Algebraic torus actions, Kempf–Ness sets and real quadrics in $\mathbb{C}^m$

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1. Categorical quotient.

G a reductive algebraic group, S an affine G-variety.

$\pi_{S,\mathsf{G}}\colon S \to S/\!\!/\mathsf{G}$ morphism dual to $\mathbb{C}[S]^{\mathsf{G}} \to \mathbb{C}[S]$.

$\pi_{S,\mathsf{G}}$ is surjective and establishes a bijection

$$\text{closed G-orbits of } S \quad \longleftrightarrow \quad \text{points of } S/\!\!/\mathsf{G}.$$

$\pi_{S,\mathsf{G}}$ is universal in the class of morphisms from S constant on G-orbits in the category of algebraic varieties.

$S/\!\!/\mathsf{G}$ is called the categorical quotient.

2. Kempf–Ness sets for affine varieties.

$\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ a representation, $\mathsf{K} \subset \mathsf{G}$ a maximal compact subgroup, $\langle\ ,\ \rangle$ a K-invariant hermitian form on V with associated norm $\|\ \|$.

Given $v \in V$, consider the function $F_v\colon \mathsf{G} \to \mathbb{R}$ sending g to $\frac{1}{2}\|gv\|^2$. It has a critical point iff $\mathsf{G}v$ is closed, and all critical points of F_v are minima. Define the subset $\mathit{KN} \subset V$ by one of the following equivalent conditions:

$$\begin{aligned} \mathit{KN} &= \{v \in V\colon (dF_v)_e = 0\} \qquad (e \in \mathsf{G} \text{ is the unit}) \\ &= \{v \in V\colon T_v\mathsf{G}v \perp v\} \\ &= \{v \in V\colon \langle \gamma v, v\rangle = 0 \text{ for all } \gamma \in \mathfrak{g}\} \\ &= \{v \in V\colon \langle \kappa v, v\rangle = 0 \text{ for all } \kappa \in \mathfrak{k}\}, \end{aligned} \tag{1}$$

where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of G (resp. K) and we consider $\mathfrak{k} \subseteq \mathfrak{g} \subseteq \mathsf{End}(V)$. Therefore, any point $v \in \mathit{KN}$ is a closest point to the origin in its orbit $\mathsf{G}v$. Then KN is called the Kempf–Ness set of V.

Assume that S is G-equivariantly embedded as a closed subvariety in a representation V of G. Then the Kempf–Ness set KN_S of S is defined as $KN \cap S$.

The importance of Kempf–Ness sets for the study of orbit quotients is due to the following result.

Thm 2. (a) [Kempf–Ness] *The composition*

$$KN_S \hookrightarrow S \to S/\!\!/\mathrm{G}$$

is proper and induces a homeomorphism

$$KN_S/\mathrm{K} \xrightarrow{\cong} S/\!\!/\mathrm{G}.$$

(b) [Neeman] *There is a* K*-equivariant deformation retraction*

$$S \to KN_S\,.$$

3. Cones and fans.

$N \cong \mathbb{Z}^n$ an integral lattice, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

A convex subset $\sigma \in N_{\mathbb{R}}$ is a cone if there are $a_1, \ldots, a_k \in N$ s. t.

$$\sigma = \{\mu_1 a_1 + \ldots + \mu_k a_k \colon \mu_i \in \mathbb{R},\ \mu_i \geqslant 0\}.$$

If the set $\{a_1, \ldots, a_k\}$ is minimal, then it is called the generator set of σ. A cone is strongly convex if it contains no line through the origin; all the cones below are assumed to be strongly convex. A cone σ is called regular (resp. simplicial) if $a_1, \ldots, a_k$ can be chosen to form a subset of a $\mathbb{Z}$-basis of N (resp. an $\mathbb{R}$-basis of $N_{\mathbb{R}}$).

A finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of cones in $N_{\mathbb{R}}$ is a fan if a face of every cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each.

A fan Σ is regular (resp. simplicial) if every cone in Σ is regular (resp. simplicial). A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is complete if $N_{\mathbb{R}} = \sigma_1 \cup \ldots \cup \sigma_s$.

