The Universal Toric Genus

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The 36th Symposium on Transformation Groups,

Osaka, 10–12 December 2009
Motivations: non-equivariant cobordism and torus actions.

**Thm** [Buchstaber-P.-Ray]. *Every complex cobordism class in dim > 2 contains a quasitoric manifold.*

In other words, every stably complex manifold is cobordant to a manifold with a nicely behaving torus action.

All manifolds are smooth and closed, unless otherwise stated.

\[ M_1^n \simeq M_2^n \text{ (co)bordant} \] if there is a manifold with boundary \( W^{n+1} \) such that \( \partial W^{n+1} = M_1 \sqcup M_2 \).
Complex bordism: work with complex manifolds.

complex mflds $\subset$ almost complex mflds $\subset$ stably complex mflds

Stably complex structure on a $2n$-dim manifold $M$ is determined by a choice of isomorphism

$$c_\tau: \tau M \oplus \mathbb{R}^{2(l-n)} \overset{\cong}{\longrightarrow} \xi$$

where $\xi$ is an $l$-dim complex vector bundle.

Complex bordism classes $[M, c_\tau]$ form the complex bordism ring $\Omega^U = \Omega^U_*(pt)$ with respect to the disjoint union and product.

$$\Omega^U \cong \mathbb{Z}[a_1, a_2, ...], \quad \text{dim } a_i = 2i \quad \text{Novikov'1960}.$$
**Quasitoric manifolds:** $2n$-dimensional manifolds $M$ with a “nice” action of the torus $T^n$ (after Davis–Januszkiewicz);

- the $T^n$-action is **locally standard** (locally looks like the standard $T^n$-representation in $\mathbb{C}^n$);
- the orbit space $M/T^n$ is an $n$-dim simple polytope $P$.

Examples include projective smooth toric varieties and symplectic manifolds $M$ with Hamiltonian actions of $T^n$ (also known as toric manifolds).

In their turn, quasitoric manifolds are examples of torus manifolds of Hattori–Masuda.
Equivariant cobordism and the universal toric genus.

$X$ a $T^k$-space. There are 3 equivariant complex cobordism theories:

- $\Omega^*_U: T^k(X)$: geometric $T^k$-cobordisms: set of cobordism classes of stably tangentially complex $T^k$-bundles over $X$ (here $X$ is a smooth manifold).

- $MU^*_T(X) = \lim [S^V \wedge X_+, MU_{T^k}(W)]_{T^k}$: homotopic $T^k$-cobordisms; here $MU_{T^k}(W)$ is the Thom $T^k$-space of the universal $|W|$-dimensional complex $T^k$-vector bundle $\gamma_{|W|}$, and $S^V$ is the unit sphere in a $T^k$-representation space $V$.

- $\Omega^*_U(ET^k \times T^k X)$: Borel $T^k$-cobordisms.
There are natural transformations of cohomology theories
\[ \Omega^*_U(T^k(X)) \xrightarrow{\nu} MU^*_T(X) \xrightarrow{\alpha} \Omega^*_U(ET^k \times_{T^k} X). \]
Restricting to \( X = pt \) we get a map
\[ \Phi := \alpha \cdot \nu : \Omega^*_U(T^k) \longrightarrow \Omega^*_U(BT^k) = \Omega^*_U[[u_1, \ldots, u_k]], \]
which we refer to as the universal toric genus. It assigns to the cobordism class \([M, c_\tau] \in \Omega^{-2n}_{U:T^k}\) of a \(2n\)-dimensional \(T^k\)-manifold \(M\) the “cobordism class” of the map \(ET^k \times_{T^k} M \to BT^k\).

We may write
\[ \Phi(M, c_\tau) = \sum \omega g_\omega(M) u^\omega, \]
where \(\omega = (\omega_1, \ldots, \omega_k) \in \mathbb{N}^k, u^\omega = u_1^{\omega_1} \cdots u_k^{\omega_k}, g_\omega(M) \in \Omega^U_2(|\omega|+n).\)

We have \(g_0(M) = [M] \in \Omega^U_{2n}\). How to express the other coefficients \(g_\omega(M)\)?
Ray's basis in $\Omega_*^{U}(BT^k)$.

