The Universal Toric Genus

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Motivations: non-equivariant cobordism and torus actions.

Thm [Buchstaber-P.-Ray]. *Every complex cobordism class in* dim > 2 *contains a quasitoric manifold.*

In other words, every stably complex manifold is cobordant to a manifold with a nicely behaving torus action.

All manifolds are smooth and closed, unless otherwise stated.

$M_1^n \simeq M_2^n$ (co)bordant if there is a manifold *with boundary* W^{n+1} such that $\partial W^{n+1} = M_1 \sqcup M_2$.

Complex bordism: work with complex manifolds.

complex mflds $\subset$ almost complex mflds $\subset$ stably complex mflds

Stably complex structure on a $2n$-dim manifold M is determined by a choice of isomorphism

$$c_{\mathcal{T}}\colon \tau M \oplus \mathbb{R}^{2(l-n)} \xrightarrow{\cong} \xi$$

where ξ is an l-dim *complex* vector bundle.

Complex bordism classes $[M, c_{\mathcal{T}}]$ form the complex bordism ring $\Omega^U = \Omega^U_*(pt)$ with respect to the disjoint union and product.

$$\Omega^U \cong \mathbb{Z}[a_1, a_2, ...], \quad \text{dim } a_i = 2i \quad \text{Novikov'1960.}$$

Quasitoric manifolds: $2n$-dimensional manifolds M with a "nice" action of the torus T^n (after Davis–Januszkiewicz);

- the T^n-action is locally standard (locally looks like the standard T^n-representation in $\mathbb{C}^n$);
- the orbit space M/T^n is an n-dim simple polytope P.

Examples include projective smooth toric varieties and symplectic manifolds M with Hamiltonian actions of T^n (also known as toric manifolds).

In their turn, quasitoric manifolds are examples of torus manifolds of Hattori–Masuda.

Equivariant cobordism and the universal toric genus.

X a T^k-space. There are 3 equivariant complex cobordism theories:

- $\Omega^*_{U:T^k}(X)$: geometric T^k-cobordisms: set of cobordism classes of stably tangentially complex T^k-bundles over X (here X is a *smooth manifold*).

- $MU^*_{T^k}(X) = \lim[S^V \wedge X_+, MU_{T^k}(W)]_{T^k}$: homotopic T^k-cobordisms; here $MU_{T^k}(W)$ is the Thom T^k-space of the universal $|W|$-dimensional complex T^k-vector bundle $\gamma_{|W|}$, and S^V is the unit sphere in a T^k-representation space V.

- $\Omega^*_U(ET^k \times_{T^k} X)$: Borel T^k-cobordisms.

There are natural transformations of cohomology theories

$$\varOmega^*_{U:T^k}(X) \xrightarrow{\nu} MU^*_{T^k}(X) \xrightarrow{\alpha} \varOmega^*_U(ET^k \times_{T^k} X).$$

Restricting to $X = pt$ we get a map

$$\varPhi := \alpha \cdot \nu \colon \varOmega^*_{U:T^k} \longrightarrow \varOmega^*_U(BT^k) = \Omega^U_*[[u_1, \ldots, u_k]],$$

which we refer to as the universal toric genus. It assigns to the cobordism class $[M, c_\mathcal{T}] \in \varOmega^{-2n}_{U:T^k}$ of a $2n$-dimensional T^k-manifold M the "cobordism class" of the map $ET^k \times_{T^k} M \to BT^k$.

We may write

$$\varPhi(M, c_\mathcal{T}) = \sum_\omega g_\omega(M) u^\omega,$$

where $\omega = (\omega_1, \ldots, \omega_k) \in \mathbb{N}^k$, $u^\omega = u_1^{\omega_1} \cdot \ldots \cdot u_k^{\omega_k}$, $g_\omega(M) \in \varOmega^U_{2(|\omega|+n)}$.

We have $g_0(M) = [M] \in \varOmega^U_{2n}$. How to express the other coefficients $g_\omega(M)$?

Ray's basis in $\Omega_*^U(BT^k)$.

