Torus Actions and Complex Cobordism

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Thm. *Every complex cobordism class in* dim > 2 *contains a quasitoric manifold.*

In other words, every stably complex manifold is cobordant to a manifold with a nicely behaving torus action.

In cobordism theory, all manifolds are smooth and closed.

$M_1^n \simeq M_2^n$ (co)bordant if there is a manifold *with boundary* W^{n+1} such that $\partial W^{n+1} = M_1 \sqcup M_2$.

Complex bordism: work with complex manifolds.

complex mflds $\subset$ almost complex mflds $\subset$ stably complex mflds

Stably complex structure on a $2n$-dim manifold M is determined by a choice of isomorphism

$$c_{\mathcal{T}} \colon \tau M \oplus \mathbb{R}^{2(l-n)} \xrightarrow{\cong} \xi$$

where ξ is an l-dim *complex* vector bundle.

Complex bordism classes $[M, c_{\mathcal{T}}]$ form the complex bordism ring Ω^U with respect to the disjoint union and product.

$$\Omega^U \cong \mathbb{Z}[a_1, a_2, ...], \quad \text{dim } a_i = 2i \quad \text{Novikov'60}.$$

Quasitoric manifolds: $2n$-dimensional manifolds M with a “nice” action of the torus T^n (after Davis–Januszkiewicz);

- the T^n-action is locally standard (locally looks like the standard T^n-representation in $\mathbb{C}^n$);
- the orbit space M/T^n is an n-dim simple polytope P.

Examples include projective smooth toric varieties and symplectic manifolds M with Hamiltonian actions of T^n (also known as toric manifolds).

Polytopes and moment-angle manifolds.

$\mathbb{R}^n$: Euclidean vector space. Consider a convex polyhedron

$$P=\{\boldsymbol{x}\in\mathbb{R}^n\colon (\boldsymbol{a}_i,\boldsymbol{x})+b_i\geqslant 0\ \text{ for } 1\leqslant i\leqslant m\},\qquad \boldsymbol{a}_i\in\mathbb{R}^n,\ b_i\in\mathbb{R}.$$

Assume $\dim P=n$, no redundant inequalities, P is bounded, and bounding hyperplanes $H_i=\{(\boldsymbol{a}_i,\boldsymbol{x})+b_i=0\}$, $1\leqslant i\leqslant m$, intersect in general position at every vertex.

Then P is an n-dim convex simple polytope with m facets

$$F_i=\{\boldsymbol{x}\in P\colon (\boldsymbol{a}_i,\boldsymbol{x})+b_i=0\}=P\cap H_i$$

and normal vectors $\boldsymbol{a}_i$, for $1\leqslant i\leqslant m$. At every vertex meets an n-tuple of facets.

Two polytopes are said to be combinatorially equivalent if their face posets are isomorphic.

We may specify P by a matrix inequality

$$P = \{\boldsymbol{x} \colon A_P\boldsymbol{x} + \boldsymbol{b}_P \geqslant 0\},$$

where $A_P = (a_{ij})$ is the $m \times n$ matrix of row vectors $\boldsymbol{a}_i$, and $\boldsymbol{b}_P$ is the column vector of scalars b_i.

The affine injection

$$i_P \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad \boldsymbol{x} \mapsto A_P\boldsymbol{x} + \boldsymbol{b}_P$$

embeds P into $\mathbb{R}^m_{\geqslant} = \{\boldsymbol{y} \in \mathbb{R}^m \colon y_i \geqslant 0\}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m \end{array} \qquad\qquad \begin{array}{c} (z_1, \ldots, z_m) \\ \big\downarrow \\ (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

Here i_Z is a T^m-equivariant embedding.

Prop 1. *$\mathcal{Z}_P$ is a smooth T^m-manifold with canonically trivialised normal bundle of $i_Z\colon \mathcal{Z}_P \to \mathbb{C}^m$.*

Idea of proof.