Consider the product of unit 3-spheres
\[(S^3)^j = \{(z_1, \ldots, z_{2j}) \in \mathbb{C}^{2j} : |z_i|^2 + |z_{i+j}|^2 = 1 \text{ for } 1 \leq i \leq j\}\]
with the free $T^j$-action by
\[(t_1, \ldots, t_j) \cdot (z_1, \ldots, z_{2j}) = (t_1^{-1}z_1, t_1^{-1}t_2^{-1}z_2, \ldots, t_{j-1}^{-1}t_j^{-1}z_j, t_1z_{j+1}, \ldots, t_jz_{2j})\]

$B_j := (S^3)^j / T^j$ : bounded flag manifold. It is a Bott manifold, i.e. the total space of a $j$-fold iterated $S^2$-bundle over $B_0 = \ast$.

For $1 \leq i \leq j$ there are complex line bundles
\[
\psi_i : (S^3)^j \times_{T^j} \mathbb{C} \longrightarrow B_j
\]
via the action $(t_1, \ldots, t_j) \cdot z = t_i z$ for $z \in \mathbb{C}$.

For any $j > 0$ have an explicit isomorphism
\[
\tau(B_j) \oplus \mathbb{C}^j \cong \psi_1 \oplus \psi_1 \psi_2 \oplus \cdots \oplus \psi_{j-1} \psi_j \oplus \overline{\psi}_1 \oplus \cdots \oplus \overline{\psi}_j,
\]
which defines a stably cplx structure $c^\partial_j$ on $B_j$ with $[B_j, c^\partial_j] = 0$ in $\Omega_*^{U}_{2j}$. 

Prop 1. The basis element $b_\omega \in \Omega^U_{2|\omega|}(BT^k)$ dual to $u^\omega \in \Omega^*_U(BT^k)$ is represented geometrically by the classifying map
\[
\psi_\omega: B_\omega \longrightarrow BT^k
\]
for the product $\psi_{\omega_1} \times \ldots \times \psi_{\omega_k}$ of line bundles over $B_\omega = B_{\omega_1} \times \ldots \times B_{\omega_k}$.

Let $T^\omega = T^{\omega_1} \times \ldots \times T^{\omega_k}$, and $(S^3)^\omega = (S^3)^{\omega_1} \times \ldots \times (S^3)^{\omega_k}$, on which $T^\omega$ acts coordinatewise. Define

\[
G_\omega(M) := (S^3)^\omega \times_{T^\omega} M,
\]
where $T^\omega$ acts on $M$ via the representation

\[
(t_{1,1}, \ldots, t_{1,\omega_1}; \ldots; t_{k,1}, \ldots, t_{k,\omega_k}) \mapsto (t_{1,\omega_1}^{-1}, \ldots, t_{k,\omega_k}^{-1}).
\]

Thm 2. The manifold $G_\omega(M)$ represents the coefficient $g_\omega(M) \in \Omega^U_{2(|\omega|+n)}$ of the universal toric genus expansion.
Hirzebruch genera and equivariant extentions.

$\mathcal{R}_*$ a (graded) commutative ring with unit.

$\ell : \Omega_*^U \to \mathcal{R}_*$ a Hirzebruch genus
(a multiplicative $\mathcal{R}_*$-valued cobordism invariant characteristic of $M$).

Every genus $\ell$ has a $T^k$-equivariant extension

$$\ell^T : = \ell \cdot \Phi : \Omega_*^U : T^k \longrightarrow \mathcal{R}_*[[u_1, \ldots, u_k]].$$

We have

$$\ell^T (M, c_T) = \ell (M) + \sum_{|\omega | > 0} \ell (g_\omega (M)) u^\omega.$$ 

In particular, the $T^k$-equivariant extension of the universal genus $ug = \text{id} : \Omega_*^U \to \Omega_*^U$ is $\Phi$; hence the name “universal toric genus”.
Rigidity and fibre multiplicativity.

Consider fibre bundles $M \to E \times_G M \xrightarrow{\pi} B$, where $M$ and $B$ are connected and stably tangentially complex, $G$ a compact Lie group of positive rank whose action preserves the stably complex structure on $M$, $E \to B$ is a principal $G$-bundle.