Consider the product of unit 3-spheres

$$(S^3)^j = \left\{(z_1,\dots,z_{2j}) \in \mathbb{C}^{2j} \colon |z_i|^2 + |z_{i+j}|^2 = 1 \text{ for } 1 \leqslant i \leqslant j\right\}$$

with the free T^j-action by

$$(t_1,\dots,t_j)\cdot(z_1,\dots,z_{2j}) = (t_1^{-1}z_1, t_1^{-1}t_2^{-1}z_2,\dots,t_{j-1}^{-1}t_j^{-1}z_j, t_1z_{j+1},\dots,t_jz_{2j})$$

$B_j := (S^3)^j/T^j$: bounded flag manifold. It is a Bott manifold, i.e. the total space of a j-fold iterated S^2-bundle over $B_0 = *$.

For $1 \leqslant i \leqslant j$ there are complex line bundles

$$\psi_i\colon (S^3)^j \times_{T^j} \mathbb{C} \longrightarrow B_j$$

via the action $(t_1,\dots,t_j)\cdot z = t_iz$ for $z \in \mathbb{C}$.

For any $j > 0$ have an explicit isomorphism

$$\tau(B_j) \oplus \mathbb{C}^j \cong \psi_1 \oplus \psi_1\psi_2 \oplus \cdots \oplus \psi_{j-1}\psi_j \oplus \bar\psi_1 \oplus \cdots \oplus \bar\psi_j,$$

which defines a stably cplx structure c_j^∂ on B_j with $[B_j, c_j^\partial] = 0$ in Ω_{2j}^U.

Prop 1. *The basis element* $b_\omega \in \Omega^U_{2|\omega|}(BT^k)$ *dual to* $u^\omega \in \Omega^*_U(BT^k)$ *is represented geometrically by the classifying map*

$$\psi_\omega\colon B_\omega \longrightarrow BT^k$$

for the product $\psi_{\omega_1}\times\ldots\times\psi_{\omega_k}$ *of line bundles over* $B_\omega = B_{\omega_1}\times\ldots\times B_{\omega_k}$.

Let $T^\omega = T^{\omega_1}\times\ldots\times T^{\omega_k}$,and $(S^3)^\omega = (S^3)^{\omega_1}\times\ldots\times(S^3)^{\omega_k}$, on which T^ω acts coordinatewise. Define

$$G_\omega(M) := (S^3)^\omega \times_{T^\omega} M,$$

where T^ω acts on M via the representation

$$(t_{1,1},\ldots,t_{1,\omega_1};\ldots;t_{k,1},\ldots,t_{k,\omega_k}) \longmapsto (t^{-1}_{1,\omega_1},\ldots,t^{-1}_{k,\omega_k}).$$

Thm 2. *The manifold* $G_\omega(M)$ *represents the coefficient* $g_\omega(M) \in \Omega^U_{2(|\omega|+n)}$ *of the universal toric genus expansion.*

Hirzebruch genera and equivariant extentions.

R_* a (graded) commutative ring with unit.

$\ell\colon \varOmega^U_* \to R_*$ a Hirzebruch genus
(a multiplicative R_*-valued cobordism invariant characteristic of M).

Every genus ℓ has a T^k-equivariant extension

$$\ell^{T^k} := \ell \cdot \varPhi\colon \varOmega^{U:T^k}_* \longrightarrow R_*[[u_1, \dots, u_k]].$$

We have

$$\ell^{T^k}(M, c_\mathcal{T}) \;=\; \ell(M) \;+\; \sum_{|\omega|>0} \ell(g_\omega(M))\, u^\omega.$$

In particular, the T^k-equivariant extension of the universal genus $ug = \mathrm{id}\colon \varOmega^U_* \to \varOmega^U_*$ is $\varPhi$; hence the name "universal toric genus".

Rigidity and fibre multiplicativity.

Consider fibre bundles $M \to E \times_G M \xrightarrow{\pi} B$,
where M and B are connected and stably tangentially complex,
G a compact Lie group of positive rank whose action preserves the stably complex structure on M,
$E \to B$ is a principal G-bundle.

Then $N := E \times_G M$ inherits a canonical stably complex structure.

