# Toric Kempf–Ness sets


Taras Panov

*Moscow State University*



Toric Geometry meeting
MFO 4-9 January 2009

## 1. Categorical quotient.

G a reductive algebraic group, $S$ an affine G-variety.

$\pi_{S,\mathsf{G}}\colon S \to S/\!\!/\mathsf{G}$ morphism dual to $\mathbb{C}[S]^{\mathsf{G}} \to \mathbb{C}[S]$.

$\pi_{S,\mathsf{G}}$ is surjective and establishes a bijection

$$\text{closed G-orbits of } S \quad \longleftrightarrow \quad \text{points of } S/\!\!/\mathsf{G}.$$

$\pi_{S,\mathsf{G}}$ is universal in the class of morphisms from $S$ constant on G-orbits in the category of algebraic varieties.

$S/\!\!/\mathsf{G}$ is called the categorical quotient.

## 2. Kempf–Ness sets for affine varieties.

$\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ a representation, $\mathsf{K} \subset \mathsf{G}$ a maximal compact subgroup, $\langle\ ,\ \rangle$ a K-invariant hermitian form on $V$ with associated norm $\|\ \|$.

Given $v \in V$, consider the function

$$F_v\colon \mathsf{G} \to \mathbb{R}, \quad g \mapsto \frac{1}{2}\|gv\|^2.$$

It has a critical point iff $\mathsf{G}v$ is closed, and all critical points of $F_v$ are minima. Define the subset $KN \subset V$ by

$$\begin{aligned} KN &= \{v \in V : (dF_v)_e = 0\} \qquad (e \in \mathsf{G} \text{ is the unit}) \\ &= \{v \in V : T_v\mathsf{G}v \perp v\} \\ &= \{v \in V : \langle \gamma v, v\rangle = 0 \text{ for all } \gamma \in \mathfrak{g}\} \\ &= \{v \in V : \langle \kappa v, v\rangle = 0 \text{ for all } \kappa \in \mathfrak{k}\}. \end{aligned} \tag{1}$$

Any $v \in KN$ is a closest point to the origin in its orbit $\mathsf{G}v$.
$KN$ is called the Kempf–Ness set of $V$.

Assume that $S$ is G-equivariantly embedded as a closed subvariety in a representation $V$ of G. Then $\mathsf{KN}_S := \mathsf{KN} \cap S$, the Kempf–Ness set of $S$.

The importance of Kempf–Ness sets for the study of orbit quotients is due to the following result.

**Thm 2.** (a) [Kempf–Ness] *The composition*

$$\mathsf{KN}_S \hookrightarrow S \to S /\!\!/ \mathsf{G}$$

*is proper and induces a homeomorphism*

$$\mathsf{KN}_S / \mathsf{K} \xrightarrow{\cong} S /\!\!/ \mathsf{G}.$$

(b) [Neeman] *There is a* K*-equivariant deformation retraction*

$$S \to \mathsf{KN}_S\,.$$

## 3. Toric varieties.

$N \cong \mathbb{Z}^n$ an integral lattice, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ the multiplicative group of complex numbers,
$\mathsf{S}^1$ the subgroup of complex numbers of absolute value one.

$\mathsf{T}_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ the algebraic torus,
$\mathsf{T} = N \otimes_{\mathbb{Z}} \mathsf{S}^1 \cong (\mathsf{S}^1)^n$ the (compact) torus.

A toric variety is a normal algebraic variety $X$ containing the algebraic torus $\mathsf{T}_{\mathbb{C}}$ as a Zariski open subset in such a way that the natural action of $\mathsf{T}_{\mathbb{C}}$ on itself extends to an action on $X$.

fans $\Sigma$ in $N_{\mathbb{R}}$ $\longleftrightarrow$ complex $n$-dim toric varieties $X_\Sigma$

regular fans $\longleftrightarrow$ non-singular varieties

complete fans $\longleftrightarrow$ compact varieties

## 4. Batyrev–Cox construction.

Assume that one-dimensional cones of $\Sigma$ span $N_{\mathbb{R}}$ as a vector space.

