Topology of Kempf–Ness sets for algebraic torus actions

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Abstract. In the theory of algebraic group actions on affine varieties, the concept of a Kempf–Ness set is used to replace the geometric quotient by the quotient with respect to a maximal compact subgroup. By making use of the recent achievements of "toric topology" we show that an appropriate notion of a Kempf–Ness set exists for a class of algebraic torus actions on quasiaffine varieties (coordinate subspace arrangement complements) arising in the "geometric invariant theory" approach to toric varieties. We proceed by studying the cohomology of these Kempf–Ness sets.

1. Kempf–Ness sets for affine varieties.

G reductive algebraic group, S affine G-variety.
$\pi_{S,\mathsf{G}}\colon S \to S/\!\!/\mathsf{G}$ dual to $\mathbb{C}[S]^{\mathsf{G}} \to \mathbb{C}[S]$. $\pi_{S,\mathsf{G}}$ establishes a bijection:

$$\text{closed G-orbits of } S \leftrightarrow \text{points of } S/\!\!/\mathsf{G}.$$

$S/\!\!/\mathsf{G}$ is called the categorical quotient.

Let $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ be a representation of G,
K be a maximal compact subgroup of G,
$\langle\ ,\ \rangle$ be a K-invariant hermitian form on V with associated norm $\|\ \|$.

Given $v \in V$, consider the function

$$F_v \colon \mathsf{G} \to \mathbb{R}, \qquad g \mapsto \|gv\|^2.$$

It has a critical point if and only if $\mathsf{G}v$ is closed, and all critical points of F_v are minima. Define $\mathit{KN} \subset V$ by one of the following equivalent conditions:

$$\begin{aligned} \mathit{KN} &= \{v \in V \colon (dF_v)_e = 0\} \qquad (e \in \mathsf{G} \text{ is the unit}) \\ &= \{v \in V \colon T_v\mathsf{G}v \perp v\} \\ &= \{v \in V \colon \langle \gamma v, v\rangle = 0 \text{ for all } \gamma \in \mathfrak{g}\} \\ &= \{v \in V \colon \langle \kappa v, v\rangle = 0 \text{ for all } \kappa \in \mathfrak{k}\}, \end{aligned} \tag{1}$$

where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of G (resp. K). Therefore, any point $v \in \mathit{KN}$ is a closest point to the origin in its orbit $\mathsf{G}v$. Then KN is called the Kempf–Ness set of V.

We may assume that the affine G-variety S is equivariantly embedded as a closed subvariety in a representation V of G. Then the Kempf–Ness set KN_S of S is defined as $KN \cap S$.

The importance of Kempf–Ness sets for the study of orbit quotients is due to the following result (Kempf–Ness, Neeman).

Thm 2. a) *The composition* $KN_S \hookrightarrow S \to S/\!\!/G$ *is proper and induces a homeomorphism* $KN_S/K \to S/\!\!/G$.
b) *There is a* K*-equivariant deformation retraction* $S \to KN_S$.

2. Algebraic torus actions.

$N \cong \mathbb{Z}^n$ integral lattice, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. A convex subset $\sigma \in N_{\mathbb{R}}$ is a cone if there are vectors $a_1, \ldots, a_k \in N$ such that

$$\sigma = \{\mu_1 a_1 + \ldots + \mu_k a_k \colon \mu_i \in \mathbb{R},\ \mu_i \geqslant 0\}.$$

A cone σ is called regular (resp. simplicial) if $a_1, \ldots, a_k$ is a subset of a $\mathbb{Z}$-basis of N (resp. an $\mathbb{R}$-basis of $N_{\mathbb{R}}$). A finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of cones in $N_{\mathbb{R}}$ is called a fan if a face of every cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is called complete if $N_{\mathbb{R}} = \sigma_1 \cup \ldots \cup \sigma_s$.

$\mathsf{T}_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ algebraic torus, $\mathsf{T} = N \otimes_{\mathbb{Z}} \mathsf{S}^1 \cong (\mathsf{S}^1)^n$, a maximal compact subgroup, the (compact) torus. A toric variety is a normal algebraic variety X containing $\mathsf{T}_{\mathbb{C}}$ as a Zariski open subset so that the natural action of $\mathsf{T}_{\mathbb{C}}$ on itself extends to an action on X.