A genus $\ell \colon \Omega_*^U \to R_*$ is multiplicative with respect to M whenever $\ell(N) = \ell(M)\ell(B)$ for any such bundle π;
if this holds for every M, then ℓ is fibre multiplicative.

The genus ℓ is T^k-rigid on M whenever $\ell^{T^k} \colon \Omega_*^{U:T^k} \longrightarrow R_*[[u_1, \ldots, u_k]]$ satisfies $\ell^{T^k}(M, c_\tau) = \ell(M)$;
if this holds for every M, then ℓ is T^k-rigid.

It follows that ℓ is T^k-rigid whenever $\ell(G_\omega(M)) = 0$ for $|\omega| > 0$.

Other definitions of rigidity:

Atiyah–Hirzebruch: Assume ℓ^{T^k} can be realised as the *equivariant index* of an elliptic complex, $\ell^{T^k}\colon \Omega_*^{U:T^k} \longrightarrow RU(T^k)$.
Then ℓ^{T^k} is rigid if it takes values in $\mathbb{Z} \subset RU(T^k)$ (trivial representations).

Krichever: Considered $\mathbb{Q}$-valued genera and equivariant extensions $\ell^{T^k}\colon \Omega_*^{U:T^k} \to K(BT^k) \otimes \mathbb{Q}$.
Then ℓ^{T^k} is rigid if it takes values in $\mathbb{Q} \subset K(BT^k) \otimes \mathbb{Q}$.

Our definition of rigidity extends both Atiyah–Hirzebruch's and Krichever's.

Thm 3. *If the genus* ℓ *is* T^k*-rigid on* M*, then it is mutiplicative with respect to* M *for bundles whose structure group* G *has the property that* $\Omega_U^*(BG)$ *is torsion-free.*

On the other hand, if ℓ *is multiplicative* M*, then it is* T^k*-rigid on* M*.*

Proof. Assume ℓ multiplicative. Apply ℓ to the bundle $M \to G_\omega(M) \to B_\omega$. Since B_ω bounds for $|\omega| > 0$, we have $\ell(G_\omega(M)) = 0$, so ℓ is T^k-rigid.

The other direction is proved by considering the pullback square

$$\begin{array}{ccccc} E \times_G M & \xrightarrow{f'} & EG \times_G M & \xleftarrow{i'} & ET^k \times_{T^k} M \\ \pi \downarrow & & \pi^G \downarrow & & \pi^{T^k} \downarrow \\ B & \xrightarrow{f} & BG & \xleftarrow{i} & BT^k. \end{array}$$

□

Ex 4. The signature is fibre multiplicative over any simply connected base, so it is a rigid genus.

Isolated fixed points.

Assume Fix(M) is isolated. Have T^k-invariant $c_\tau \colon \tau M \oplus \mathbb{R}^{2(l-n)} \to \xi$.

For any $x \in$ Fix(M), the sign $\varsigma(x)$ is $+1$ if the isomorphism

$$\tau_x(M) \xrightarrow{i} \tau_x(M) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_{\tau,x}} \xi_x \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{p} \mathbb{C}^n$$

respects the canonical orientations, and -1 if it does not.

If M is almost complex then $\varsigma(x) = 1$ for every $x \in$ Fix(M).

The non-trivial T^k-representation $\mathbb{C}^n$ decomposes into 1-dimensional representations as $r_{x,1} \oplus \ldots \oplus r_{x,n}$.

$w_j(x) := (w_{j,1}(x), \ldots, w_{j,k}(x))$ the integral weight vector of $r_{x,j}$.

We refer to the collection of signs $\varsigma(x)$ and weight vectors $w_j(x)$ as the fixed point data for (M, c_τ).

Each weight vector determines a line bundle

$$\zeta^{w_j(x)} := \zeta_1^{w_{j,1}(x)} \otimes \cdots \otimes \zeta_k^{w_{j,k}(x)}$$

over BT^k, whose first Chern class is a formal power series

$$[w_j(x)](u) := \sum_{\omega} a_\omega [w_{j,1}(x)](u_1)^{\,\omega_1} \cdots [w_{j,k}(x)](u_k)^{\,\omega_k}$$

in $\Omega_U^2(BT^k)$. Here $[m](u_j)$ denotes the power series $c_1^{MU}(\zeta_j^m)$ in $\Omega_U^2(\mathbb{C}P^\infty)$, and the a_ω are the coefficients of $c_1^{MU}(\zeta_1 \otimes \cdots \otimes \zeta_k)$.

