Moment-angle manifolds: recent developments and perspectives

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Moment-angle manifolds from simple polytopes.

$\mathbb{R}^n$ Euclidean vector space. Consider a convex polyhedron

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0 \ \text{ for } 1 \leqslant i \leqslant m\}, \quad \boldsymbol{a}_i \in \mathbb{R}^n,\ b_i \in \mathbb{R}.$$

Assume:

a) $\dim P = n$;

b) no redundant inequalities (cannot remove any inequality without changing P);

c) P is bounded;

d) bounding hyperplanes $H_i = \{(\boldsymbol{a}_i, \boldsymbol{x}) + b_i = 0\}$, $1 \leqslant i \leqslant m$, intersect in general position at every vertex, i.e. there are exactly n facets of P meeting at each vertex.

Then P is an n-dim convex simple polytope with m facets

$$F_i = \{\boldsymbol{x} \in P \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i = 0\} = P \cap H_i$$

and normal vectors $\boldsymbol{a}_i$, for $1 \leqslant i \leqslant m$.

The faces of P form a poset with respect to the inclusion. Two polytopes are said to be combinatorially equivalent if their face posets are isomorphic. The corresponding equivalence classes are called combinatorial polytopes.

We may specify P by a matrix inequality

$$P = \{\boldsymbol{x} \colon A_P\boldsymbol{x} + \boldsymbol{b}_P \geqslant 0\},$$

where $A_P = (a_{ij})$ is the $m \times n$ matrix of row vectors $\boldsymbol{a}_i$, and $\boldsymbol{b}_P$ is the column vector of scalars b_i.

The affine injection

$$i_P \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad \boldsymbol{x} \mapsto A_P\boldsymbol{x} + \boldsymbol{b}_P$$

embeds P into $\mathbb{R}^m_{\geqslant} = \{\boldsymbol{y} \in \mathbb{R}^m \colon y_i \geqslant 0\}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \downarrow & & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m \end{array} \qquad\qquad \begin{array}{c} (z_1, \ldots, z_m) \\ \downarrow \\ (|z_1|^2, \ldots, |z_m|^2) \end{array}$$

Here i_Z is a T^m-equivariant embedding.

Prop 1. *$\mathcal{Z}_P$ is a smooth T^m-manifold with canonically trivialised normal bundle of $i_Z\colon \mathcal{Z}_P \to \mathbb{C}^m$.*

Idea of proof.

1) Write the image $i_P(\mathbb{R}^n) \subset \mathbb{R}^m$ as the set of common solutions of $m-n$ linear equations $\sum_{k=1}^m c_{jk}(y_k - b_k) = 0$, $1 \leqslant j \leqslant m-n$;

2) replace every y_k by $|z_k|^2$ to get a representation of $\mathcal{Z}_P$ as an intersection of $m-n$ real quadratic hypersurfaces:

$$\sum_{k=1}^{m} c_{jk}\left(|z_k|^2 - b_k\right) = 0, \quad \text{for } 1 \le j \le m-n.$$

3) check that 2) is a non-degenerate intersection, i.e. the gradient vectors are linearly independent at each point of $\mathcal{Z}_P$. $\square$

$\mathcal{Z}_P$ is called the moment-angle manifold corresponding to P.

Original Davis–Januszkiewicz construction.

Given $\boldsymbol{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m_{\geqslant}$, set

$$T(\boldsymbol{y}) = \{\boldsymbol{t} = (t_1, \ldots, t_m) \in T^m \colon t_i = 1 \text{ if } y_i = 0\} \subset T^m.$$

Regard $\mathbb{C}^m$ as the identification space $\mathbb{R}^m_{\geqslant} \times T^m/\sim$, where

$$(\boldsymbol{y}, \boldsymbol{t}) \cong (\boldsymbol{y}', \boldsymbol{t}') \text{ iff } \boldsymbol{y} = \boldsymbol{y}' \text{ and } \boldsymbol{t}^{-1}\boldsymbol{t}' \in T(\boldsymbol{y}).$$

Then $i_Z \colon \mathcal{Z}_P \to \mathbb{C}^m$ embeds $\mathcal{Z}_P$ as a subspace $P \times T^m/\sim$ in $\mathbb{R}^m_{\geqslant} \times T^m/\sim$.

Cor 1. *The topological type of $\mathcal{Z}_P$ is determined by the combinatorial type of P.*

In fact, the T^m-equivariant smooth structure on $\mathcal{Z}_P$ is also unique [Bosio–Meersseman].

Simplicial complexes.

