Algebraic torus actions, Kempf–Ness sets and real quadrics in $\mathbb{C}^m$

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1. **Categorical quotient.**

$G$ a reductive algebraic group, $S$ an affine $G$-variety.

$\pi_{S,G}: S \to S//G$ morphism dual to $\mathbb{C}[S]^G \to \mathbb{C}[S]$.

$\pi_{S,G}$ is surjective and establishes a bijection

$$\text{closed } G\text{-orbits of } S \longleftrightarrow \text{ points of } S//G.$$  

$\pi_{S,G}$ is universal in the class of morphisms from $S$ constant on $G$-orbits in the category of algebraic varieties.

$S//G$ is called the **categorical quotient**.

\[ \rho : G \to \text{GL}(V) \] a representation, \( K \subset G \) a maximal compact subgroup, \( \langle , \rangle \) a \( K \)-invariant hermitian form on \( V \) with associated norm \( \| \| \).

Given \( v \in V \), consider the function \( F_v : G \to \mathbb{R} \) sending \( g \) to \( \frac{1}{2} \| gv \|^2 \). It has a critical point iff \( Gv \) is closed, and all critical points of \( F_v \) are minima. Define the subset \( KN \subset V \) by one of the following equivalent conditions:

\[
KN = \{ v \in V : (dF_v)_e = 0 \} \quad (e \in G \text{ is the unit})
\]

\[
= \{ v \in V : T_v Gv \perp v \}
\]

\[
= \{ v \in V : \langle \gamma v, v \rangle = 0 \text{ for all } \gamma \in g \}
\]

\[
= \{ v \in V : \langle \kappa v, v \rangle = 0 \text{ for all } \kappa \in \mathfrak{k} \},
\]

where \( g \) (resp. \( \mathfrak{t} \)) is the Lie algebra of \( G \) (resp. \( K \)) and we consider \( \mathfrak{t} \subseteq g \subseteq \text{End}(V) \). Therefore, any point \( v \in KN \) is a closest point to the origin in its orbit \( Gv \). Then \( KN \) is called the Kempf–Ness set of \( V \).
Assume that $S$ is $G$-equivariantly embedded as a closed subvariety in a representation $V$ of $G$. Then the Kempf–Ness set $KN_S$ of $S$ is defined as $KN \cap S$.

The importance of Kempf–Ness sets for the study of orbit quotients is due to the following result.

**Thm 2.** (a) [Kempf–Ness] The composition

$$KN_S \hookrightarrow S \twoheadrightarrow S/G$$

is proper and induces a homeomorphism

$$KN_S/K \overset{\cong}{\longrightarrow} S/G.$$ 

(b) [Neeman] There is a $K$-equivariant deformation retraction

$$S \to KN_S.$$
3. Cones and fans.

\( N \cong \mathbb{Z}^n \) an integral lattice, \( N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R} \).

A convex subset \( \sigma \in N_\mathbb{R} \) is a cone if there are \( a_1, \ldots, a_k \in N \) s. t.

\[ \sigma = \{ \mu_1 a_1 + \ldots + \mu_k a_k : \mu_i \in \mathbb{R}, \mu_i \geq 0 \} . \]

If the set \( \{a_1, \ldots, a_k\} \) is minimal, then it is called the generator set of \( \sigma \). A cone is strongly convex if it contains no line through the origin; all the cones below are assumed to be strongly convex. A cone \( \sigma \) is called regular (resp. simplicial) if \( a_1, \ldots, a_k \) can be chosen to form a subset of a \( \mathbb{Z} \)-basis of \( N \) (resp. an \( \mathbb{R} \)-basis of \( N_\mathbb{R} \)).

A finite collection \( \Sigma = \{ \sigma_1, \ldots, \sigma_s \} \) of cones in \( N_\mathbb{R} \) is a fan if a face of every cone in \( \Sigma \) belongs to \( \Sigma \) and the intersection of any two cones in \( \Sigma \) is a face of each.

A fan \( \Sigma \) is regular (resp. simplicial) if every cone in \( \Sigma \) is regular (resp. simplicial). A fan \( \Sigma = \{ \sigma_1, \ldots, \sigma_s \} \) is complete if \( N_\mathbb{R} = \sigma_1 \cup \ldots \cup \sigma_s \).
4. Algebraic torus actions.

