Toric Topology

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1. From polytopes to quadrics.

$\mathbb{R}^n$: Euclidean vector space. Consider a convex polyhedron

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0 \ \text{ for } 1 \leqslant i \leqslant m\}, \quad \boldsymbol{a}_i \in \mathbb{R}^n,\ b_i \in \mathbb{R}.$$

Assume $\dim P = n$, no redundant inequalities, P is bounded, and bounding hyperplanes $H_i = \{(\boldsymbol{a}_i, \boldsymbol{x}) + b_i = 0\}$, $1 \leqslant i \leqslant m$, intersect in general position at every vertex.

Then P is an n-dim convex simple polytope with m facets

$$F_i = \{\boldsymbol{x} \in P \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i = 0\} = P \cap H_i$$

and normal vectors $\boldsymbol{a}_i$, for $1 \leqslant i \leqslant m$. At every vertex meets an n-tuple of facets.

Two polytopes are said to be combinatorially equivalent if their face posets are isomorphic.

We may specify P by a matrix inequality

$$P = \{\boldsymbol{x} \colon A_P\boldsymbol{x} + \boldsymbol{b}_P \geqslant 0\},$$

where $A_P = (a_{ij})$ is the $m \times n$ matrix of row vectors $\boldsymbol{a}_i$, and $\boldsymbol{b}_P$ is the column vector of scalars b_i.

The affine injection

$$i_P \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad \boldsymbol{x} \mapsto A_P\boldsymbol{x} + \boldsymbol{b}_P$$

embeds P into $\mathbb{R}^m_{\geqslant} = \{\boldsymbol{y} \in \mathbb{R}^m \colon y_i \geqslant 0\}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m \end{array} \qquad\qquad \begin{array}{c} (z_1, \ldots, z_m) \\ \big\downarrow \\ (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

Here i_Z is a T^m-equivariant embedding.

Prop 1. *$\mathcal{Z}_P$ is a smooth T^m-manifold with canonically trivialised normal bundle of $i_Z\colon \mathcal{Z}_P \to \mathbb{C}^m$.*

Idea of proof.

1) Write the image $i_P(\mathbb{R}^n) \subset \mathbb{R}^m$ as the set of common solutions of $m-n$ linear equations $\sum_{k=1}^m c_{jk}(y_k - b_k) = 0$, $1 \leqslant j \leqslant m-n$;
2) replace every y_k by $|z_k|^2$ to get a representation of $\mathcal{Z}_P$ as an intersection of $m-n$ real quadratic hypersurfaces:

$$\sum_{k=1}^{m} c_{jk}\left(|z_k|^2 - b_k\right) = 0, \quad \text{for } 1 \leqslant j \leqslant m-n.$$

3) check that 2) is a non-degenerate intersection, i.e. the gradient vectors are linearly independent at each point of $\mathcal{Z}_P$. □

$\mathcal{Z}_P$ is called the moment-angle manifold corresponding to P.

In fact, the topological type of $\mathcal{Z}_P$ depends only on the combinatorial type of P (the original construction of [Davis–Januszkiewicz]).

Write the system $\sum_{k=1}^{m} c_{jk}(|z_k|^2 - b_k) = 0$, $1 \leqslant j \leqslant m - n$, as

$$\mathcal{Z}_P = \{\boldsymbol{z} \in \mathbb{C}^m \colon C|\boldsymbol{z}|^2 = C\boldsymbol{b}_P\}$$

with $C = (c_{jk})$ $(m-n) \times m$-matrix, $|\boldsymbol{z}|^2$ column of $|z_k|^2$.

Rows of C constitute a basis in the space of linear relations between the $\boldsymbol{a}_i$'s. That is, $CA_P = 0$ and $\operatorname{rank} C = m - n$ (note $\operatorname{rank} A_P = n$).

Given P, how to choose C?

1st method:

Assume the first n facets $F_1, \ldots, F_n$ meet at a vertex, and take their normals as the basis for $\mathbb{R}^n$ (after linear transformation). Then

$$A_P = \begin{pmatrix} E \\ A_P^* \end{pmatrix}$$

with E unit $n \times n$-matrix and A_P^* an $(m-n) \times n$-matrix. Then we may take

$$C = \begin{pmatrix} -A_P^* & E \end{pmatrix}.$$

(Remember $CA_P = 0$!)