K : an (abstract) simplicial complex on the set $[m] = \{1, \dots, m\}$.

$\sigma = \{i_1, \dots, i_k\} \in K$ a simplex; always assume $\varnothing \in K$.

Ex 1. Given P as above, set

$$K_P = \Big\{\sigma = \{i_1, \dots, i_k\} \colon F_{i_1} \cap \ldots \cap F_{i_k} \neq \varnothing \text{ in } P\Big\},$$

the boundary complex of the dual (or polar) polytope of P. It is a sphere triangulation, i.e. $|K_P| \cong S^{n-1}$.

Moment-angle complexes.

$D^2 \subset \mathbb{C}$ unit disk. Given $\omega \subset \{1,\ldots,m\}$, set

$$B_\omega := \{(z_1,\ldots,z_m) \in (D^2)^m \colon |z_i| = 1 \text{ if } i \notin \omega\} \cong (D^2)^{|\omega|} \times (S^1)^{m-|\omega|}.$$

The moment-angle complex

$$\mathcal{Z}_K := \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^m.$$

Prop 2. *$\mathcal{Z}_K$ has a T^m-action with quotient* cone K'*:*

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & (D^2)^m \\ \big\downarrow & & \big\downarrow \\ \text{cone}\, K' & \longrightarrow & I^m \end{array},$$

where K' is the barycentric subdivision of K;

$$\sigma = \{i_1,\ldots,i_k\} \mapsto e_\sigma = (\varepsilon_1,\ldots,\varepsilon_m),$$

where $\varepsilon_i = 0$ if $i \in \sigma$ and $\varepsilon_i = 1$ if $i \notin \sigma$.

If $K = K_P$ for a simple polytope P, then $\operatorname{cone} K'$ can be identified with P, and $\mathcal{Z}_{K_P}$ with $\mathcal{Z}_P$!

Moreover,

Prop 3. *a) Assume $|K| \cong S^{n-1}$ (a sphere triangulation with m vertices). Then $\mathcal{Z}_K$ is an $(m+n)$-manifold;*

б) Assume K is a triangulated manifold. Then $\mathcal{Z}_K \setminus \mathcal{Z}_\varnothing$ is an open manifold, where $\mathcal{Z}_\varnothing \cong T^m$.

Ex 2. $\mathcal{Z}_{\partial\Delta^n} \cong S^{2n+1}$. For $n = 1$,

$$S^3 = D^2 \times S^1 \cup S^1 \times D^2 \subset D^2 \times D^2.$$

First summary.

So far we had

- real quadratic complete intersection determined by P;
- identification spaces $P \times T^m/\sim$ and $|\operatorname{cone} K'| \times T^m/\sim$;
- polydisk subspace $\bigcup_{\sigma\in K} B_\sigma \subset (D^2)^m$.

The three spaces agree when $K = K_P$, but there is no quadratic description of $\mathcal{Z}_K$ for non-polytopal K.

Question 1. *Is there something similar to the real quadratic description of $\mathcal{Z}_P$ in the case of non-polytopal sphere triangulations K?*

In fact, despite $\mathcal{Z}_P$ is defined as a *real* complete intersection, it is a *complex* manifold (we need to multiply it by S^1 if $\dim \mathcal{Z}_P$ is odd). In this way we get a family of non-Kähler complex manifolds, generalising those of Hopf and Calabi–Eckmann [Bosio–Meersseman].

Partial product space.

As was noticed by N. Strickland, by replacing (D^2, S^1) in the definition of $\mathcal{Z}_K$ by an arbitrary pair of spaces (X, W), we get a generalised m.-a. complex, or partial product space $\mathcal{Z}_K(X, W)$.

In more detail, given $\omega \subset \{1, \dots, m\}$, set

$$B_\omega(X, W) := \{(x_1, \dots, x_m) \in X^m \colon x_i \in W \text{ if } i \notin \omega\},$$

and

$$\mathcal{Z}_K(X, W) := \bigcup_{\sigma \in K} B_\sigma(X, W) \subset X^m.$$

Coordinate subspace arrangement complements.

A coordinate subspace in $\mathbb{C}^m$ may be written as

$$L_\omega = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\},$$

where $\omega = \{i_1, \ldots, i_k\}$. Coordinate subspace arrangements in $\mathbb{C}^m$ are parameterised by simplicial complexes K on m vertices. Their complements are then given by

$$U(K) = \mathbb{C}^m \setminus \bigcup_{\omega \notin K} L_\omega.$$

Prop 4. *There is a T^m-equivariant deformation retraction*

$$U(K) \xrightarrow{\simeq} \mathcal{Z}_K.$$

Proof. $U(K) = \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$, $\mathcal{Z}_K = \mathcal{Z}_K(D^2, S^1)$, and $(D^2, S^1) \sim (\mathbb{C}, \mathbb{C}^*)$. This gives a homotopy equivalence. □

Homotopy fibre realisation of $\mathcal{Z}_K$.