Modulo decomposables we have that

$$[w_j(x)](u_1, \ldots, u_k) \equiv w_{j,1}u_1 + \cdots + w_{j,k}u_k.$$

Thm 5 (Localisation formula). *For any stably tangentially complex M^{2n} with isolated fixed points, the equation*

$$\Phi(M) = \sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{1}{[w_j(x)](u)}$$

is satisfied in $\Omega_U^{-2n}(BT^k)$.

Quasitoric manifolds revisited.

Quasitoric manifolds M provide a vast source of examples of stably complex T^n-manifolds with isolated fixed points, for which calculations with the fixed point data and Hirzebruch genera can be made explicit.

Every such M is determined by the characteristic pair (P, Λ), where
P is a simple n-polytope with m facets $F_1, \ldots, F_m$,
Λ is an integral $n \times m$ matrix.

Given a fixed point $x = F_{j_1} \cap \ldots \cap F_{j_n}$ denote
$N(P)_x$ the matrix of column vectors normal to $F_{j_1}, \ldots, F_{j_n}$,
Λ_x the square submatrix of Λ of column vectors $j_1, \ldots, j_n$,
W_x the matrix determined by $W_x^t = \Lambda_x^{-1}$.

Prop 6. *1. the sign* $\varsigma(x)$ *is given by* sign $\Big(\det(\Lambda_x N(P)_x)\Big)$

2. the weight vectors $w_1(x)$, $\ldots w_n(x)$ *are the columns of* W_x.

Elliptic genera.

Buchstaber considered the formal group law

$$F_b(u_1,u_2) \;=\; u_1c(u_2)+u_2c(u_1)-au_1u_2-\frac{d(u_1)-d(u_2)}{u_1c(u_2)-u_2c(u_1)}u_1^2u_2^2$$

over the graded ring $R_*=\mathbb{Z}[a,c_j,d_k\colon j\geq 2,k\geq 1]/J$, where $\deg a=2$, $\deg c_j=2j$ and $\deg d_k=2(k+2)$;
J is the ideal of associativity relations, and

$$c(u) \;:=\; 1+\sum_{j\geq 2} c_ju^j, \qquad d(u) \;:=\; \sum_{k\geq 1} d_ku^k.$$

Thm 7. *The exponential series $f_b(x)$ of F_b may be written analytically as $e^{ax}/\phi(x,z)$, where*

$$\varphi(x,z) \;=\; \frac{\sigma(z-x)}{\sigma(z)\sigma(x)}\,e^{\zeta(z)\,x},$$

$\sigma(z)$ is the Weierstrass sigma function, and $\zeta(z)=(\ln\sigma(z))'$.

Moreover, $R_*\otimes\mathbb{Q}$ is isomorphic to $\mathbb{Q}[a,c_2,c_3,c_4]$ as graded algebras.

The function $\varphi(x,z)$ is known as the Baker–Akhiezer function associated to the elliptic curve $y^2 = 4x^3 - g_2x - g_3$. It satisfies the Lamé equation, and is important in the theory of nonlinear integrable equations.

Krichever studied the genus $kv\colon \Omega_*^U \to R_*$ corresponding to the exponential series f_b, which therefore classifies the formal group law F_b. Analytically, it depends on the four complex variables z, a, g_2 and g_3.

Cor 8. *Krichever's generalised elliptic genus* $kv\colon \Omega_*^U \to R_*$ *induces an isomorphism of graded abelian groups in dimensions* < 10.

Thm 9. *Let* M^{2n} *be an* SU*-quasitoric manifold (i.e.* $c_1(M) = 0$*); then*
(1) *the Krichever genus* kv *vanishes on* M^{2n}*,*
(2) M^{2n} *represents* 0 *in* Ω_{2n}^U *whenever* $n < 5$*.*

Conjecture 10. *Theorem* 9(2) *holds for all* n*.*

Ex 11.