$m$ the number of one-dimensional cones.

$a_i \in N$ the primitive generator of the $i$th one-dim cone, $1 \leqslant i \leqslant m$.

Consider the map

$$\mathbb{Z}^m \to N, \quad e_i \mapsto a_i.$$

The corresponding maps of tori fit into exact sequences

$$1 \longrightarrow \mathsf{G} \longrightarrow (\mathbb{C}^*)^m \longrightarrow \mathsf{T}_{\mathbb{C}} \longrightarrow 1, \tag{3}$$

$$1 \longrightarrow \mathsf{K} \longrightarrow T^m \longrightarrow \mathsf{T} \longrightarrow 1 \tag{4}$$

where $\mathsf{G}$ is isomorphic to a product of $(\mathbb{C}^*)^{m-n}$ and a finite group. If $\Sigma$ is a regular fan and has at least one $n$-dimensional cone, then $\mathsf{G} \cong (\mathbb{C}^*)^{m-n}$, and similarly for $\mathsf{K}$.

We say that a subset $\{i_1, \ldots, i_k\} \in [m] = \{1, \ldots, m\}$ is a g-subset if $\{a_{i_1}, \ldots, a_{i_k}\}$ is a subset of the generator set of a cone in $\Sigma$.

The collection of $g$-subsets is closed with respect to the inclusion, and therefore forms an (abstract) simplicial complex on the set $[m]$, which we denote $\mathcal{K}_\Sigma$.

If $\Sigma$ is a complete simplicial fan, then $\mathcal{K}_\Sigma$ is a triangulation of an $(n-1)$-dimensional sphere.

Given a cone $\sigma \in \Sigma$, we denote by $g(\sigma) \subseteq [m]$ the set of its generators. Now set

$$A(\Sigma) = \bigcup_{\{i_1,\ldots,i_k\} \text{ is not a } g\text{-subset}} \{z \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\}$$

and

$$U(\Sigma) = \mathbb{C}^m \setminus A(\Sigma).$$

Unlike G and K, both $A(\Sigma)$ and $U(\Sigma)$ depend only on the combinatorial structure of the simplicial complex $\mathcal{K}_\Sigma$; the set $U(\Sigma)$ coincides with the coordinate subspace arrangement complement $U(\mathcal{K}_\Sigma)$.

The set $A(\Sigma)$ is an affine variety, while its complement $U(\Sigma)$ admits a simple affine cover, as described in the following statement.

**Prop 5.** *Given a cone* $\sigma \in \Sigma$, *set* $z^{\widehat{\sigma}} = \prod_{j \notin g(\sigma)} z_j$ *and define*

$$V(\Sigma) = \{z \in \mathbb{C}^m \colon z^{\widehat{\sigma}} = 0 \text{ for all } \sigma \in \Sigma\}$$

*and*

$$U(\sigma) = \{z \in \mathbb{C}^m \colon z_j \neq 0 \text{ if } j \notin g(\sigma)\}.$$

*Then* $A(\Sigma) = V(\Sigma)$ *and*

$$U(\Sigma) = \mathbb{C}^m \setminus V(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma).$$

The complement $U(\Sigma) \subset \mathbb{C}^m$ is $(\mathbb{C}^*)^m$-invariant.

If $\Sigma$ is simplicial, the subgroup $\mathrm{G} \subset (\mathbb{C}^*)^m$ acts on $U(\Sigma)$ with finite isotropy subgroups (or freely if $\Sigma$ is regular). The quotient can be identified with the toric variety $X_\Sigma$ determined by $\Sigma$:

**Thm 6.** [Cox] (a) *The toric variety $X_\Sigma$ is isomorphic to the categorical quotient of $U(\Sigma)$ by* G*.*
(b) *$X_\Sigma$ is the geometric quotient of $U(\Sigma)$ by* G *if and only if $\Sigma$ is simplicial.*

Therefore, if $\Sigma$ is a simplicial, then all the orbits of the G-action on $U(\Sigma)$ are closed and we have $U(\Sigma)/\!\!/\mathrm{G} = U(\Sigma)/\mathrm{G}$.