The original Davis–Januszkiewicz space is the Borel construction

$$DJ(K) := ET^m \times_{T^m} \mathcal{Z}_K.$$

Prop 5. *There is a canonical homotopy equivalence*

$$DJ(K) \xrightarrow{\simeq} \mathcal{Z}_K(\mathbb{C}P^\infty, *),$$

where $\mathcal{Z}_K(\mathbb{C}P^\infty, *) = \bigcup_{\sigma\in K} BT^\sigma \subset BT^m = (\mathbb{C}P^\infty)^m$.

Cor 2. *(a)* $\mathcal{Z}_K \simeq \mathrm{hofibre}\Bigl(\bigcup_{\sigma\in K} BT^\sigma \hookrightarrow BT^m\Bigr)$;

(b) $H^*(DJ(K)) \cong H^*_{T^m}(\mathcal{Z}_K) \cong \mathbb{Z}[K]$, *where*

$$\mathbb{Z}[K] = \mathbb{Z}[v_1,\ldots,v_m]\big/\bigl(v_{i_1}\cdots v_{i_k} \colon \{i_1,\ldots,i_k\} \notin K\bigr)$$

is the face ring (or the Stanley–Reisner ring) of K, $\deg v_i = 2$.

Cohomology calculation.

Thm 1. [Baskakov-Buchstaber-P, Franz] *There is an isomorphism of (bi)graded algebras*

$$\begin{aligned} H^*(\mathcal{Z}_K;\mathbb{Z}) &\cong \mathrm{Tor}^{*,*}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z}) \\ &\cong H\Big[\Lambda[u_1,\dots,u_m]\otimes\mathbb{Z}[K];d\Big], \end{aligned}$$

where $du_i = v_i$, $dv_i = 0$ *for* $1 \leqslant i \leqslant m$. *In particular,*

$$H^p(\mathcal{Z}_K) \cong \sum_{-i+2j=p} \mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z}).$$

Cor 3. [Hochster'1975]

$$\mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z}) \cong \bigoplus_{|\omega|=j} \widetilde{H}^{j-i-1}(K_\omega),$$

where K_ω *is the full subcomplex (the restriction of* K *to the subset* $\omega \subset \{1,\dots,m\}$*).*

You can rewrite the above in terms of P instead of K:

Cor 4.

$$H^{-i,2j}(\mathcal{Z}_P) = \mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[P],\mathbb{Z}) \cong \bigoplus_{|\omega|=j} \widetilde{H}^{j-i-1}(P_\omega),$$

where $P_\omega = \bigcup_{i\in\omega} F_i$*, the union of facets of* P *belonging to* ω*.*

Cor 5. [Goresky–MacPherson]

$$\widetilde{H}_i\bigl(U(K)\bigr) = \bigoplus_{\omega\in\widehat{K}} \widetilde{H}^{2m-2|\omega|-i-2}(\mathrm{link}_{\widehat{K}}\,\omega),$$

where $\widehat{K} = \{\omega\colon [m]\backslash\omega \notin K\}$ *is the Alexander dual complex of* K*.*

The above cohomology ring calculation for $H^*(\mathcal{Z}_K)$ translates into explicit product formula in terms of Hochster's full subcomplexes [Baskakov].

Also, de Longueville's description of the product in the cohomology of coordinate subspace arrangement complements in terms of links follows from Baskakov's result by applying the Alexander duality.

Thm 2. [Bahri–Bendersky–Cohen–Gitler]

$$\Sigma \mathcal{Z}_K \simeq \bigvee_{\omega \notin K} \Sigma^{|\omega|+2} |K_\omega|.$$

This generalises to $\mathcal{Z}_K(X, W)$.

Back in $\sim$ 2001 we were able to make the following calculations.