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ the multiplicative group of complex numbers, $S^1$ the subgroup of complex numbers of absolute value one.

$T_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ the algebraic torus,
$T = N \otimes_{\mathbb{Z}} S^1 \cong (S^1)^n$ the (compact) torus.

A toric variety is a normal algebraic variety $X$ containing the algebraic torus $T_{\mathbb{C}}$ as a Zariski open subset in such a way that the natural action of $T_{\mathbb{C}}$ on itself extends to an action on $X$.

There is a classical construction establishing bijections

fans $\Sigma$ in $N_{\mathbb{R}}$ $\longleftrightarrow$ complex $n$-dim toric varieties $X_{\Sigma}$

regular fans $\longleftrightarrow$ non-singular varieties

complete fans $\longleftrightarrow$ compact varieties
5. Batyrev–Cox construction.

Assume that one-dimensional cones of $\Sigma$ span $N_\mathbb{R}$ as a vector space. $m$ the number of one-dimensional cones.

$a_i \in N$ the primitive generator of the $i$th one-dim cone, $1 \leq i \leq m$.

Consider the map

$$\mathbb{Z}^m \to N, \quad e_i \mapsto a_i.$$ 

The corresponding map of tori fit into an exact sequences

$$1 \to G \to (\mathbb{C}^*)^m \to T_\mathbb{C} \to 1, \quad (3)$$

$$1 \to K \to T^m \to T \to 1 \quad (4)$$

where $G$ is isomorphic to a product of $(\mathbb{C}^*)^{m-n}$ and a finite group. If $\Sigma$ is a regular fan and has at least one $n$-dimensional cone, then $G \cong (\mathbb{C}^*)^{m-n}$, and similarly for $K$. 

We say that a subset \( \{i_1, \ldots, i_k\} \in [m] = \{1, \ldots, m\} \) is a g-subset if \( \{a_{i_1}, \ldots, a_{i_k}\} \) is a subset of the generator set of a cone in \( \Sigma \).

The collection of g-subsets is closed with respect to the inclusion, and therefore forms an (abstract) simplicial complex on the set \([m]\), which we denote \( \mathcal{K}_\Sigma \).

If \( \Sigma \) is a complete simplicial fan, then \( \mathcal{K}_\Sigma \) is a triangulation of an \((n - 1)\)-dimensional sphere.

Given a cone \( \sigma \in \Sigma \), we denote by \( g(\sigma) \subseteq [m] \) the set of its generators. Now set

\[
A(\Sigma) = \bigcup \{ z \in \mathbb{C}^m : z_{i_1} = \ldots = z_{i_k} = 0 \}
\]

for \( \{i_1, \ldots, i_k\} \) is not a g-subset

and

\[
U(\Sigma) = \mathbb{C}^m \setminus A(\Sigma).
\]
Unlike $G$ and $K$, both $A(\Sigma)$ and $U(\Sigma)$ depend only on the combinatorial structure of the simplicial complex $K_\Sigma$; the set $U(\Sigma)$ coincides with the \textit{coordinate subspace arrangement complement} $U(K_\Sigma)$.

The set $A(\Sigma)$ is an affine variety, while its complement $U(\Sigma)$ admits a simple affine cover, as described in the following statement.

\textbf{Prop 5.} Given a cone $\sigma \in \Sigma$, set $z^{\hat{\sigma}} = \prod_{j \notin g(\sigma)} z_j$ and define

$$V(\Sigma) = \{ z \in \mathbb{C}^m : z^{\hat{\sigma}} = 0 \text{ for all } \sigma \in \Sigma \}$$

and

$$U(\sigma) = \{ z \in \mathbb{C}^m : z_j \neq 0 \text{ if } j \notin g(\sigma) \}. $$

Then $A(\Sigma) = V(\Sigma)$ and

$$U(\Sigma) = \mathbb{C}^m \setminus V(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma).$$
The complement $U(\Sigma) \subset \mathbb{C}^m$ is $(\mathbb{C}^*)^m$-invariant.