This is convenient for applications in cobordism (finding quasitoric representatives in complex cobordism classes).

2^{nd} method:

Have $\alpha_1 \boldsymbol{a}_1 + \ldots + \alpha_m \boldsymbol{a}_m = 0$ with $\alpha_i > 0$.
In fact, $\alpha_i = \operatorname{Vol} F_i$ if $|\boldsymbol{a}_i| = 1$.

By scaling the $\boldsymbol{a}_i$'s can always achieve $\boldsymbol{a}_1 + \ldots + \boldsymbol{a}_m = 0$ and so take $C = \begin{pmatrix} C_1 \\ 1 \cdots 1 \end{pmatrix}$ where C_1 is $(m-n-1) \times m$.

By moving the origin 0 into $\operatorname{Int} P$, get $b_i > 0$. By scaling P get $b_1 + \ldots + b_m = 1$, so the last quadratic equation defining $\mathcal{Z}_P$ is

$$|z_1|^2 + \ldots + |z_m|^2 = 1.$$

Subtracting this from the first $m-n-1$ equations, finally get

$$\mathcal{Z}_P = \begin{cases} \boldsymbol{z} \in \mathbb{C}^m \colon & C_* |\boldsymbol{z}|^2 = 0, \\ & |z_1|^2 + \ldots + |z_m|^2 = 1 \end{cases}$$

where C_* is $(m-n-1) \times m$.

Ex 1.

$$P = \left\{ \begin{array}{ll} \boldsymbol{x} \in \mathbb{R}^n\colon & x_i \geqslant 0,\ i = 1, \ldots, n, \\ & -x_1 - \ldots - x_n + 1 \geqslant 0 \end{array} \right\}: \qquad n\text{-simplex.}$$

So $m = n + 1$, $\boldsymbol{a}_i = \boldsymbol{e}_i$ for $i = 1, \ldots, n$, $\boldsymbol{a}_{n+1} = -\boldsymbol{e}_1 - \ldots - \boldsymbol{e}_n$.

$A_P = \begin{pmatrix} E \\ -1 \cdots -1 \end{pmatrix}$, $C = (1 \cdots 1)$,

$$\mathcal{Z}_P = \{\boldsymbol{z}\colon |z_1|^2 + \ldots + |z_m|^2 = 1\} = S^{2n+1},$$

and $C_* = \varnothing$.

2. From quadrics to polytopes.

After Lopes de Medrano, Bosio–Meersseman.

$C_* = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$: $p \times m$ matrix, $0 \leqslant p < m$ (later set $p = m - n - 1$).

Set

$$\mathcal{L}_C = \left\{ \begin{array}{ll} \mathbf{z} \in \mathbb{C}^m : & \mathbf{c}_1|z_1|^2 + \ldots + \mathbf{c}_m|z_m|^2 = 0 \\ & |z_1|^2 + \ldots + |z_m|^2 = 1 \end{array} \right\} :$$

"link of p special real quadrics in $\mathbb{C}^m$".

C_* is admissible if $\mathcal{L}_C$ is nonempty and nondegenerate.
(So $\mathcal{L}_C$ is a $(2m - p - 1)$-dimensional manifold.)

$\mathrm{Conv}(C_*) :=$ convex hull of $\mathbf{c}_1, \ldots, \mathbf{c}_m$ in $\mathbb{R}^p$.

Lemma 1. *C_* is admissible iff*
1) $0 \in \mathrm{Conv}(C_*)$, *and*
2) $0 \in \mathrm{Conv}(\mathbf{c}_i,\ i \in I)$ *implies* $|I| > p$.

Ex 2. $p = 1$, so $\boldsymbol{c}_i \in \mathbb{R}$. Then (2) implies $\boldsymbol{c}_i \neq 0$.
Assume k of $\boldsymbol{c}_i$ are positive and $l = m - k$ are negative.
Then (1) implies $k > 0$, $l > 0$.
Get like $|z_1|^2 + \ldots + |z_k|^2 - |z_{k+1}|^2 - \ldots - |z_m|^2 = 0$:
cone over $S^{2k-1} \times S^{2l-1}$,
and $|z_1|^2 + \ldots + |z_m|^2 = 1$, so $\mathcal{L}_C = S^{2k-1} \times S^{2l-1}$.