4. Algebraic torus actions.

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ the multiplicative group of complex numbers,
S^1 the subgroup of complex numbers of absolute value one.

$\mathsf{T}_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ the algebraic torus,
$\mathsf{T} = N \otimes_{\mathbb{Z}} \mathsf{S}^1 \cong (\mathsf{S}^1)^n$ the (compact) torus.

A toric variety is a normal algebraic variety X containing the algebraic torus $\mathsf{T}_{\mathbb{C}}$ as a Zariski open subset in such a way that the natural action of $\mathsf{T}_{\mathbb{C}}$ on itself extends to an action on X.

There is a classical construction establishing bijections

fans Σ in $N_{\mathbb{R}}$ $\longleftrightarrow$ complex n-dim toric varieties X_Σ

regular fans $\longleftrightarrow$ non-singular varieties

complete fans $\longleftrightarrow$ compact varieties

5. Batyrev–Cox construction.

Assume that one-dimensional cones of Σ span $N_{\mathbb{R}}$ as a vector space.

m the number of one-dimensional cones.

$a_i \in N$ the primitive generator of the ith one-dim cone, $1 \leqslant i \leqslant m$.

Consider the map

$$\mathbb{Z}^m \to N, \quad e_i \mapsto a_i.$$

The corresponding map of tori fit into an exact sequences

$$1 \longrightarrow \mathsf{G} \longrightarrow (\mathbb{C}^*)^m \longrightarrow \mathsf{T}_{\mathbb{C}} \longrightarrow 1, \tag{3}$$

$$1 \longrightarrow \mathsf{K} \longrightarrow T^m \longrightarrow \mathsf{T} \longrightarrow 1 \tag{4}$$

where G is isomorphic to a product of $(\mathbb{C}^*)^{m-n}$ and a finite group. If Σ is a regular fan and has at least one n-dimensional cone, then $\mathsf{G} \cong (\mathbb{C}^*)^{m-n}$, and similarly for K.

We say that a subset $\{i_1, \ldots, i_k\} \in [m] = \{1, \ldots, m\}$ is a g-subset if $\{a_{i_1}, \ldots, a_{i_k}\}$ is a subset of the generator set of a cone in Σ.

The collection of g-subsets is closed with respect to the inclusion, and therefore forms an (abstract) simplicial complex on the set $[m]$, which we denote $\mathcal{K}_\Sigma$.

If Σ is a complete simplicial fan, then $\mathcal{K}_\Sigma$ is a triangulation of an $(n-1)$-dimensional sphere.

Given a cone $\sigma \in \Sigma$, we denote by $g(\sigma) \subseteq [m]$ the set of its generators. Now set

$$A(\Sigma) = \bigcup_{\{i_1,\ldots,i_k\} \text{ is not a } g\text{-subset}} \{z \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\}$$

and

$$U(\Sigma) = \mathbb{C}^m \setminus A(\Sigma).$$

Unlike G and K, both $A(\Sigma)$ and $U(\Sigma)$ depend only on the combinatorial structure of the simplicial complex $\mathcal{K}_\Sigma$; the set $U(\Sigma)$ coincides with the coordinate subspace arrangement complement $U(\mathcal{K}_\Sigma)$.

The set $A(\Sigma)$ is an affine variety, while its complement $U(\Sigma)$ admits a simple affine cover, as described in the following statement.

Prop 5. *Given a cone $\sigma \in \Sigma$, set $z^{\widehat{\sigma}} = \prod_{j \notin g(\sigma)} z_j$ and define*

$$V(\Sigma) = \{z \in \mathbb{C}^m \colon z^{\widehat{\sigma}} = 0 \text{ for all } \sigma \in \Sigma\}$$

and

$$U(\sigma) = \{z \in \mathbb{C}^m \colon z_j \neq 0 \text{ if } j \notin g(\sigma)\}.$$

Then $A(\Sigma) = V(\Sigma)$ and

$$U(\Sigma) = \mathbb{C}^m \setminus V(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma).$$

The complement $U(\Sigma) \subset \mathbb{C}^m$ is $(\mathbb{C}^*)^m$-invariant.