If $\Sigma$ is simplicial, the subgroup $G \subset (\mathbb{C}^*)^m$ acts on $U(\Sigma)$ with finite isotropy subgroups (or freely if $\Sigma$ is regular). The quotient can be identified with the toric variety $X_\Sigma$ determined by $\Sigma$:

**Thm 6.** [Cox] (a) The toric variety $X_\Sigma$ is isomorphic to the categorical quotient of $U(\Sigma)$ by $G$.

(b) $X_\Sigma$ is the geometric quotient of $U(\Sigma)$ by $G$ if and only if $\Sigma$ is simplicial.

Therefore, if $\Sigma$ is a simplicial, then all the orbits of the $G$-action on $U(\Sigma)$ are closed and we have $U(\Sigma)//G = U(\Sigma)/G$.

However, the corresponding Kempf–Ness set cannot constructed in the standard way, as $U(\Sigma)$ is not an affine variety in $\mathbb{C}^m$ (it is only quasiaffine in general)!
6. The moment-angle complex.

Consider the unit polydisc

$$(D^2)^m = \{ z \in \mathbb{C}^m : |z_j| \leq 1 \text{ for all } j \}. $$

Given a cone $\sigma \in \Sigma$, define

$$\mathcal{Z}(\sigma) = \{ z \in (D^2)^m : |z_j| = 1 \text{ if } j \notin g(\sigma) \},$$

and the moment-angle complex

$$\mathcal{Z}(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathcal{Z}(\sigma) \subseteq (D^2)^m.$$ 

$\mathcal{Z}(\Sigma)$ is $T^m$-invariant. Also, $\mathcal{Z}(\Sigma) \subset U(\Sigma)$.

**Prop 7.** Assume $\Sigma$ is complete simplicial. Then $\mathcal{Z}(\Sigma)$ is a compact $T^m$-manifold of dimension $m + n$. 


\( \mathcal{Z}(\Sigma) \) has the same properties with respect to the \( G \)-action on \( U(\Sigma) \) as \( CNS \) with respect to the \( G \)-action on an affine variety \( S \):

**Thm 8 (Buchstaber-P.'00).** Assume \( \Sigma \) is simplicial.
(a) *If \( \Sigma \) is complete, then the composition*

\[ \mathcal{Z}(\Sigma) \hookrightarrow U(\Sigma) \rightarrow U(\Sigma)/G \]

*induces a homeomorphism*

\[ \mathcal{Z}(\Sigma)/K \rightarrow U(\Sigma)/G. \]

(b) *There is a \( T^m \)-equivariant deformation retraction \( U(\Sigma) \rightarrow \mathcal{Z}(\Sigma) \).*

We therefore refer to \( \mathcal{Z}(\Sigma) \) as the **toric Kempf–Ness set** of \( U(\Sigma) \).
Ex 9. Let $n = 2$ and $e_1, e_2$ be a basis in $N_{\mathbb{R}}$.

1. Consider a complete fan $\Sigma$ having the following three 2-dimensional cones: the first is spanned by $e_1$ and $e_2$, the second spanned by $e_2$ and $-e_1 - e_2$, and the third spanned by $-e_1 - e_2$ and $e_1$. The simplicial complex $K_\Sigma$ is a complete graph on 3 vertices (or the boundary of a triangle). We have

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z : z_1 = z_2 = z_3 = 0\} = \mathbb{C}^3 \setminus \{0\}$$

and

$$Z(\Sigma) = D^2 \times D^2 \times S^1 \cup D^2 \times S^1 \times D^2 \cup S^1 \times D^2 \times D^2 = \partial((D^2)^3) \cong S^5.$$ 

Then $G$ is the diagonal subtorus in $(\mathbb{C}^*)^3$, and $K$ is the diagonal subcircle in $T^3$. Therefore,

$$X_\Sigma = U(\Sigma)/G = Z(\Sigma)/K = \mathbb{C}P^2.$$
2. Now consider the fan $\Sigma$ consisting of three 1-dimensional cones generated by vectors $e_1$, $e_2$ and $-e_1-e_2$. This fan is not complete, but its 1-dimensional cones span $N_\mathbb{R}$ as a vector space. So Cox' Thm 6 applies, but Thm 8 (a) does not. We have

$$K_\Sigma = 3 \text{ disjoint points},$$

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z_1 = z_2 = 0, z_1 = z_3 = 0, z_2 = z_3 = 0\},$$

and

$$Z(\Sigma) = D^2 \times S^1 \times S^1 \cup S^1 \times D^2 \times S^1 \cup S^1 \times S^1 \times S^1 \times D^2.$$ Both spaces are homotopy equivalent to $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$.