1) Write the image $i_P(\mathbb{R}^n) \subset \mathbb{R}^m$ as the set of common solutions of $m-n$ linear equations $\sum_{k=1}^m c_{jk}(y_k - b_k) = 0$, $1 \leqslant j \leqslant m-n$;
2) replace every y_k by $|z_k|^2$ to get a representation of $\mathcal{Z}_P$ as an intersection of $m-n$ real quadratic hypersurfaces:

$$\sum_{k=1}^m c_{jk}\left(|z_k|^2 - b_k\right) = 0, \quad \text{for } 1 \leqslant j \leqslant m-n.$$

3) check that 2) is a non-degenerate intersection, i.e. the gradient vectors are linearly independent at each point of $\mathcal{Z}_P$. □

$\mathcal{Z}_P$ is called the moment-angle manifold corresponding to P.

In fact, the topological type of $\mathcal{Z}_P$ depends only on the combinatorial type of P (the original construction of Davis–Januszkiewicz).

Quasitoric manifolds from combinatorial data.

Assume given P as above, and an integral $n \times m$ matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}$$

satisfying the condition

> the columns of $\lambda_{j_1}, \dots, \lambda_{j_n}$ corresponding to any vertex $p = F_{j_1} \cap \dots \cap F_{j_n}$ form a basis of $\mathbb{Z}^n$.

We refer to (P, Λ) as the combinatorial quasitoric pair.

Define $K = K(\Lambda) := \ker(\Lambda\colon T^m \to T^n) \cong T^{m-n}$.

Prop 2. *$K(\Lambda)$ acts freely on $\mathcal{Z}_P$.*

The quotient

$$M = M(P, \Lambda) := \mathcal{Z}_P / K(\Lambda)$$

is the quasitoric manifold corresponding to (P, Λ). It has a residual T^n-action ($T^m/K(\Lambda) \cong T^n$) satisfying the two Davis–Januszkiewicz conditions:

a) the T^n-action is locally standard;

b) there is a projection $\pi\colon M \to P$ whose fibres are orbits of the T^n-action.

Algebraic and symplectic geometers would recognise in the above construction of a quasitoric manifold M from $\mathcal{Z}_P$ a generalisation of the symplectic reduction construction of a Hamiltonian toric manifold. In the latter case we take $\Lambda = A_P^t$; then M is a toric manifold corresponding to the Delzant polytope

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0 \ \text{ for } 1 \leqslant i \leqslant m\}, \qquad \boldsymbol{a}_i \in \mathbb{Z}^n,\ b_i \in \mathbb{R}.$$

Here we additionally assume the normal vectors $\boldsymbol{a}_i$ to be *integer*, and the Delzant condition:

> for every vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P, the corresponding normal vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ form a basis of $\mathbb{Z}^n$

to be satisfied.

Then $\mathcal{Z}_P$ is the level set for the moment map $\mu \colon \mathbb{C}^m \to \mathbb{R}^{m-n}$ corresponding to the Hamiltonian action of $K = \operatorname{Ker} \Lambda = \operatorname{Ker} A^t$ on $\mathbb{C}^m$.

Quasitoric representatives in cobordism classes.

Define complex line bundles

$$\rho_i\colon \mathcal{Z}_P \times_K \mathbb{C}_i \to M, \quad 1 \leqslant i \leqslant m,$$

where $\mathbb{C}_i$ is the 1-dim complex T^m-representation defined via the quotient projection $\mathbb{C}^m \to \mathbb{C}_i$ onto the ith factor.

Thm 3 (Davis–Januszkiewicz). *There is an isomorphism of real vector bundles*

$$\tau M \oplus \mathbb{R}^{2(m-n)} \xrightarrow{\cong} \rho_1 \oplus \cdots \oplus \rho_m.$$

This endows M with the canonical equivariant stably complex structure. So we may consider its complex cobordism class $[M] \in \Omega^U$.

Thm 4 (Buchstaber–P–Ray). *Every complex cobordism class in* $\dim > 2$ *contains a quasitoric manifold.*

The complex cobordism ring Ω^U is multiplicatively generated by the cobordism classes $[H_{ij}]$, $0 \leqslant i \leqslant j$, of Milnor hypersurfaces

$$H_{ij} = \{(z_0 : \ldots : z_i) \times (w_0 : \ldots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : z_0 w_0 + \ldots z_i w_i = 0\}.$$

However, H_{ij} is *not* a quasitoric manifold if $i > 1$.