If Σ is simplicial, the subgroup $\mathrm{G} \subset (\mathbb{C}^*)^m$ acts on $U(\Sigma)$ with finite isotropy subgroups (or freely if Σ is regular). The quotient can be identified with the toric variety X_Σ determined by Σ:

Thm 6. [Cox] (a) *The toric variety X_Σ is isomorphic to the categorical quotient of $U(\Sigma)$ by* G.
(b) X_Σ *is the geometric quotient of $U(\Sigma)$ by* G *if and only if Σ is simplicial.*

Therefore, if Σ is a simplicial, then all the orbits of the G-action on $U(\Sigma)$ are closed and we have $U(\Sigma)/\!\!/\mathrm{G} = U(\Sigma)/\mathrm{G}$.

However, the corresponding Kempf–Ness set cannot constructed in the standard way, as $U(\Sigma)$ is *not* an affine variety in $\mathbb{C}^m$ (it is only quasiaffine in general)!

6. The moment-angle complex.

Consider the unit polydisc

$$(\mathsf{D}^2)^m = \{z \in \mathbb{C}^m \colon |z_j| \leqslant 1 \text{ for all } j\}.$$

Given a cone $\sigma \in \Sigma$, define

$$\mathcal{Z}(\sigma) = \{z \in (\mathsf{D}^2)^m \colon |z_j| = 1 \text{ if } j \notin g(\sigma)\},$$

and the moment-angle complex

$$\mathcal{Z}(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathcal{Z}(\sigma) \subseteq (\mathsf{D}^2)^m.$$

$\mathcal{Z}(\Sigma)$ is T^m-invariant. Also, $\mathcal{Z}(\Sigma) \subset U(\Sigma)$.

Prop 7. *Assume Σ is complete simplicial. Then $\mathcal{Z}(\Sigma)$ is a compact T^m-manifold of dimension $m+n$.*

7. Toric Kempf–Ness sets.

$\mathcal{Z}(\Sigma)$ has the same properties with respect to the G-action on $U(\Sigma)$ as KN_S with respect to the G-action on an affine variety S:

Thm 8 (Buchstaber-P.'00)**.** *Assume* Σ *is simplicial.*
(a) *If* Σ *is complete, then the composition*

$$\mathcal{Z}(\Sigma) \hookrightarrow U(\Sigma) \to U(\Sigma)/\mathrm{G}$$

induces a homeomorphism

$$\mathcal{Z}(\Sigma)/\mathrm{K} \to U(\Sigma)/\mathrm{G}.$$

(b) *There is a* T^m*-equivariant deformation retraction* $U(\Sigma) \to \mathcal{Z}(\Sigma)$.

We therefore refer to $\mathcal{Z}(\Sigma)$ as the toric Kempf–Ness set of $U(\Sigma)$.

Ex 9. Let $n = 2$ and e_1, e_2 be a basis in $N_{\mathbb{R}}$.

1. Consider a complete fan Σ having the following three 2-dimensional cones: the first is spanned by e_1 and e_2, the second spanned by e_2 and $-e_1 - e_2$, and the third spanned by $-e_1 - e_2$ and e_1. The simplicial complex $\mathcal{K}_\Sigma$ is a complete graph on 3 vertices (or the boundary of a triangle). We have

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z \colon z_1 = z_2 = z_3 = 0\} = \mathbb{C}^3 \setminus \{0\}$$

and

$$\begin{aligned}\mathcal{Z}(\Sigma) = \mathsf{D}^2 \times \mathsf{D}^2 \times \mathsf{S}^1 \cup \mathsf{D}^2 \times \mathsf{S}^1 \times \mathsf{D}^2 \cup \mathsf{S}^1 \times \mathsf{D}^2 \times \mathsf{D}^2& \\ = \partial((\mathsf{D}^2)^3) \cong \mathsf{S}^5.&\end{aligned}$$