$G$ is again a diagonal subtorus in $(\mathbb{C}^*)^3$. By Thm 6,

$$X_\Sigma = U(\Sigma)/G = \mathbb{C}P^2 \setminus \{3 \text{ points}\}.$$ This in non-compact, and cannot be identified with $Z(\Sigma)/K$. 
8. Polytopes and normal fans.

$N^*_R$ the dual vector space. Given primitive vectors $a_1, \ldots, a_m \in N$ and integer numbers $b_1, \ldots, b_m \in \mathbb{Z}$, consider

$$P = \{x \in N^*_R: \langle a_i, x \rangle + b_i \geq 0 \text{ for } 1 \leq i \leq m\}.$$

Assume:

- $P$ is bounded;
- the affine hull of $P$ is the whole $N^*_R$;
- no redundant inequalities;
- no $(n + 1)$ hyperlanes $\langle a_i, x \rangle + b_i = 0$ meet at a point.

Then $P$ is a **convex simple polytope with $m$ facets**

$$F_i = \{x \in P: \langle a_i, x \rangle + b_i = 0\}$$

with normal vectors $a_i$, for $1 \leq i \leq m$. 

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We may specify $P$ by a matrix inequality

$$A_Px + b_P \geq 0,$$

where $A_P$ is the $m \times n$ matrix of row vectors $a_i$, and $b_P$ is the column vector of scalars $b_i$.

The affine injection

$$i_P: N_{\mathbb{R}}^* \rightarrow \mathbb{R}^m, \quad x \mapsto A_Px + b_P$$

embeds $P$ into $\mathbb{R}^m_{\succ} = \{y \in \mathbb{R}^m: y_i \geq 0\}$. 
Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$
\begin{array}{ccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\
\mu_P & & \mu \\
\downarrow & & \downarrow \\
P & \xrightarrow{i_P} & \mathbb{R}^m
\end{array}
$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. Here $i_Z$ is a $T^m$-equivariant embedding.

The normal fan $\Sigma_P$ consists of the cones spanned by the sets $\{a_{i_1}, \ldots, a_{i_k}\}$ such that the intersection $F_{i_1} \cap \ldots \cap F_{i_k}$ of the corresponding facets is non-empty. $\Sigma_P$ is a simplicial fan.

**Prop 10.** (a) We have $\mathcal{Z}_P \subset U(\Sigma_P)$.

(b) There is a $T^m$-homeomorphism $\mathcal{Z}_P \cong Z(\Sigma_P)$. 

The linear transformation $A_P : N_R^* \rightarrow \mathbb{R}^m$ is exactly the one obtained from $T^m \rightarrow T$ by applying $\text{Hom}_\mathbb{Z}(\cdot, S^1) \otimes \mathbb{Z} \mathbb{R}$.

Applying $\text{Hom}_\mathbb{Z}(\cdot, S^1) \otimes \mathbb{Z} \mathbb{R}$ to the whole exact sequence of tori, we obtain

$$0 \rightarrow N_R^* \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \rightarrow 0,$$

where $\mathbb{R}^{m-n} = \text{Hom}_\mathbb{Z}(G, S^1) \otimes \mathbb{Z} \mathbb{R}$.