Ex 3. $p = 2$, so $\boldsymbol{c}_i \in \mathbb{R}^2$. Then by (1), $0 \in \mathrm{Conv}(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$.
(2) says that no segment joining $\boldsymbol{c}_i$ contains 0.
Lemma 2. *Can always achieve odd number of points on a circle with positive weights assigned.*

Set $k(C) =$ number of i such that $\mathcal{L}_C \cap \{z_i = 0\} = \varnothing$.
Lemma 3. $\mathcal{L}_C \cong \mathcal{L}_{C'} \times T^{k(C)}$, *where* $\mathcal{L}_{C'} \subset \mathbb{C}^{m-k(C)}$ *intersects every coordinate hyperplane.*

How to get a polytope out of $\mathcal{L}_C$?

$\mathcal{L}_C/T^m$ is given by the nonnegative solutions of

$$C_*\boldsymbol{y} = 0, \quad y_1 + \ldots + y_m = 1.$$

By nondegeneracy condition (2), this system has maximal rank. So may write its general solution as

$$y_i = (\boldsymbol{a}_i, \boldsymbol{x}) + b_i, \quad \boldsymbol{x} \in \mathbb{R}^{m-p-1}, \text{ with } b_i > 0.$$

Therefore,

$$\mathcal{L}_C/T^m = \{\boldsymbol{x} \in \mathbb{R}^{m-p-1} \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0\}.$$

Lemma 4. *This is a simple* $(m-p-1)$*-dimensional polytope* P *with* $m - k(C_*)$ *facets.*

So every $\{z_i = 0\} \cap \mathcal{L}_C = \varnothing$ gives a redundant inequality.

$P^* = \text{Conv}\left(\frac{\boldsymbol{a}_1}{b_1}, \ldots, \frac{\boldsymbol{a}_m}{b_m}\right)$: polar (or dual) simplicial polytope.
Denote $\boldsymbol{a}'_i = \frac{\boldsymbol{a}_i}{b_i}$.

Lemma 5. $0 \in \text{Int}\,\text{Conv}(\boldsymbol{c}_i,\ i \in I)$
$\iff \text{Conv}(\boldsymbol{a}'_i,\ i \in [m] \setminus I)$ *is a proper face of* P^*.

In other words, $(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$ is the Gale diagram of $(\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_m)$.

Finally, we get
Thm 1. *Every* $\mathcal{L}_C$ *with* $k(C) = 0$ *is* $\mathcal{Z}_P$ *for some* P.

3. Topology of $\mathcal{Z}_P$.

$F_1, \ldots, F_m$: facets of P.

Given $I \subset [m]$, set $P_I = \bigcup_{i \in I} F_i \subset P$.

Thm 2. $H^k(\mathcal{Z}_P) = \bigoplus_{I \subset [m]} \widetilde{H}^{k-|I|-1}(P_I)$.

Ex 4. P a 5-gon. Then $\dim \mathcal{Z}_P = 7$, and the Betti vector is

$$(1, 0, 0, 5, 5, 0, 0, 1).$$

In fact, $\mathcal{Z}_P = (S^3 \times S^4)^{\# 5}$.

Proof of Thm 2. $\mathcal{Z}_P$ generalises to $\mathcal{Z}_K$, the moment-angle complex defined for an arbitrary simplicial complex K on m vertices.

Given P as above, set

$$K_P = \Big\{\sigma = \{i_1, \dots, i_k\} \colon F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing \text{ in } P\Big\},$$

the boundary complex of P^*. It is a sphere triangulation: $|K_P| \cong S^{n-1}$.

Then $\mathcal{Z}_{K_P} = \mathcal{Z}_P$.

By [Buchstaber-P], there is an isomorphism of (bi)graded algebras

$$\begin{aligned} H^*(\mathcal{Z}_K) &\cong \mathrm{Tor}^{*,*}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K], \mathbb{Z}) \\ &\cong H\Big[\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]; d\Big], \end{aligned}$$

where $\mathbb{Z}[K]$ is the face ring (or the Stanley–Reisner ring) of K, $du_i = v_i$, $dv_i = 0$ for $1 \leqslant i \leqslant m$.