Idea of proof of Thm 4.

1) Replace each H_{ij} by a quasitoric manifold B_{ij} so that $\{B_{ij}\}$ is still a multiplicative generator set for Ω^U. Therefore, every stably complex manifold is cobordant to the disjoint union of products of B_{ij}'s. Every such product is a q-t manifold, but their disjoint union is not.
2) Replace the disjoint unions by certain connected sums. This is tricky, because we need to take account of both the torus action and the stably complex structure.

□

Equivariant cobordism and the universal toric genus.

X a T^k-space. There are 3 equivariant complex cobordism theories:

- $\Omega^*_{U:T^k}(X)$: geometric T^k-cobordisms: set of cobordism classes of stably tangentially complex T^k-bundles over X.

- $MU^*_{T^k}(X) = \lim[S^V \wedge X_+, MU_{T^k}(W)]_{T^k}$: homotopic T^k-cobordisms; here $MU_{T^k}(W)$ is the Thom T^k-space of the universal $|W|$-dimensional complex T^k-vector bundle $\gamma_{|W|}$, and S^V is the unit sphere in a T^k-representation space V.

- $\Omega^*_U(ET^k \times_{T^k} X)$: Borel T^k-cobordisms.

There are natural transformations of cohomology theories

$$\varOmega^*_{U:T^k}(X) \xrightarrow{\nu} MU^*_{T^k}(X) \xrightarrow{\alpha} \varOmega^*_U(ET^k \times_{T^k} X).$$

Restricting to $X = pt$ we get a map

$$\varPhi := \alpha \cdot \nu \colon \varOmega^*_{U:T^k} \longrightarrow \varOmega^*_U(BT^k) = \Omega^U_*[[u_1, \dots, u_k]],$$

which we refer to as the universal toric genus. It assigns to the cobordism class $[M, c_\tau] \in \varOmega^{-2n}_{U:T^k}$ of a $2n$-dimensional T^k-manifold M the "cobordism class" of the map $ET^k \times_{T^k} M \to BT^k$.

We may write

$$\varPhi(M, c_\tau) = \sum_\omega g_\omega(M) u^\omega,$$

where $\omega = (\omega_1, \dots, \omega_k) \in \mathbb{N}^k$, $u^\omega = u_1^{\omega_1} \cdot \dots \cdot u_k^{\omega_k}$, $g_\omega(M) \in \varOmega^U_{2(|\omega|+n)}$.

We have $g_0(M) = [M] \in \varOmega^U_{2n}$. How to express the other coefficients $g_\omega(M)$?

Ray's basis in $\Omega_*^U(BT^k)$.

Consider the product of unit 3-spheres

$$(S^3)^j = \left\{(z_1, \dots, z_{2j}) \in \mathbb{C}^{2j} \colon |z_i|^2 + |z_{i+j}|^2 = 1 \text{ for } 1 \leqslant i \leqslant j\right\}$$

with the free T^j-action by

$$(t_1, \dots, t_j) \cdot (z_1, \dots, z_{2j}) = (t_1^{-1} z_1, t_1^{-1} t_2^{-1} z_2, \dots, t_{j-1}^{-1} t_j^{-1} z_j, t_1 z_{j+1}, \dots, t_j z_{2j})$$

The quotient $B_j := (S^3)^j / T^j$ is the bounded flag manifold. It is a "Bott tower", i.e. a j-fold iterated 2-sphere bundle over $B_0 = *$.

For $1 \leqslant i \leqslant j$ there are complex line bundles

$$\psi_i \colon (S^3)^j \times_{T^j} \mathbb{C} \longrightarrow B_j$$

via the action $(t_1, \dots, t_j) \cdot z = t_i z$ for $z \in \mathbb{C}$.

For any $j > 0$ have an explicit isomorphism

$$\tau(B_j) \oplus \mathbb{C}^j \cong \psi_1 \oplus \psi_1 \psi_2 \oplus \cdots \oplus \psi_{j-1} \psi_j \oplus \bar{\psi}_1 \oplus \cdots \oplus \bar{\psi}_j,$$

which defines a stably cplx structure c_j^∂ on B_j with $[B_j, c_j^\partial] = 0$ in Ω_{2j}^U.

