Torus Actions and Complex Cobordism

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Thm 1. *Every complex cobordism class in* dim > 2 *contains a quasitoric manifold.*

In other words, every stably complex manifold is cobordant to a manifold with a nicely behaving torus action.

In cobordism theory, all manifolds are smooth and closed.

$M_1^n \simeq M_2^n$ (co)bordant if there is a manifold *with boundary* W^{n+1} such that $\partial W^{n+1} = M_1 \sqcup M_2$.

Complex cobordism: work with complex manifolds.

complex mflds $\subset$ almost complex mflds $\subset$ stably complex mflds

Stably complex structure on M is determined by a choice of isomorphism

$$\tau M \oplus \mathbb{R}^n \xrightarrow{\cong} \xi$$

where ξ is a *complex* vector bundle.

Complex cobordism classes $[M]$ form the complex cobordism ring Ω_U with respect to the disjoint union and product.

$$\Omega_U \cong \mathbb{Z}[a_1, a_2, ...], \quad \text{dim } a_i = 2i \quad \text{Novikov'60}.$$

Quasitoric manifolds: manifolds M^{2n} with a "nice" action of the torus T^n;

- the T^n-action is locally standard (locally looks like the standard T^n-representation in $\mathbb{C}^n$);

- the orbit space M^{2n}/T^n is an n-dim simple polytope P^n.

Examples include projective smooth toric varieties and symplectic manifolds M^{2n} with Hamiltonian actions of T^n (also known as toric manifolds).

Quasitoric manifolds from combinatorial data.

$\mathbb{R}^n$ Euclidean vector space. Consider a convex polyhedron

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m\}, \quad \boldsymbol{a}_i \in \mathbb{R}^n,\ b_i \in \mathbb{R}.$$

Assume:

a) $\dim P = n$;

b) no redundant inequalities (cannot remove any inequality without changing P);

c) P is bounded;

d) bounding hyperplanes $H_i = \{(\boldsymbol{a}_i, \boldsymbol{x}) + b_i = 0\}$, $1 \leqslant i \leqslant m$, intersect in general position at every vertex, i.e. there are exactly n facets of P meeting at each vertex.

Then P is an n-dim convex simple polytope with m facets

$$F_i = \{\boldsymbol{x} \in P \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i = 0\} = P \cap H_i$$

and normal vectors $\boldsymbol{a}_i$, for $1 \leqslant i \leqslant m$.

The faces of P form a poset with respect to the inclusion. Two polytopes are said to be combinatorially equivalent if their face posets are isomorphic. The corresponding equivalence classes are called combinatorial polytopes.

We may specify P by a matrix inequality

$$P = \{\boldsymbol{x} \colon A_P\boldsymbol{x} + \boldsymbol{b}_P \geqslant 0\},$$

where $A_P = (a_{ij})$ is the $m \times n$ matrix of row vectors $\boldsymbol{a}_i$, and $\boldsymbol{b}_P$ is the column vector of scalars b_i.

The affine injection

$$i_P \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad \boldsymbol{x} \mapsto A_P\boldsymbol{x} + \boldsymbol{b}_P$$

embeds P into $\mathbb{R}^m_{\geqslant} = \{\boldsymbol{y} \in \mathbb{R}^m \colon y_i \geqslant 0\}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m \end{array} \qquad\qquad \begin{array}{c} (z_1, \dots, z_m) \\ \big\downarrow \\ (|z_1|^2, \dots, |z_m|^2) \end{array}$$

Here i_Z is a T^m-equivariant embedding.

Prop 2. *$\mathcal{Z}_P$ is a smooth T^m-manifold with the canonical trivialisation of the normal bundle of $i_Z\colon \mathcal{Z}_P \to \mathbb{C}^m$.*

Idea of proof.

1) Write the image $i_P(\mathbb{R}^n) \subset \mathbb{R}^m$ as the set of common solutions of $m-n$ linear equations in y_i, $1 \leqslant i \leqslant m$;

2) replace every y_i by $|z_i|^2$ to get a representation of $\mathcal{Z}_P$ as an intersection of $m-n$ real quadratic hypersurfaces;

3) check that 2) is a "complete" intersection, i.e. the gradients are linearly independent at each point of $\mathcal{Z}_P$. □

$\mathcal{Z}_P$ is called the moment-angle manifold corresponding to P.

It can be proved that the equivariant smooth structure on $\mathcal{Z}_P$ depends only on the combinatorial type of P.

Assume given P as above, and an integral $n \times m$ matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}$$

satisfying the condition

> the columns of $\lambda_{j_1}, \dots, \lambda_{j_n}$ corresponding to any vertex $p = F_{j_1} \cap \dots \cap F_{j_n}$ form a basis of $\mathbb{Z}^n$.

We refer to (P, Λ) as the combinatorial quasitoric pair.

Define $K = K(\Lambda) := \ker(\Lambda\colon T^m \to T^n) \cong T^{m-n}$.

Prop 3. *$K(\Lambda)$ acts freely on $\mathcal{Z}_P$.*

The quotient

$$M = M(P, \Lambda) := \mathcal{Z}_P / K(\Lambda)$$

is the quasitoric manifold corresponding to (P, Λ). It has a residual T^n-action ($T^m/K(\Lambda) \cong T^n$) satisfying the two Davis–Januszkiewicz conditions:

a) the T^n-action is locally standard;

b) there is a projection $\pi\colon M \to P$ whose fibres are orbits of the T^n-action.

Define complex line bundles

$$\rho_i\colon \mathcal{Z}_P \times_K \mathbb{C}_i \to M, \quad 1 \leqslant i \leqslant m,$$

where $\mathbb{C}_i$ is the 1-dim complex T^m-representation defined via the quotient projection $\mathbb{C}^m \to \mathbb{C}_i$ onto the ith factor.

Thm 4. *There is an isomorphism of real vector bundles*

$$\tau M \oplus \mathbb{R}^{m-n} \xrightarrow{\cong} \rho_1 \oplus \cdots \oplus \rho_m.$$

This endows M with the canonical equivariant stably complex structure. So we may consider its complex cobordism class $[M] \in \Omega_U$.

Thm 1. *Every complex cobordism class in* dim > 2 *contains a quasitoric manifold.*

The complex cobordism ring Ω_U is multiplicatively generated by the cobordism classes $[H_{ij}]$, $0 \leqslant i \leqslant j$, of Milnor hypersurfaces

$H_{ij} = \{(z_0 : \ldots : z_i) \times (w_0 : \ldots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : z_0 w_0 + \ldots z_i w_i = 0\}.$

But H_{ij} is *not* a quasitoric manifold if $i > 1$.

Idea of proof of the main theorem.

1) Replace each H_{ij} by a quasitoric manifold B_{ij} so that $\{B_{ij}\}$ is still a multiplicative generator set for Ω_U. Therefore, every stably complex manifold is cobordant to the disjoint union of products of B_{ij}'s. Every such product is a q-t manifold, but their disjoint union is not.
2) Replace the disjoint unions by the connected sums. This is tricky, because you need to take account of both the torus action and the stably complex structure.

$\square$

[1] Victor M Buchstaber and Taras E Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* Volume 24 of *University Lecture Series*, Amer. Math. Soc., Providence, R.I., 2002.

[2] В. М. Бухштабер, Т. Е. Панов. *Торические действия в топологии и комбинаторике*. Издательство МЦНМО, Москва, 2004 (in Russian).

[3] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Spaces of polytopes and cobordism of quasitoric manifolds*. Moscow Math. J. **7** (2007), no. 2, 219–242; arXiv:math.AT/0609346.