Ex 3. Let $K = m$ disjoint points. Then

$$U(K) = \mathbb{C}^m \Big\backslash \bigcup_{1 \leqslant i < j \leqslant m} \{z_i = z_j = 0\},$$

the complement of the union of all codim 2 coordinate planes, and

$$H^*(U(K)) = H^*\Big(\bigvee_{k=2}^{m} (S^{k+1})^{\vee (k-1)\binom{m}{k}} \Big).$$

Ex 4. Let P be an m-gon, so K_P is the boundary of m-gon. Then

$$U(K_P) = \mathbb{C}^m \Big\backslash \bigcup_{i-j \neq 0,1 \mod m} \{z_i = z_j = 0\};$$

$\mathcal{Z}_P$ is an $(m+2)$-dim manifold, and

$$H^*(\mathcal{Z}_P) = H^*(U(K_P)) = H^*\Big(\mathop{\#}_{k=2}^{m-2} (S^{k+1} \times S^{m-k+1})^{\#(k-1)\binom{m-2}{k}} \Big).$$

Thm 3. [Grbić–Theriault] *If K is a shifted complex (e.g., a k-skeleton of Δ^{m-1}, or m disjoint points), then $\mathcal{Z}_K$ (and $U(K)$) is homotopy equivalent to a wedge of spheres.*

The proof uses the *homotopy fibre* realisation of $\mathcal{Z}_K$ and elaborated unstable homotopy techniques. The number of spheres in the wedge and their dimensions are also given.

Thm 4. [Bosio–Meersseman] *If P is obtained by applying a "vertex cut" operation to Δ^n several times (e.g., P is an m-gon), then $\mathcal{Z}_P$ is diffeomorphic to a connected sum of spaces of the form $S^i \times S^j$.*

The proof uses *real quadratic realisation* of $\mathcal{Z}_P$ and equivariant surgery techniques. The number of spheres is also given.

Polytopes P described in the above theorem achieve the lower bound for the number of faces in a polytope with the given number of facets. The dual complexes K_P are known to combinatorialists as stacked.

Ex 5. Let $K = 4$ points. Then

$$\mathcal{Z}_K \simeq (S^3)^{\vee 6} \vee (S^4)^{\vee 8} \vee (S^5)^{\vee 3}.$$

Ex 6. Let P be a polytope obtained by applying a vertex cut to Δ^3 trice. Then

$$\mathcal{Z}_P \cong (S^3 \times S^7)^{\# 6} \# (S^4 \times S^6)^{\# 8} \# (S^5 \times S^5)^{\# 3}.$$

There should be some general principle underlying both calculations!

Warning. In general, topology of $\mathcal{Z}_P$ is much more complicated than that of the previous examples. E.g., if P is obtained from a 3-cube by cutting two non-adjacent edges, then $\mathcal{Z}_P$ has non-trivial triple Massey products in cohomology [Baskakov].

Quasitoric manifolds.

Assume given P as above, and an integral $n \times m$ matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}$$

satisfying the condition

> the columns of $\lambda_{j_1}, \dots, \lambda_{j_n}$ corresponding to any vertex $v = F_{j_1} \cap \dots \cap F_{j_n}$ form a basis of $\mathbb{Z}^n$.

We refer to (P, Λ) as a combinatorial quasitoric pair.

Define $K = K(\Lambda) := \ker(\Lambda\colon T^m \to T^n) \cong T^{m-n}$.

Prop 6. $K(\Lambda)$ *acts freely on* $\mathcal{Z}_P$.

The quotient

$$M = M(P, \Lambda) := \mathcal{Z}_P / K(\Lambda)$$

is the quasitoric manifold corresponding to (P, Λ). It has a residual T^n-action ($T^m/K(\Lambda) \cong T^n$) satisfying the two Davis–Januszkiewicz conditions:

a) the T^n-action is locally standard;

b) there is a projection $\pi\colon M \to P$ whose fibres are orbits of the T^n-action.

Algebraic and Hamiltonian toric manifolds.

Algebraic and symplectic geometers would recognise in the above construction of a quasitoric manifold M from $\mathcal{Z}_P$ a generalisation of the symplectic reduction construction of a Hamiltonian toric manifold. In the latter case we take $\Lambda = A_P^t$; then M is a toric manifold corresponding to the Delzant polytope

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0 \ \text{ for } 1 \leqslant i \leqslant m\}, \quad \boldsymbol{a}_i \in \mathbb{Z}^n,\ b_i \in \mathbb{R}.$$

Here we additionally assume the normal vectors $\boldsymbol{a}_i$ to be *integer*, and the Delzant condition:

> for every vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P, the corresponding normal vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_n}$ form a basis of $\mathbb{Z}^n$

to be satisfied.

Then $\mathcal{Z}_P$ is the level set for the moment map $\mu\colon \mathbb{C}^m \to \mathbb{R}^{m-n}$ corresponding to the Hamiltonian action of $K = \operatorname{Ker} \Lambda = \operatorname{Ker} A^t$ on $\mathbb{C}^m$.