1. The 2-parameter Todd genus $t2$ may be identified with the case

$$c(u) = 1 - yzu^2, \ \ d(u) = -yz(y+z)u - y^2z^2u^2.$$

The corresponding formal group law is

$$F_{t2} = \frac{u_1 + u_2 - (y+z)u_1u_2}{1 - yzu_1u_2}\,.$$

It generalises the χ_y-genus ($z = -1$) and the Todd genus ($y = 0$).

2. The elliptic genus Ell corresponds to Euler's formal group law

$$F_{Ell}(u_1, u_2) = \frac{u_1c(u_2) + u_2c(u_1)}{1 - \varepsilon u_1^2u_2^2}$$

$$= u_1c(u_2) + u_2c(u_1) + \varepsilon\frac{u_1^2 - u_2^2}{u_1c(u_2) - u_2c(u_1)}u_1^2u_2^2\,,$$

and may therefore be identified with the case

$$a = 0, \ \ d(u) = -\varepsilon u^2, \quad \text{and} \quad c^2(u) = R(u) := 1 - 2\delta u^2 + \varepsilon u^4.$$

Further applications to rigidity.

Prop 12. *For any series f over a $\mathbb{Q}$-algebra A, the corresponding Hirzebruch genus ℓ_f is T^k-rigid on M only if the functional equation*

$$\sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{1}{f(w_j(x) \cdot u)} = c$$

is satisfied in $A[[u_1, \ldots, u_k]]$, for some constant $c \in A$.

The quasitoric examples $\mathbb{C}P^1$, $\mathbb{C}P^2$, and the T^2-manifold S^6 are all instructive.

Ex 13. A genus ℓ_f is T-rigid on $\mathbb{C}P^1$ only if the equation

$$\frac{1}{f(u)} + \frac{1}{f(-u)} = c,$$

holds in $A[[u]]$. The general analytic solution is

$$f(u) = \frac{u}{q(u^2) + cu/2}, \quad \text{where } q(0) = 1.$$

An example is provided by the Todd genus, $f_{td}(u) = (e^{zu} - 1)/z$. In fact td is multiplicative with respect to $\mathbb{C}P^1$.

Ex 14. A genus ℓ_f is T^2-rigid on the stably complex manifold $\mathbb{C}P^2_{(1,-1)}$ only if the equation

$$\frac{1}{f(u_1)f(u_2)} - \frac{1}{f(u_1)f(u_1+u_2)} + \frac{1}{f(-u_2)f(u_1+u_2)} \;=\; c$$

holds in $A[[u_1, u_2]]$. The general analytic solution satisfies

$$f(u_1+u_2) \;=\; \frac{f(u_1)+f(u_2)-c'f(u_1)f(u_2)}{1-cf(u_1)f(u_2)}\,.$$

So f is the exponential series of the 2-parameter Todd genus, with $c' = y+z$ and $c = yz$.

Cor 15 (Musin). *The* 2*-parameter Todd genus $t2$ is universal for rigid genera.*

Ex 16. A genus ℓ_f is T^2-rigid on the almost complex manifold S^6 only if the equation

$$\frac{1}{f(u_1)f(u_2)f(-u_1-u_2)}+\frac{1}{f(-u_1)f(-u_2)f(u_1+u_2)} = c$$

holds in $A[[u_1,u_2]]$, for some constant c.

The general analytic solution is of the form $e^{ax}/\varphi(x,z)$, and f coincides with Krichever's exponential series f_b.

Thm 17. *Krichever's generalised elliptic genus kv is universal for genera that are rigid on SU-manifolds.*

[1] Victor M. Buchstaber and Taras E. Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* Volume 24 of *University Lecture Series*, Amer. Math. Soc., Providence, R.I., 2002.

[2] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Spaces of polytopes and cobordism of quasitoric manifolds*. Moscow Math. J. **7** (2007), no. 2, 219–242; arXiv:math.AT/0609346.

[3] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Toric genera*. Preprint (2009); arXiv:0908.3298.