Then $N := E \times_G M$ inherits a canonical stably complex structure.

A genus $\ell: \Omega^U_\ast \to R_\ast$ is multiplicative with respect to $M$ whenever $\ell(N) = \ell(M)\ell(B)$ for any such bundle $\pi$; if this holds for every $M$, then $\ell$ is fibre multiplicative.

The genus $\ell$ is $T^k$-rigid on $M$ whenever $\ell^{T^k}: \Omega^U_{\ast:T^k} \to R_\ast[[u_1, \ldots, u_k]]$ satisfies $\ell^{T^k}(M, c_T) = \ell(M)$; if this holds for every $M$, then $\ell$ is $T^k$-rigid.

It follows that $\ell$ is $T^k$-rigid whenever $\ell(G\omega(M)) = 0$ for $|\omega| > 0$. 
Other definitions of rigidity:

**Atiyah–Hirzebruch:** Assume $\ell^{T^k}$ can be realised as the *equivariant index* of an elliptic complex, $\ell^{T^k} : \Omega^*_U : T^k \to RU(T^k)$. Then $\ell^{T^k}$ is *rigid* if it takes values in $\mathbb{Z} \subset RU(T^k)$ (trivial representations).

**Krichever:** Considered $\mathbb{Q}$-valued genera and equivariant extensions $\ell^{T^k} : \Omega^*_U : T^k \to K(BT^k) \otimes \mathbb{Q}$. Then $\ell^{T^k}$ is *rigid* if it takes values in $\mathbb{Q} \subset K(BT^k) \otimes \mathbb{Q}$.

Our definition of rigidity extends both Atiyah–Hirzebruch’s and Krichever’s.
**Thm 3.** If the genus $\ell$ is $T^k$-rigid on $M$, then it is multiplicative with respect to $M$ for bundles whose structure group $G$ has the property that $\Omega_U^*(BG)$ is torsion-free.

On the other hand, if $\ell$ is multiplicative $M$, then it is $T^k$-rigid on $M$.

**Proof.** Assume $\ell$ multiplicative. Apply $\ell$ to the bundle $M \to G_\omega(M) \to B_\omega$. Since $B_\omega$ bounds for $|\omega| > 0$, we have $\ell(G_\omega(M)) = 0$, so $\ell$ is $T^k$-rigid.

The other direction is proved by considering the pullback square

$$
\begin{array}{ccc}
E \times_G M & \xrightarrow{f'} & EG \times_G M & \leftarrow & ET^k \times_{T^k} M \\
\pi & & \pi^G & & \pi^{T^k} \\
B & \xrightarrow{f} & BG & \leftarrow & BT^k.
\end{array}
$$


**Ex 4.** The signature is fibre multiplicative over any simply connected base, so it is a rigid genus.
**Isolated fixed points.**

Assume $\text{Fix}(M)$ is isolated. Have $T_k^k$-invariant $c_\tau : \tau M \oplus \mathbb{R}^{2(l-n)} \to \xi$. For any $x \in \text{Fix}(M)$, the sign $\varsigma(x)$ is $+1$ if the isomorphism

$$
\tau_x(M) \xrightarrow{i} \tau_x(M) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_{\tau,x}} \xi_x \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{p} \mathbb{C}^n
$$

respects the canonical orientations, and $-1$ if it does not.

If $M$ is almost complex then $\varsigma(x) = 1$ for every $x \in \text{Fix}(M)$.

The non-trivial $T_k^k$-representation $\mathbb{C}^n$ decomposes into 1-dimensional representations as $r_{x,1} \oplus \ldots \oplus r_{x,n}$.

$$w_j(x) := (w_{j,1}(x), \ldots, w_{j,k}(x))$$

the integral weight vector of $r_{x,j}$.

We refer to the collection of signs $\varsigma(x)$ and weight vectors $w_j(x)$ as the fixed point data for $(M, c_\tau)$. 