However, the corresponding Kempf–Ness set cannot constructed in the standard way, as $U(\Sigma)$ is *not* an affine variety in $\mathbb{C}^m$.

## 5. The moment-angle complex.

Consider the unit polydisc

$$(\mathsf{D}^2)^m = \{z \in \mathbb{C}^m \colon |z_j| \leqslant 1 \text{ for all } j\}.$$

Given a cone $\sigma \in \Sigma$, define

$$\mathcal{Z}(\sigma) = \{z \in (\mathsf{D}^2)^m \colon |z_j| = 1 \text{ if } j \notin g(\sigma)\},$$

and the moment-angle complex

$$\mathcal{Z}(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathcal{Z}(\sigma) \subseteq (\mathsf{D}^2)^m.$$

$\mathcal{Z}(\Sigma)$ is $T^m$-invariant. Also, $\mathcal{Z}(\Sigma) \subset U(\Sigma)$.

**Prop 7.** *Assume $\Sigma$ is complete simplicial. Then $\mathcal{Z}(\Sigma)$ is a compact $T^m$-manifold of dimension $m+n$.*

## 6. Toric Kempf–Ness sets.

$\mathcal{Z}(\Sigma)$ has the same properties with respect to the G-action on $U(\Sigma)$ as $KN_S$ with respect to the G-action on an affine variety $S$:

**Thm 8** (Buchstaber-P.'00)**.** *Assume $\Sigma$ is simplicial.*
(a) *If $\Sigma$ is complete, then the composition*

$$\mathcal{Z}(\Sigma) \hookrightarrow U(\Sigma) \to U(\Sigma)/\mathrm{G}$$

*induces a homeomorphism*

$$\mathcal{Z}(\Sigma)/\mathrm{K} \to U(\Sigma)/\mathrm{G}.$$

(b) *There is a $T^m$-equivariant deformation retraction $U(\Sigma) \to \mathcal{Z}(\Sigma)$.*

We therefore refer to $\mathcal{Z}(\Sigma)$ as the toric Kempf–Ness set of $U(\Sigma)$.

**Ex 9.** Let $n = 2$ and $e_1, e_2$ be a basis in $N_{\mathbb{R}}$.

1. Consider a complete fan $\Sigma$ having the following three 2-dimensional cones: the first is spanned by $e_1$ and $e_2$, the second spanned by $e_2$ and $-e_1 - e_2$, and the third spanned by $-e_1 - e_2$ and $e_1$. The simplicial complex $\mathcal{K}_\Sigma$ is a complete graph on 3 vertices (or the boundary of a triangle). We have

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z \colon z_1 = z_2 = z_3 = 0\} = \mathbb{C}^3 \setminus \{0\}$$

and

$$\mathcal{Z}(\Sigma) = \mathsf{D}^2 \times \mathsf{D}^2 \times \mathsf{S}^1 \cup \mathsf{D}^2 \times \mathsf{S}^1 \times \mathsf{D}^2 \cup \mathsf{S}^1 \times \mathsf{D}^2 \times \mathsf{D}^2 \\ = \partial((\mathsf{D}^2)^3) \cong \mathsf{S}^5.$$

Then G is the diagonal subtorus in $(\mathbb{C}^*)^3$, and K is the diagonal subcircle in $T^3$. Therefore,

$$X_\Sigma = U(\Sigma)/\mathsf{G} = \mathcal{Z}(\Sigma)/\mathsf{K} = \mathbb{C}P^2.$$