Then G is the diagonal subtorus in $(\mathbb{C}^*)^3$, and K is the diagonal subcircle in T^3. Therefore,

$$X_\Sigma = U(\Sigma)/\mathsf{G} = \mathcal{Z}(\Sigma)/\mathsf{K} = \mathbb{C}P^2.$$

2. Now consider the fan Σ consisting of three 1-dimensional cones generated by vectors e_1, e_2 and $-e_1-e_2$. This fan is not complete, but its 1-dimensional cones span $N_{\mathbb{R}}$ as a vector space. So Cox' Thm 6 applies, but Thm 8 (a) does not. We have

$$\mathcal{K}_\Sigma = \text{ 3 disjoint points},$$

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z_1 = z_2 = 0, z_1 = z_3 = 0, z_2 = z_3 = 0\},$$

and

$$\mathcal{Z}(\Sigma) = \mathsf{D}^2 \times \mathsf{S}^1 \times \mathsf{S}^1 \cup \mathsf{S}^1 \times \mathsf{D}^2 \times \mathsf{S}^1 \cup \mathsf{S}^1 \times \mathsf{S}^1 \times \mathsf{D}^2.$$

Both spaces are homotopy equivalent to $\mathsf{S}^3 \vee \mathsf{S}^3 \vee \mathsf{S}^3 \vee \mathsf{S}^4 \vee \mathsf{S}^4$.

G is again a diagonal subtorus in $(\mathbb{C}^*)^3$. By Thm 6,

$$X_\Sigma = U(\Sigma)/\mathsf{G} = \mathbb{C}P^2 \setminus \{\text{3 points}\}.$$

This in non-compact, and cannot be identified with $\mathcal{Z}(\Sigma)/\mathsf{K}$.

8. Polytopes and normal fans.

$N^*_{\mathbb{R}}$ the dual vector space. Given primitive vectors $a_1, \ldots, a_m \in N$ and integer numbers $b_1, \ldots, b_m \in \mathbb{Z}$, consider

$$P = \{x \in N^*_{\mathbb{R}} \colon \langle a_i, x \rangle + b_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m\}.$$

Assume:

- P is bounded;
- the affine hull of P is the whole $N^*_{\mathbb{R}}$;
- no redundant inequalities;
- no $(n+1)$ hyperlanes $\langle a_i, x \rangle + b_i = 0$ meet at a point.

Then P is a convex simple polytope with m facets

$$F_i = \{x \in P \colon \langle a_i, x \rangle + b_i = 0\}$$

with normal vectors a_i, for $1 \leqslant i \leqslant m$.

We may specify P by a matrix inequality

$$A_P x + b_P \geqslant 0,$$

where A_P is the $m \times n$ matrix of row vectors a_i, and b_P is the column vector of scalars b_i.

The affine injection

$$i_P \colon N^*_{\mathbb{R}} \longrightarrow \mathbb{R}^m, \quad x \mapsto A_P x + b_P$$

embeds P into $\mathbb{R}^m_{\geqslant} = \{y \in \mathbb{R}^m \colon y_i \geqslant 0\}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \mu_P \big\downarrow & & \big\downarrow \mu \\ P & \xrightarrow{i_P} & \mathbb{R}^m \end{array}$$

where $\mu(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$. Here i_Z is a T^m-equivariant embedding.

The normal fan Σ_P consists of the cones spanned by the sets $\{a_{i_1}, \dots, a_{i_k}\}$ such that the intersection $F_{i_1} \cap \dots \cap F_{i_k}$ of the corresponding facets is non-empty. Σ_P is a simplicial fan.

Prop 10. (a) *We have $\mathcal{Z}_P \subset U(\Sigma_P)$.*
(b) *There is a T^m-homeomorphism $\mathcal{Z}_P \cong \mathcal{Z}(\Sigma_P)$.*

9. Complete intersections of real quadrics.

The linear transformation $A_P \colon N^*_{\mathbb{R}} \to \mathbb{R}^m$ is exactly the one obtained from $T^m \to \mathsf{T}$ by applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathsf{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$.

Applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathsf{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$ to the whole exact sequence of tori, we obtain

$$0 \longrightarrow N^*_{\mathbb{R}} \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \longrightarrow 0,$$

where $\mathbb{R}^{m-n} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{G}, \mathsf{S}^1) \otimes_{\mathbb{Z}} \mathbb{R}$.

Assume the first n normal vectors $a_1, \ldots, a_n$ span a cone of Σ_P, and take these vectors as a basis of $N^*_{\mathbb{R}}$. In this basis, we may take

$$C = (c_{ij}) = \begin{pmatrix} -a_{n+1,1} & \ldots & -a_{n+1,n} & 1 & 0 & \ldots & 0 \\ -a_{n+2,1} & \ldots & -a_{n+2,n} & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{m,1} & \ldots & -a_{m,n} & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Then $\mathcal{Z}_P$ embeds in $\mathbb{C}^m$ as the space of common solutions of $m-n$ real quadratic equations

$$\sum_{k=1}^{m} c_{jk}\left(|z_k|^2 - b_k\right) = 0, \quad \text{for } 1 \le j \le m-n.$$

This intersection is non-degenerate, so $\mathcal{Z}_P \subset \mathbb{C}^m$ is a smooth submanifold with trivial normal bundle (Buchstaber-P-Ray'07).

The projective toric variety $X_P = X_{\Sigma_P}$ can be obtained from the action of K on $U(\Sigma_P) \subset \mathbb{C}^m$ via the process of symplectic reduction. The moment map μ_{Σ_P} is given by the composition

$$\mathbb{C}^m \xrightarrow{\mu} \mathbb{R}^m \xrightarrow{C} \text{Lie(K)} \cong \mathbb{R}^{m-n},$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$ and $C = (c_{jk})$, so

$$\mathcal{Z}_P = \mu_{\Sigma_P}^{-1}(Cb_P).$$

is its level surface. Then $X_P = \mathcal{Z}_P/\text{K}$.

Problem 11. *There are many complete regular fans* Σ *which cannot be realised as normal fans of convex polytopes. The corresponding toric varieties* X_Σ *are not projective, although being non-singular. In this case the Kempf–Ness set* $\mathcal{Z}(\Sigma)$ *is still defined. Is there a description of* $\mathcal{Z}(\Sigma)$ *similar to that of* $\mathcal{Z}(\Sigma_P)$ *as a complete intersection of real quadrics?*

14. Cohomology of Kempf–Ness sets.

Given an abstract simplicial complex $\mathcal{K}$ on the set $[m]$, the face ring (or the Stanley–Reisner ring) $\mathbb{Z}[\mathcal{K}]$ is the quotient

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} \colon \{i_1, \ldots, i_k\} \notin \mathcal{K}).$$