From this description follows Hochster's calculation of Tor modules in terms of full subcomplexes of K:

$$\mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K],\mathbb{Z}) \cong \bigoplus_{|J|=j} \widetilde{H}^{j-i-1}(K_J),$$

where K_J is the restriction of K to the subset $J \subset [m]$.

This dualises to the required description of the cohomology in the case $K = K_P$ (because P_J retracts onto K_J). $\square$

4. Quasitoric manifolds and cobordism.

Assume given P as above, and an integral $n \times m$ matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}$$

satisfying the condition

> the column vectors $\lambda_{j_1}, \dots, \lambda_{j_n}$ corresponding to any vertex $v = F_{j_1} \cap \dots \cap F_{j_n}$ form a basis of $\mathbb{Z}^n$.

We refer to (P, Λ) as a combinatorial quasitoric pair.

Define $K = K(\Lambda) := \ker(\Lambda\colon T^m \to T^n) \cong T^{m-n}$.

Prop 2. $K(\Lambda)$ *acts freely on* $\mathcal{Z}_P$.

The quotient

$$M = M(P, \Lambda) := \mathcal{Z}_P / K(\Lambda)$$

is the quasitoric manifold corresponding to (P, Λ). It has a residual T^n-action ($T^m/K(\Lambda) \cong T^n$) satisfying the two Davis–Januszkiewicz conditions:

a) the T^n-action is locally standard;

b) there is a projection $\pi\colon M \to P$ whose fibres are orbits of the T^n-action.

Algebraic and symplectic geometers would recognise in the above construction of a quasitoric manifold M from $\mathcal{Z}_P$ a generalisation of the symplectic reduction construction of a Hamiltonian toric manifold. In the latter case we take $\Lambda = A_P^t$; then M is a toric manifold corresponding to the Delzant polytope

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m\}, \quad \boldsymbol{a}_i \in \mathbb{Z}^n,\ b_i \in \mathbb{R}.$$

Here we additionally assume the normal vectors $\boldsymbol{a}_i$ to be *integer*, and the Delzant condition:

> for every vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P, the corresponding normal vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ form a basis of $\mathbb{Z}^n$

to be satisfied.

Then $\mathcal{Z}_P$ is the level set for the moment map $\mu \colon \mathbb{C}^m \to \mathbb{R}^{m-n}$ corresponding to the Hamiltonian action of $K = \operatorname{Ker} \Lambda = \operatorname{Ker} A^t$ on $\mathbb{C}^m$.

Define complex line bundles

$$\rho_i\colon \mathcal{Z}_P \times_K \mathbb{C}_i \to M, \quad 1 \leqslant i \leqslant m,$$

where $\mathbb{C}_i$ is the 1-dim complex T^m-representation defined via the quotient projection $\mathbb{C}^m \to \mathbb{C}_i$ onto the ith factor.

Thm 3 (Davis–Januszkiewicz). *There is an isomorphism of real vector bundles*

$$\tau M \oplus \mathbb{R}^{2(m-n)} \xrightarrow{\cong} \rho_1 \oplus \cdots \oplus \rho_m.$$

This endows M with the canonical equivariant stably complex structure. So we may consider its complex cobordism class $[M] \in \Omega_U$.

Thm 4 (Buchstaber-P-Ray). *Every complex cobordism class in* dim > 2 *contains a quasitoric manifold.*

The complex cobordism ring Ω_U is multiplicatively generated by the cobordism classes $[H_{ij}]$, $0 \leqslant i \leqslant j$, of Milnor hypersurfaces

$H_{ij} = \{(z_0 : \ldots : z_i) \times (w_0 : \ldots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : z_0 w_0 + \ldots + z_i w_i = 0\}$.

But H_{ij} is not a quasitoric manifold if $i > 1$.

Idea of proof of Thm 4

1) Replace each H_{ij} by a quasitoric (in fact, toric) manifold B_{ij} so that $\{B_{ij}\}$ is still a multiplicative generator set for Ω_U. Therefore, every stably complex manifold is cobordant to the disjoint union of products of B_{ij}'s. Every such product is a q-t manifold, but their disjoint union is not.
2) Replace disjoint unions by certain connected sums. This is tricky, because you need to take account of both the torus action and the stably complex structure.

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