2. Now consider the fan $\Sigma$ consisting of three 1-dimensional cones generated by vectors $e_1$, $e_2$ and $-e_1-e_2$. This fan is not complete, but its 1-dimensional cones span $N_{\mathbb{R}}$ as a vector space. So Cox' Thm 6 applies, but Thm 8 (a) does not. We have

$$\mathcal{K}_\Sigma = \text{ 3 disjoint points},$$

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z_1 = z_2 = 0, z_1 = z_3 = 0, z_2 = z_3 = 0\},$$

and

$$\mathcal{Z}(\Sigma) = \mathsf{D}^2 \times \mathsf{S}^1 \times \mathsf{S}^1 \cup \mathsf{S}^1 \times \mathsf{D}^2 \times \mathsf{S}^1 \cup \mathsf{S}^1 \times \mathsf{S}^1 \times \mathsf{D}^2.$$

Both spaces are homotopy equivalent to $\mathsf{S}^3 \vee \mathsf{S}^3 \vee \mathsf{S}^3 \vee \mathsf{S}^4 \vee \mathsf{S}^4$.

G is again a diagonal subtorus in $(\mathbb{C}^*)^3$. By Thm 6,

$$X_\Sigma = U(\Sigma)/\mathsf{G} = \mathbb{C}P^2 \setminus \{3 \text{ points}\}.$$

This in non-compact, and cannot be identified with $\mathcal{Z}(\Sigma)/\mathsf{K}$.

## 7. Polytopes and normal fans.

$N_{\mathbb{R}}^*$ the dual vector space. Given primitive vectors $a_1, \dots, a_m \in N$ and integer numbers $b_1, \dots, b_m \in \mathbb{Z}$, consider

$$P = \{x \in N_{\mathbb{R}}^* \colon \langle a_i, x \rangle + b_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m\}.$$

Assume:

- $P$ is bounded;
- the affine hull of $P$ is the whole $N_{\mathbb{R}}^*$;
- no redundant inequalities;
- no $(n+1)$ hyperlanes $\langle a_i, x \rangle + b_i = 0$ meet at a point.

Then $P$ is a convex simple polytope with $m$ facets

$$F_i = \{x \in P \colon \langle a_i, x \rangle + b_i = 0\}$$

with normal vectors $a_i$, for $1 \leqslant i \leqslant m$.

We may specify $P$ by a matrix inequality

$$A_P x + b_P \geqslant 0,$$

where $A_P$ is the $m \times n$ matrix of row vectors $a_i$, and $b_P$ is the column vector of scalars $b_i$.

The affine injection

$$i_P \colon N^*_{\mathbb{R}} \longrightarrow \mathbb{R}^m, \quad x \mapsto A_P x + b_P$$

embeds $P$ into $\mathbb{R}^m_{\geqslant} = \{y \in \mathbb{R}^m \colon y_i \geqslant 0\}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \stackrel{i_Z}{\longrightarrow} & \mathbb{C}^m \\ {\scriptstyle \mu_P}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu} \\ P & \stackrel{i_P}{\longrightarrow} & \mathbb{R}^m \end{array}$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. Here $i_Z$ is a $T^m$-equivariant embedding.

The normal fan $\Sigma_P$ consists of the cones spanned by the sets $\{a_{i_1}, \ldots, a_{i_k}\}$ such that the intersection $F_{i_1} \cap \ldots \cap F_{i_k}$ of the corresponding facets is non-empty. $\Sigma_P$ is a simplicial fan.

**Prop 10.** (a) *We have* $\mathcal{Z}_P \subset U(\Sigma_P)$.
(b) *There is a* $T^m$*-homeomorphism* $\mathcal{Z}_P \cong \mathcal{Z}(\Sigma_P)$.

## 8. Complete intersections of real quadrics.

The linear transformation $A_P\colon N^*_{\mathbb{R}} \to \mathbb{R}^m$ is exactly the one obtained from $T^m \to \mathsf{T}$ by applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathsf{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$.

Applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathsf{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$ to the whole exact sequence of tori, we obtain

$$0 \longrightarrow N^*_{\mathbb{R}} \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \longrightarrow 0,$$

where $\mathbb{R}^{m-n} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{G}, \mathsf{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$.

Assume the first $n$ normal vectors $a_1, \ldots, a_n$ span a cone of $\Sigma_P$, and take these vectors as a basis of $N^*_{\mathbb{R}}$. In this basis, we may take

$$C = (c_{ij}) = \begin{pmatrix} -a_{n+1,1} & \ldots & -a_{n+1,n} & 1 & 0 & \ldots & 0 \\ -a_{n+2,1} & \ldots & -a_{n+2,n} & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{m,1} & \ldots & -a_{m,n} & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Then $\mathcal{Z}_P$ embeds in $\mathbb{C}^m$ as the space of common solutions of $m-n$ real quadratic equations

$$\sum_{k=1}^{m} c_{jk}\left(|z_k|^2 - b_k\right) = 0, \quad \text{for } 1 \le j \le m-n.$$

This intersection is non-degenerate, so $\mathcal{Z}_P \subset \mathbb{C}^m$ is a smooth submanifold with trivial normal bundle (Buchstaber-P-Ray'07).

The projective toric variety $X_P = X_{\Sigma_P}$ can be obtained from the action of K on $U(\Sigma_P) \subset \mathbb{C}^m$ via the process of symplectic reduction. The moment map $\mu_{\Sigma_P}$ is given by the composition

$$\mathbb{C}^m \xrightarrow{\mu} \mathbb{R}^m \xrightarrow{C} \text{Lie(K)} \cong \mathbb{R}^{m-n},$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$ and $C = (c_{jk})$, so

$$\mathcal{Z}_P = \mu_{\Sigma_P}^{-1}(Cb_P).$$

is its level surface. Then $X_P = \mathcal{Z}_P/\text{K}$.

**Question 11.** *There are many complete regular fans* $\Sigma$ *which cannot be realised as normal fans of convex polytopes. The corresponding toric varieties* $X_\Sigma$ *are non-singular, but not projective. In this case the Kempf–Ness set* $\mathcal{Z}(\Sigma)$ *is still defined. Is there a description of* $\mathcal{Z}(\Sigma)$ *similar to that of* $\mathcal{Z}(\Sigma_P)$ *as a complete intersection of real quadrics?*

## 9. Cohomology of Kempf–Ness sets.

Given an abstract simplicial complex $\mathcal{K}$ on the set $[m]$, the face ring (or the Stanley–Reisner ring) $\mathbb{Z}[\mathcal{K}]$ is the quotient

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m]/(v_{i_1} \cdots v_{i_k} : \{i_1, \dots, i_k\} \notin \mathcal{K}).$$