Prop 5. *The basis element* $b_\omega \in \Omega^U_{2|\omega|}(BT^k)$ *dual to* $u^\omega \in \Omega^*_U(BT^k)$ *is represented geometrically by the classifying map*

$$\psi_\omega \colon B_\omega \longrightarrow BT^k$$

for the product $\psi_{\omega_1} \times \cdots \times \psi_{\omega_k}$ *of line bundles over* $B_\omega = B_{\omega_1} \times \ldots \times B_{\omega_k}$.

Let $T^\omega = T^{\omega_1} \times \ldots \times T^{\omega_k}$,and $(S^3)^\omega = (S^3)^{\omega_1} \times \ldots \times (S^3)^{\omega_k}$, on which T^ω acts coordinatewise. Define

$$G_\omega(M) := (S^3)^\omega \times_{T^\omega} M,$$

where T^ω acts on M via the representation

$$(t_{1,1}, \ldots, t_{1,\omega_1}; \ldots; t_{k,1}, \ldots, t_{k,\omega_k}) \longmapsto (t^{-1}_{1,\omega_1}, \ldots, t^{-1}_{k,\omega_k}).$$

The stably complex structure c_ω on $G_\omega(M)$ is induced by the structures c_τ and c^∂_ω on the base and fibre of the bundle $M \to G_\omega(M) \to B_\omega$.

Thm 6. *The manifold* $G_\omega(M)$ *represents* $g_\omega(M)$ *in* $\Omega^U_{2(|\omega|+n)}$.

Hirzebruch genera and equivariant extentions.

R_* a (graded) commutative ring with unit.

$\ell\colon \Omega_*^U \to R_*$ a Hirzebruch genus.

Every genus ℓ has a T^k-equivariant extension

$$\ell^{T^k} := \ell \cdot \Phi\colon \Omega_*^{U:T^k} \longrightarrow R_*[[u_1, \dots, u_k]].$$

We have

$$\ell^{T^k}(M, c_\tau) \;=\; \ell(M) \;+ \sum_{|\omega|>0} \ell(g_\omega(M))\, u^\omega.$$

In particular, the T^k-equivariant extension of the universal genus $ug =$ id$\colon \Omega_*^U \to \Omega_*^U$ is Φ; hence the name “universal toric genus”.

Rigidity and fibre multiplicativity.

A genus ℓ is multiplicative with respect to N when $\ell(E) = \ell(N)\ell(B)$ holds for every bundle $E \to B$ of stably complex manifolds with compact connected structure group and fibre N. If ℓ is multiplicative with respect to every N, then it is fibre multiplicative.

The genus ℓ is T^k-rigid on M when $\ell^{T^k}\colon \Omega_*^{U:T^k} \longrightarrow R_*[[u_1, \ldots, u_k]]$ is constant, i.e. satisfies $\ell^{T^k}(M) = \ell(M)$.

If ℓ^{T^k} is rigid on every M, then ℓ is T^k-rigid.

In fact, T^1-rigidity suffices to imply G-rigidity for any compact Lie group G. We therefore refer simply to rigidity in case $k = 1$.

It follows that ℓ is rigid whenever $\ell(G_\omega(M)) = 0$ for $|\omega| > 0$.

Prop 7. *If a genus ℓ is multiplicative with respect to M, then it is T^k-rigid on M.*

Proof. The B_ω bound for $|\omega| > 0$, so apply ℓ to the bundle $M \to G_\omega(M) \to B_\omega$. □

Ex 8. The signature is fibre multiplicative over any simply connected base, so it is a rigid genus.

Isolated fixed points.

For any $x \in \mathrm{Fix}(M)$, have the representation $r_x \colon T^k \to GL(l, \mathbb{C})$ associated to the T^k-invariant structure $c_\tau \colon \tau M \oplus \mathbb{R}^{2(l-n)} \to \xi$. The fibre ξ_x decomposes as $\mathbb{C}^n \oplus \mathbb{C}^{l-n}$, where r_x has no trivial summands on $\mathbb{C}^n$, and is trivial on $\mathbb{C}^{l-n}$. Also, $c_{\tau,x}$ induces an orientation of $\tau_x(M)$.

