as a product of Calabi–Eckmann manifolds, namely, $CE(1, 1) \times CE(2, 2)$ and $CE(1, 2) \times CE(1, 2)$.

In the first case $h^{2,1} = 1$, and $h^{2,1} = 0$ in the second.

- [BM] Frédéric Bosio and Laurent Meersseman. *Real quadrics in \mathbb{C}^n , complex manifolds and convex polytopes*. Acta Math. **197** (2006), no. 1, 53–127.
- [BP] Victor Buchstaber and Taras Panov. *Torus Actions and Their Applications in Topology and Combinatorics*. University Lecture Series, vol. **24**, Amer. Math. Soc., Providence, R.I., 2002.
- [PU] Taras Panov and Yuri Ustinovsky. *Complex-analytic structures on moment-angle manifolds*. Preprint (2010), no. 2; arXiv:1008.4764.