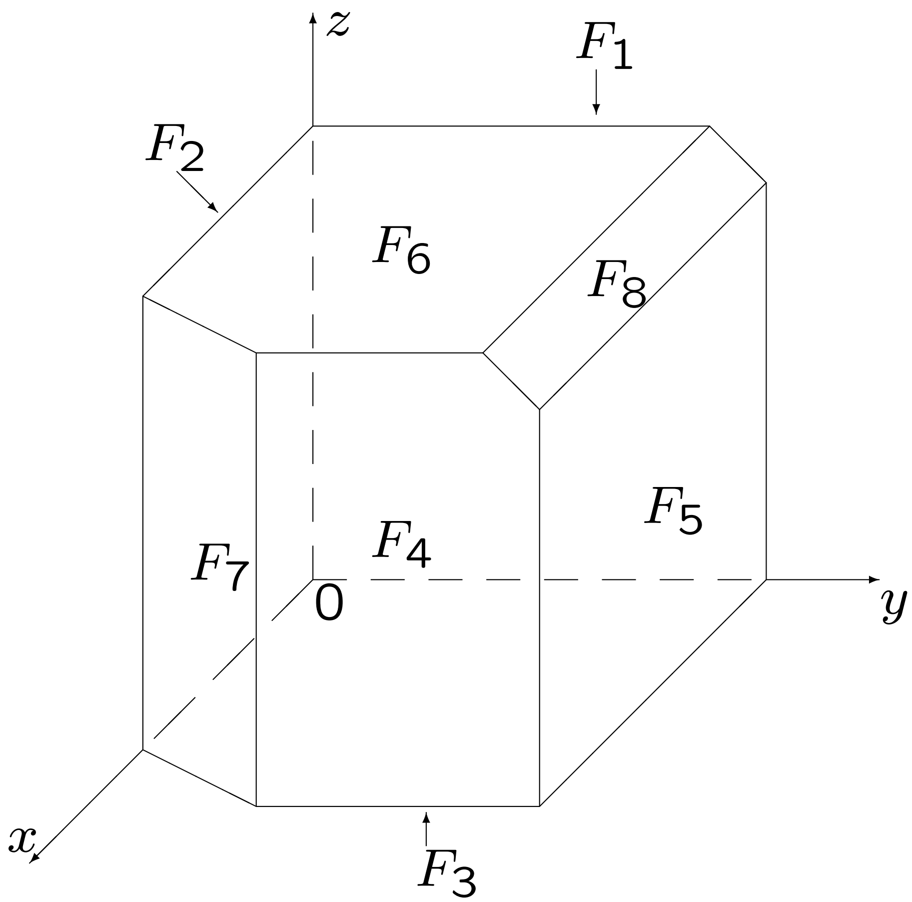
**Thm 12.** [Buchstaber-P., Franz] *For every simplicial fan $\Sigma$ there are algebra isomorphisms*

$$H^*\bigl(\mathcal{Z}(\Sigma); \mathbb{Z}\bigr) \cong \mathrm{Tor}^*_{\mathbb{Z}[v_1,\dots,v_m]}\bigl(\mathbb{Z}[\mathcal{K}_\Sigma], \mathbb{Z}\bigr) \cong H\Bigl[\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}_\Sigma], d\Bigr],$$

*where* $\deg u_i = 1$, $\deg v_i = 2$, $du_i = v_i$, $dv_i = 0$, *for* $1 \leqslant i \leqslant m$.

**Ex 13.** Let $P$ be the simple polytope obtained by cutting two non-adjacent edges off a cube in $N^*_{\mathbb{R}} \cong \mathbb{R}^3$. We may specify $P$ by 8 inequalities:

$$x \geqslant 0, \quad y \geqslant 0, \quad z \geqslant 0,$$
$$-x + 3 \geqslant 0, \quad -y + 3 \geqslant 0,$$
$$-z + 3 \geqslant 0,$$
$$-x + y + 2 \geqslant 0, \quad -y - z + 5 \geqslant 0.$$



Toric variety $X_P$ is obtained by blowing up the product $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ at two complex 1-dimensional subvarieties $\{\infty\} \times \{0\} \times \mathbb{C}P^1$ and $\mathbb{C}P^1 \times \{\infty\} \times \{\infty\}$.

The Kempf–Ness set $\mathcal{Z}_P$ is given by 5 real quadratic equations:

$$|z_1|^2 + |z_4|^2 - 3 = 0, \quad |z_2|^2 + |z_5|^2 - 3 = 0,$$

$$|z_3|^2 + |z_6|^2 - 3 = 0, \quad |z_1|^2 - |z_2|^2 + |z_7|^2 - 2 = 0,$$

$$|z_2|^2 + |z_3|^2 + |z_8|^2 - 5 = 0.$$

It is an 11-dimensional manifold with Betti vector

$$(1, 0, 0, 10, 16, 5, 5, 16, 10, 0, 0, 1)$$

and non-trivial Massey products of 3-dimensional classes (Baskakov'03).