13
Each weight vector determines a line bundle

\[ \zeta_{w_j}(x) := \zeta_{w_j,1}(x) \otimes \cdots \otimes \zeta_{w_j,k}(x) \]

over \( BT^k \), whose first Chern class is a formal power series

\[ [w_j(x)](u) := \sum_\omega a_\omega [w_{j,1}(x)](u_1)^{\omega_1} \cdots [w_{j,k}(x)](u_k)^{\omega_k} \]

in \( \Omega_U^2(BT^k) \). Here \([m](u_j)\) denotes the power series \( c_1^{MU}(\zeta_j^m) \) in \( \Omega_U^2(\mathbb{C}P^\infty) \), and the \( a_\omega \) are the coefficients of \( c_1^{MU}(\zeta_1 \otimes \cdots \otimes \zeta_k) \).

Modulo decomposables we have that

\[ [w_j(x)](u_1, \ldots, u_k) \equiv w_{j,1}u_1 + \cdots + w_{j,k}u_k. \]

**Thm 5** (Localisation formula). For any stably tangentially complex \( M^{2n} \) with isolated fixed points, the equation

\[ \Phi(M) = \sum_{\text{Fix}(M)} \varsigma(x) \prod_{j=1}^n \frac{1}{[w_j(x)](u)} \]

is satisfied in \( \Omega_U^{-2n}(BT^k) \).
Quasitoric manifolds revisited.

Quasitoric manifolds $M$ provide a vast source of examples of stably complex $T^n$-manifolds with isolated fixed points, for which calculations with the fixed point data and Hirzebruch genera can be made explicit.

Every such $M$ is determined by the characteristic pair $(P, \Lambda)$, where $P$ is a simple $n$-polytope with $m$ facets $F_1, \ldots, F_m$, $\Lambda$ is an integral $n \times m$ matrix.

Given a fixed point $x = F_{j_1} \cap \ldots \cap F_{j_n}$ denote $N(P)_x$ the matrix of column vectors normal to $F_{j_1}, \ldots, F_{j_n}$, $\Lambda_x$ the square submatrix of $\Lambda$ of column vectors $j_1, \ldots, j_n$, $W_x$ the matrix determined by $W^t_x = \Lambda^{-1}_x$.

Prop 6. 1. the sign $\varsigma(x)$ is given by $\text{sign}\left(\det(\Lambda_x N(P)_x)\right)$

2. the weight vectors $w_1(x), \ldots w_n(x)$ are the columns of $W_x$. 
Elliptic genera.

Buchstaber considered the formal group law

\[ F_b(u_1, u_2) = u_1 c(u_2) + u_2 c(u_1) - a u_1 u_2 - \frac{d(u_1) - d(u_2)}{u_1 c(u_2) - u_2 c(u_1)} u_1^2 u_2^2 \]

over the graded ring \( R_* = \mathbb{Z}[a, c_j, d_k : j \geq 2, k \geq 1]/J \), where \( \deg a = 2 \), \( \deg c_j = 2j \) and \( \deg d_k = 2(k + 2) \); \( J \) is the ideal of associativity relations, and

\[ c(u) := 1 + \sum_{j \geq 2} c_j u^j, \quad d(u) := \sum_{k \geq 1} d_k u^k. \]

**Thm 7.** The exponential series \( f_b(x) \) of \( F_b \) may be written analytically as \( e^{ax}/\varphi(x, z) \), where

\[ \varphi(x, z) = \frac{\sigma(z - x)}{\sigma(z) \sigma(x)} e^{\zeta(z) x}, \]

\( \sigma(z) \) is the Weierstrass sigma function, and \( \zeta(z) = (\ln \sigma(z))' \).
Moreover, \( R_* \otimes \mathbb{Q} \) is isomorphic to \( \mathbb{Q}[a, c_2, c_3, c_4] \) as graded algebras.
The function $\varphi(x, z)$ is known as the Baker–Akhiezer function associated to the elliptic curve $y^2 = 4x^3 - g_2x - g_3$. It satisfies the Lamé equation, and is important in the theory of nonlinear integrable equations.

Krichever studied the genus $kv: \Omega^U \rightarrow R_*$ corresponding to the exponential series $f_b$, which therefore classifies the formal group law $F_b$. Analytically, it depends on the four complex variables $z$, $a$, $g_2$ and $g_3$.

**Cor 8.** Krichever’s generalised elliptic genus $kv: \Omega^U \rightarrow R_*$ induces an isomorphism of graded abelian groups in dimensions $< 10$.