Cohomological rigidity phenomenon.

Problem 1. *Does a graded isomorphism $H^*(M) \cong H^*(M')$ imply a homeomorphism of quasitoric manifolds M and M'?*

Rem 1. Equivariant cohomology (as an algebra over $H^*(BT^n)$) does determine the topological type of a quasitoric manifold [Masuda].

A q-t manifold M is cohomologically rigid if its homeomorphism type is determined by the cohomology ring. A non-rigid q-t manifold would provide a counterexample to Problem 1.

A simple polytope P is cohomologically rigid if its combinatorial type is determined by the cohomology ring of any q-t manifold over P. In other words, P is rigid if for any q-t $M \to P$ and $M' \to P'$ isomorphism $H^*(M) \cong H^*(M')$ implies $P \sim P'$.

There are examples of non-rigid polytopes [Masuda–Suh]. These are obtained by applying a "vertex cut" to a 3-simplex trice. The corresponding m-a manifolds $\mathcal{Z}_P$ are also diffeomorphic!

In positive direction, the following is known:

- product $\mathbb{C}P^1 \times \ldots \times \mathbb{C}P^1$ is rigid [Masuda-P];

- n-dim cube I^n is rigid [Masuda-P];

- product $\mathbb{C}P^{i_1} \times \ldots \times \mathbb{C}P^{i_k}$ is rigid [Masuda-Suh];

- product of simplices $\Delta^{i_1} \times \ldots \times \Delta^{i_k}$ is rigid [Masuda-Suh].

The proofs use a result of Dobrinskaya on decomposability of a quasitoric manifold over a product of simplices into a tower of fibrations.

Also, most 3-dim simple polytopes with few facets are rigid [Choi-Suh]; the only known non-rigid polytopes are obtained as multiple vertex-cuts.

How to establish rigidity of polytopes?

Face vector of P is easily recovered from $H^*(M)$; so if there is only one combinatorial type P with the given face vector, then P is rigid. But this is a rare situation; usually more subtle combinatorial invariants are required.

Set $\beta^{-i,2j}(P) := \beta^{-i,2j}(\mathcal{Z}_P) = \dim \operatorname{Tor}^{-i,2j}_{\mathbb{Q}[v_1,\ldots,v_m]}(\mathbb{Q}[P],\mathbb{Q})$.

Prop 7 ([Choi-P-Suh]). *Assume M and M' are q-t manifolds over P and P' respectively. Then $H^*(M) \cong H^*(M')$ implies $\beta^{-i,2j}(P) = \beta^{-i,2j}(P')$ for all i,j.*

It follows that if there is only one combinatorial type P with given bigraded Betti numbers, then P is rigid. In this way the rigidity of many 3-dim polytopes with few facets (e.g. a dodecahedron) is established.

Application to complex cobordism.

Define complex line bundles

$$\rho_i \colon \mathcal{Z}_P \times_K \mathbb{C}_i \to M, \quad 1 \leqslant i \leqslant m,$$

where $\mathbb{C}_i$ is the 1-dim complex T^m-representation defined via the quotient projection $\mathbb{C}^m \to \mathbb{C}_i$ onto the ith factor.

Thm 5. *There is an isomorphism of real vector bundles*

$$\tau M \oplus \mathbb{R}^{m-n} \xrightarrow{\cong} \rho_1 \oplus \cdots \oplus \rho_m.$$

This endows M with the canonical equivariant stably complex structure. So we may consider its complex cobordism class $[M] \in \Omega_U$.

Thm 6. [Buchstaber–Ray–P] *Every complex cobordism class in* dim > 2 *contains a quasitoric manifold.*

The complex cobordism ring Ω_U is multiplicatively generated by the cobordism classes $[H_{ij}]$, $0 \leqslant i \leqslant j$, of Milnor hypersurfaces

$H_{ij} = \{(z_0 : \ldots : z_i) \times (w_0 : \ldots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : z_0 w_0 + \ldots + z_i w_i = 0\}.$
But H_{ij} is *not* a quasitoric manifold if $i > 1$.

Idea of proof

1) Replace each H_{ij} by a quasitoric (in fact, toric) manifold B_{ij} so that $\{B_{ij}\}$ is still a multiplicative generator set for Ω_U. Therefore, every stably complex manifold is cobordant to the disjoint union of products of B_{ij}'s. Every such product is a q-t manifold, but their disjoint union is not.
2) Replace disjoint unions by certain connected sums. This is tricky, because you need to take account of both the torus action and the stably complex structure.

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