Assume the first $n$ normal vectors $a_1, \ldots, a_n$ span a cone of $\Sigma_P$, and take these vectors as a basis of $N_R^*$. In this basis, we may take

$$C = (c_{ij}) = \begin{pmatrix}
-a_{n+1,1} & \cdots & -a_{n+1,n} & 1 & 0 & \cdots & 0 \\
-a_{n+2,1} & \cdots & -a_{n+2,n} & 0 & 1 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{m,1} & \cdots & -a_{m,n} & 0 & 0 & \cdots & 1
\end{pmatrix}.$$
Then $Z_P$ embeds in $\mathbb{C}^m$ as the space of common solutions of $m - n$ real quadratic equations

$$\sum_{k=1}^{m} c_{jk} (|z_k|^2 - b_k) = 0, \text{ for } 1 \leq j \leq m - n.$$ 

This intersection is non-degenerate, so $Z_P \subset \mathbb{C}^m$ is a smooth submanifold with trivial normal bundle (Buchstaber-P-Ray’07).

The projective toric variety $X_P = X_{\Sigma_P}$ can be obtained from the action of $K$ on $U(\Sigma_P) \subset \mathbb{C}^m$ via the process of symplectic reduction. The moment map $\mu_{\Sigma_P}$ is given by the composition

$$\mathbb{C}^m \xrightarrow{\mu} \mathbb{R}^m \xrightarrow{C} \text{Lie}(K) \cong \mathbb{R}^{m-n},$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$ and $C = (c_{jk})$, so

$$Z_P = \mu_{\Sigma_P}^{-1}(Cb_P).$$

is its level surface. Then $X_P = Z_P/K$. 

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Problem 11. There are many complete regular fans $\Sigma$ which cannot be realised as normal fans of convex polytopes. The corresponding toric varieties $X_\Sigma$ are not projective, although being non-singular. In this case the Kempf–Ness set $Z(\Sigma)$ is still defined. Is there a description of $Z(\Sigma)$ similar to that of $Z(\Sigma_P)$ as a complete intersection of real quadrics?

Given an abstract simplicial complex $\mathcal{K}$ on the set $[m]$, the face ring (or the Stanley–Reisner ring) $\mathbb{Z}[\mathcal{K}]$ is the quotient

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} : \{i_1, \ldots, i_k\} \notin \mathcal{K}).$$

**Thm 12.** [Buchstaber-P’02] For every simplicial fan $\Sigma$ there are algebra isomorphisms

$$H^*(\mathbb{Z}(\Sigma); \mathbb{Z}) \cong \text{Tor}^*_{\mathbb{Z}[v_1, \ldots, v_m]}(\mathbb{Z}[\mathcal{K}_\Sigma], \mathbb{Z})$$

$$\cong H[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}_\Sigma], d],$$

where $\deg u_i = 1$, $\deg v_i = 2$, $du_i = v_i$, $dv_i = 0$, for $1 \leq i \leq m$. 

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**Ex 13.** Let $P$ be the simple polytope obtained by cutting two non-adjacent edges off a cube in $\mathbb{N}^*_\mathbb{R} \cong \mathbb{R}^3$. We may specify $P$ by 8 inequalities:

\[
\begin{align*}
  x &\geq 0, \quad y \geq 0, \quad z \geq 0, \\
  -x + 3 &\geq 0, \quad -y + 3 \geq 0, \\
  -z + 3 &\geq 0, \\
  -x + y + 2 &\geq 0, \quad -y - z + 5 \geq 0.
\end{align*}
\]
Toric variety $X_P$ is obtained by blowing up the product $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ at two complex 1-dimensional subvarieties $\{\infty\} \times \{0\} \times \mathbb{CP}^1$ and $\mathbb{CP}^1 \times \{\infty\} \times \{\infty\}$.

The Kempf–Ness set $Z_P$ is given by 5 real quadratic equations:

\[
|z_1|^2 + |z_4|^2 - 3 = 0, \quad |z_2|^2 + |z_5|^2 - 3 = 0, \\
|z_3|^2 + |z_6|^2 - 3 = 0, \quad |z_1|^2 - |z_2|^2 + |z_7|^2 - 2 = 0, \\
|z_2|^2 + |z_3|^2 + |z_8|^2 - 5 = 0.
\]

It is an 11-dimensional manifold with Betti vector

\[(1, 0, 0, 10, 16, 5, 5, 16, 10, 0, 0, 1)\]

and non-trivial Massey products of 3-dimensional classes (Baskakov'03).