**Thm 9.** Let $M^{2n}$ be an SU-quasitoric manifold (i.e. $c_1(M) = 0$); then
(1) the Krichever genus $kv$ vanishes on $M^{2n}$,
(2) $M^{2n}$ represents 0 in $\Omega^U_{2n}$ whenever $n < 5$.

**Conjecture 10.** Theorem 9(2) holds for all $n$. 
Ex 11.
1. The 2-parameter Todd genus $t_2$ may be identified with the case

$$c(u) = 1 - yzu^2, \quad d(u) = -yz(y + z)u - y^2z^2u^2.$$ 

The corresponding formal group law is

$$F_{t_2} = \frac{u_1 + u_2 - (y + z)u_1u_2}{1 - yzu_1u_2}.$$ 

It generalises the $\chi_y$-genus ($z = -1$) and the Todd genus ($y = 0$).

2. The elliptic genus $Ell$ corresponds to Euler’s formal group law

$$F_{Ell}(u_1, u_2) = \frac{u_1c(u_2) + u_2c(u_1)}{1 - \varepsilon u_1^2u_2^2}$$

$$= u_1c(u_2) + u_2c(u_1) + \varepsilon\frac{u_1^2 - u_2^2}{u_1c(u_2) - u_2c(u_1)}u_1^2u_2^2,$$

and may therefore be identified with the case

$$a = 0, \quad d(u) = -\varepsilon u^2, \quad \text{and} \quad c^2(u) = R(u) := 1 - 2\delta u^2 + \varepsilon u^4.$$
Further applications to rigidity.

**Prop 12.** For any series $f$ over a $\mathbb{Q}$-algebra $A$, the corresponding Hirzebruch genus $\ell_f$ is $T^k$-rigid on $M$ only if the functional equation

$$\sum_{\text{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{1}{f(w_j(x) \cdot u)} = c$$

is satisfied in $A[[u_1,\ldots,u_k]]$, for some constant $c \in A$.

The quasitoric examples $\mathbb{C}P^1$, $\mathbb{C}P^2$, and the $T^2$-manifold $S^6$ are all instructive.

**Ex 13.** A genus $\ell_f$ is $T$-rigid on $\mathbb{C}P^1$ only if the equation

$$\frac{1}{f(u)} + \frac{1}{f(-u)} = c,$$

holds in $A[[u]]$. The general analytic solution is

$$f(u) = \frac{u}{q(u^2) + cu/2}, \quad \text{where } q(0) = 1.$$

An example is provided by the Todd genus, $f_{td}(u) = (e^{z u} - 1)/z$. In fact $td$ is multiplicative with respect to $\mathbb{C}P^1$. 


**Ex 14.** A genus $\ell_f$ is $T^2$-rigid on the stably complex manifold $\mathbb{C}P^2_{(1,-1)}$ only if the equation

$$\frac{1}{f(u_1)f(u_2)} - \frac{1}{f(u_1)f(u_1 + u_2)} + \frac{1}{f(-u_2)f(u_1 + u_2)} = c$$

holds in $A[[u_1,u_2]]$. The general analytic solution satisfies

$$f(u_1 + u_2) = \frac{f(u_1) + f(u_2) - c' f(u_1)f(u_2)}{1 - cf(u_1)f(u_2)}.$$ 

So $f$ is the exponential series of the 2-parameter Todd genus, with $c' = y + z$ and $c = yz$.

**Cor 15 (Musin).** *The 2-parameter Todd genus $t^2$ is universal for rigid genera.*
Ex 16. A genus $\ell_f$ is $T^2$-rigid on the almost complex manifold $S^6$ only if the equation

$$\frac{1}{f(u_1)f(u_2)f(-u_1-u_2)} + \frac{1}{f(-u_1)f(-u_2)f(u_1+u_2)} = c$$

holds in $A[[u_1,u_2]]$, for some constant $c$.

The general analytic solution is of the form $e^{\alpha x}/\varphi(x,z)$, and $f$ coincides with Krichever’s exponential series $f_b$.

Thm 17. Krichever’s generalised elliptic genus $k_v$ is universal for genera that are rigid on $SU$-manifolds.