For any $x \in \mathrm{Fix}(M)$, the sign $\varsigma(x)$ is $+1$ if the isomorphism

$$\tau_x(M) \xrightarrow{\;i\;} \tau_x(M) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{\;c_{\tau,x}\;} \xi_x \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{\;p\;} \mathbb{C}^n$$

respects the canonical orientations, and -1 if it does not.

So $\varsigma(x)$ compares the orientations induced by r_x and $c_{\tau,x}$ on $\tau_x(M)$, and if M is almost complex then $\varsigma(x) = 1$ for every $x \in \mathrm{Fix}(M)$.

The non-trivial summand of r_x decomposes into 1-dimensional representations as $r_{x,1} \oplus \ldots \oplus r_{x,n}$, and we write the integral weight vector of $r_{x,j}$ as $w_j(x) := (w_{j,1}(x), \ldots, w_{j,k}(x))$, for $1 \le j \le n$.

We refer to the collection of signs $\varsigma(x)$ and weight vectors $w_j(x)$ as the fixed point data for (M, c_τ).

Each weight vector determines a line bundle

$$\zeta^{w_j(x)} := \zeta_1^{w_{j,1}(x)} \otimes \cdots \otimes \zeta_k^{w_{j,k}(x)}$$

over BT^k, whose first Chern class is a formal power series

$$[w_j(x)](u) := \sum_\omega a_\omega [w_{j,1}(x)](u_1)^{\,\omega_1} \cdots [w_{j,k}(x)](u_k)^{\,\omega_k}$$

in $\Omega_U^2(BT^k)$. Here $[m](u_j)$ denotes the power series $c_1^{MU}(\zeta_j^m)$ in $\Omega_U^2(\mathbb{C}P^\infty)$, and the a_ω are the coefficients of $c_1^{MU}(\zeta_1 \otimes \cdots \otimes \zeta_k)$.

Modulo decomposables we have that

$$[w_j(x)](u_1, \ldots, u_k) \equiv w_{j,1}u_1 + \cdots + w_{j,k}u_k.$$

Thm 9 (Localisation formula). *For any stably tangentially complex M^{2n} with isolated fixed points, the equation*

$$\Phi(M) = \sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{1}{[w_j(x)](u)}$$

is satisfied in $\Omega_U^{-2n}(BT^k)$.

Quasitoric manifolds revisited.

Quasitoric manifolds provide a vast source of examples of stably complex T^n-manifolds with isolated fixed points, for which calculations with the fixed point data and Hirzebruch genera can be made explicit.

Thm 10. *For any quasitoric manifold M with combinatorial data (P, Λ) and fixed point $x = F_{j_1} \cap \ldots \cap F_{j_n}$, let $N(P)_x$ be a matrix of column vectors normal to $F_{j_1}, \ldots, F_{j_n}$, let Λ_x be square submatrix of Λ of column vectors $j_1, \ldots, j_n$, and W_x be the matrix determined by $W_x^t \Lambda_x = I_n$ (unit n-matrix). Then*

1. *the sign $\varsigma(x)$ is given by* $\operatorname{sign}\Big(\det(\Lambda_x N(P)_x)\Big)$
2. *the weight vectors $w_1(x)$, $\ldots w_n(x)$ are the columns of W_x.*

Elliptic genera.

Buchstaber introduced the formal group law

$$F_b(u_1, u_2) \ = \ u_1 c(u_2) + u_2 c(u_1) - a u_1 u_2 - \frac{d(u_1) - d(u_2)}{u_1 c(u_2) - u_2 c(u_1)} u_1^2 u_2^2$$

over the graded ring $R_* = \mathbb{Z}[a, c_j, d_k \colon j \geq 2, k \geq 1]/J$, where $\deg a = 2$, $\deg c_j = 2j$ and $\deg d_k = 2(k+2)$; also, J is the ideal of associativity relations, and

$$c(u) \ := \ 1 + \sum_{j \geq 2} c_j u^j, \quad d(u) \ := \ \sum_{k \geq 1} d_k u^k.$$

Thm 11. *The exponential series $f_b(x)$ of F_b may be written analytically as* $\exp(ax)/\phi(x, z)$*, where*

$$\phi(x, z) \ = \ \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} \exp(\zeta(z)\, x),$$

$\sigma(z)$ is the Weierstrass sigma function, and $\zeta(z) = (\ln \sigma(z))'$.