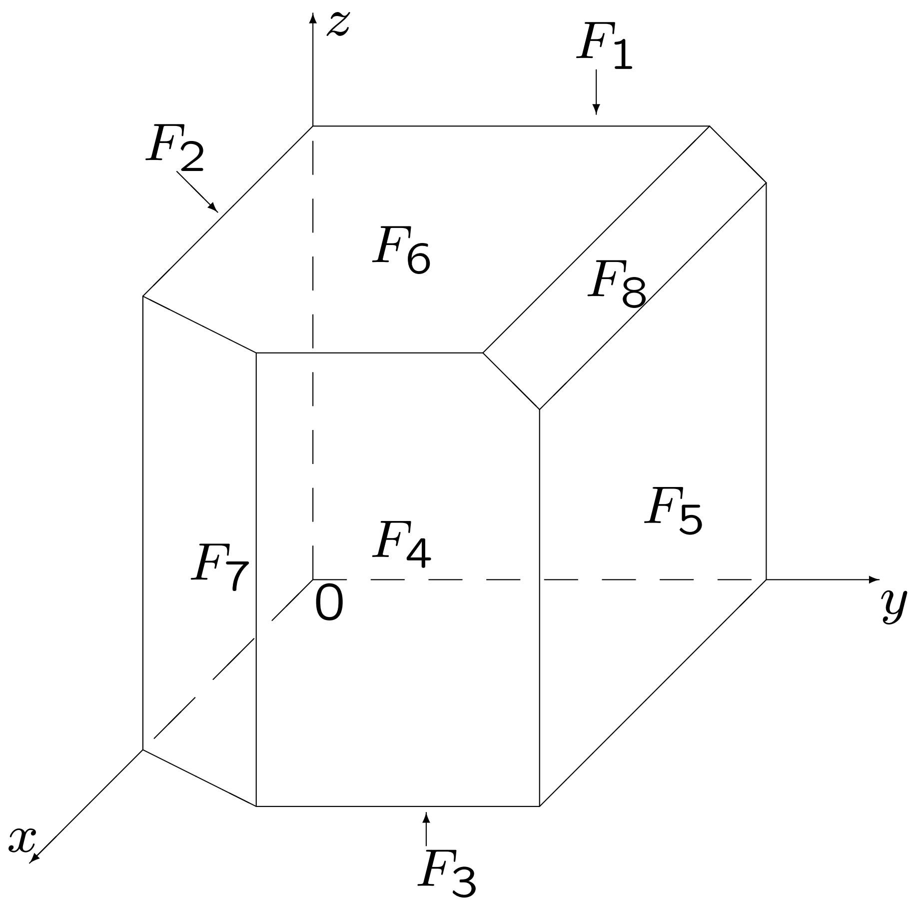
Thm 12. [Buchstaber-P'02] *For every simplicial fan Σ there are algebra isomorphisms*

$$H^*\Big(\mathcal{Z}(\Sigma); \mathbb{Z}\Big) \cong \mathrm{Tor}^*_{\mathbb{Z}[v_1,\ldots,v_m]}\Big(\mathbb{Z}[\mathcal{K}_\Sigma], \mathbb{Z}\Big)$$
$$\cong H\Big[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}_\Sigma], d\Big],$$

where $\deg u_i = 1$, $\deg v_i = 2$, $du_i = v_i$, $dv_i = 0$, *for* $1 \leqslant i \leqslant m$.

Ex 13. Let P be the simple polytope obtained by cutting two non-adjacent edges off a cube in $N^*_{\mathbb{R}} \cong \mathbb{R}^3$. We may specify P by 8 inequalities:

$$x \geqslant 0, \quad y \geqslant 0, \quad z \geqslant 0,$$
$$-x+3 \geqslant 0, \quad -y+3 \geqslant 0,$$
$$-z+3 \geqslant 0,$$
$$-x+y+2 \geqslant 0, \quad -y-z+5 \geqslant 0.$$



Toric variety X_P is obtained by blowing up the product $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ at two complex 1-dimensional subvarieties $\{\infty\} \times \{0\} \times \mathbb{C}P^1$ and $\mathbb{C}P^1 \times \{\infty\} \times \{\infty\}$.

The Kempf–Ness set $\mathcal{Z}_P$ is given by 5 real quadratic equations:

$$|z_1|^2 + |z_4|^2 - 3 = 0, \quad |z_2|^2 + |z_5|^2 - 3 = 0,$$

$$|z_3|^2 + |z_6|^2 - 3 = 0, \quad |z_1|^2 - |z_2|^2 + |z_7|^2 - 2 = 0,$$

$$|z_2|^2 + |z_3|^2 + |z_8|^2 - 5 = 0.$$

It is an 11-dimensional manifold with Betti vector

$$(1, 0, 0, 10, 16, 5, 5, 16, 10, 0, 0, 1)$$

and non-trivial Massey products of 3-dimensional classes (Baskakov'03).

[1] Victor M Buchstaber and Taras E Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* Volume 24 of *University Lecture Series*, Amer. Math. Soc., Providence, R.I., 2002.

[2] Taras E. Panov. *Topology of Kempf–Ness sets for algebraic torus actions*; arXiv:math.AG/0603556.

[3] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Spaces of polytopes and cobordism of quasitoric manifolds*. Moscow Math. J. **7** (2007), no. 2; arXiv:math.AT/0609346.