Moreover, $R_* \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}[a, c_2, c_3, c_4]$ as graded algebras.

The function $\varphi(x, z)$ is known as the Baker–Akhiezer function associated to the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$. It satisfies the Lamé equation, and is important in the theory of nonlinear integrable equations. Krichever studies the genus kv corresponding to the exponential series f_b, which therefore classifies the formal group law F_b. Analytically, it depends on the four complex variables z, a, g_2 and g_3.

Cor 12. *The genus $kv\colon \Omega_*^U \to R_*$ induces an isomorphism of graded abelian groups in dimensions < 10.*

Thm 13. *Let M^{2n} be an SU quasitoric manifold; then*
(1) *the Krichever genus kv vanishes on M^{2n}*
(2) *M^{2n} represents 0 in Ω_{2n}^U whenever $n < 5$.*

Conjecture 14. *Theorem 13(2) holds for all n.*

Further applications to rigidity.

Prop 15. *For any series* f *over a* $\mathbb{Q}$*-algebra* A*, the corresponding Hirzebruch genus* ℓ_f *is* T^k*-rigid on* M *only if the functional equation*

$$\sum_{\mathrm{Fix}(M)} \varsigma(x) \prod_{j=1}^{n} \frac{1}{f(w_j(x) \cdot u)} = c$$

is satisfied in $A[[u_1, \ldots, u_k]]$*, for some constant* $c \in A$*.*

The quasitoric examples $\mathbb{C}P^1$, $\mathbb{C}P^2$, and the T^2-manifold S^6 are all instructive.

Ex 16. A genus ℓ_f is T-rigid on $\mathbb{C}P^1$ only if the equation

$$\frac{1}{f(u)} + \frac{1}{f(-u)} = c,$$

holds in $A[[u]]$. The general analytic solution is

$$f(u) = \frac{u}{q(u^2) + cu/2}, \quad \text{where } q(0) = 1.$$

An example is provided by the Todd genus, $f_{td}(u) = (e^{zu} - 1)/z$. In fact td is multiplicative with respect to $\mathbb{C}P^1$.

Ex 17. A genus ℓ_f is T^2-rigid on the stably complex manifold $\mathbb{C}P^2_{(1,-1)}$ only if the equation

$$\frac{1}{f(u_1)f(u_2)} - \frac{1}{f(u_1)f(u_1+u_2)} + \frac{1}{f(-u_2)f(u_1+u_2)} = c$$

holds in $A[[u_1, u_2]]$. The general analytic solution satisfies

$$f(u_1+u_2) = \frac{f(u_1)+f(u_2)-c'f(u_1)f(u_2)}{1-cf(u_1)f(u_2)}.$$

So f is the exponential series of 2-parameter Todd genus $t2$ (also known as the $T_{x,y}$-genus), with $c' = y+z$ and $c = yz$.

Cor 18 (Musin). *The* 2*-parameter Todd genus* $t2$ *is universal for rigid genera.*

Ex 19. A genus ℓ_f is T^2-rigid on the almost complex manifold S^6 only if the equation

$$\frac{1}{f(u_1)f(u_2)f(-u_1-u_2)}+\frac{1}{f(-u_1)f(-u_2)f(u_1+u_2)} = c$$

holds in $A[[u_1,u_2]]$, for some constant c. The general analytic solution is of the form $\exp(ax)/\phi(x,z)$, and f coincides with Krichever's exponential series f_b.

Thm 20. *Krichever's generalised elliptic genus kv is universal for genera that are rigid on SU-manifolds.*

[1] Victor M. Buchstaber and Taras E. Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* Volume 24 of *University Lecture Series*, Amer. Math. Soc., Providence, R.I., 2002.

[2] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Spaces of polytopes and cobordism of quasitoric manifolds*. Moscow Math. J. **7** (2007), no. 2, 219–242; arXiv:math.AT/0609346.

[3] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Toric genera*. Preprint (2009); arXiv:0908.3298.