A classical construction establishes a one-to-one correspondence:

$$\begin{aligned} \text{fans in } N_{\mathbb{R}} \quad &\longleftrightarrow \quad \text{complex } n\text{-dimensional toric varieties,} \\ \text{regular fans} \quad &\longleftrightarrow \quad \text{non-singular varieties,} \\ \text{complete fans} \quad &\longleftrightarrow \quad \text{compact varieties.} \end{aligned}$$

Assume Σ has m one-dimensional cones, and consider the map $\mathbb{Z}^m \to N$ sending the ith generator of $\mathbb{Z}^m$ to the integer primitive vector a_i generating the ith one-dimensional cone. Get an exact sequence

$$1 \longrightarrow \mathsf{G} \longrightarrow (\mathbb{C}^*)^m \longrightarrow \mathsf{T}_{\mathbb{C}} \longrightarrow 1,$$

where $\mathsf{G} \cong (\mathbb{C}^*)^{m-n} \times$ (finite group), and

$$1 \longrightarrow \mathsf{K} \longrightarrow T^m \longrightarrow \mathsf{T} \longrightarrow 1 \quad (3)$$

(here and below we denote $T^m = (\mathsf{S}^1)^m$).

Say that $\{i_1, \dots, i_k\} \in [m] = \{1, \dots, m\}$ is a g-subset if $\{a_{i_1}, \dots, a_{i_k}\}$ is a subset of the generator set of a cone in Σ.

$K_\Sigma = \{g\text{-subsets}\}$ an abstract simplicial complex on the set $[m]$.

Σ is complete simplicial $\Rightarrow K_\Sigma$ is a triangulation of S^{n-1}.

Now set

$$A(\Sigma) = \bigcup_{\{i_1,\dots,i_k\} \text{ is not a } g\text{-subset}} \{z \in \mathbb{C}^m \colon z_{i_1} = \dots = z_{i_k} = 0\}$$

and

$$U(\Sigma) = \mathbb{C}^m \setminus A(\Sigma).$$

Both sets depend only on the combinatorial structure of the simplicial complex K_Σ; and $U(\Sigma)$ is the coordinate subspace arrangement complement $U(K_\Sigma)$.

The set $A(\Sigma)$ is an affine variety, while its complement $U(\Sigma)$ admits a simple affine cover, as described in the following statement.

Prop 4. *Given a cone* $\sigma \in \Sigma$, *denote* $g(\sigma) \subseteq [m]$ *the set of its generators,* $z^{\widehat{\sigma}} = \prod_{j \notin g(\sigma)} z_j$, *and define*

$$V(\Sigma) = \{z \in \mathbb{C}^m \colon z^{\widehat{\sigma}} = 0 \text{ for all } \sigma \in \Sigma\}$$

and

$$U(\sigma) = \{z \in \mathbb{C}^m \colon z_j \neq 0 \text{ if } j \notin g(\sigma)\}.$$

Then $A(\Sigma) = V(\Sigma)$ *and*

$$U(\Sigma) = \mathbb{C}^m \setminus V(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma).$$

$U(\Sigma) \subset \mathbb{C}^m$ is $(\mathbb{C}^*)^m$-invariant, and $\mathrm{G} \subset (\mathbb{C}^*)^m$ acts on $U(\Sigma)$ with finite isotropy subgroups if Σ is simplicial. The corresponding quotient is identified with the toric variety X_Σ determined by Σ. The more precise statement is as follows (Batyrev–Cox).

Thm 5. (a) $X_\Sigma \cong U(\Sigma)/\!\!/\mathrm{G}$ *(categorical quotient).*

(b) X_Σ *is the geometric quotient* $\Leftrightarrow$ Σ *is simplicial.*

However, the analysis of the previous section does not apply here, as $U(\Sigma)$ is *not* an affine variety in $\mathbb{C}^m$ (it is only quasiaffine in general). E.g., if Σ is a complete fan, then the G-action on the whole $\mathbb{C}^m$ has only one closed orbit 0, and the quotient $\mathbb{C}^m/\!\!/\mathrm{G}$ consists of a single point. Below we show that an appropriate notion of the Kempf–Ness set still exists for this class of torus actions.

Consider the unit polydisc

$$(\mathsf{D}^2)^m = \{z \in \mathbb{C}^m \colon |z_j| \leqslant 1 \text{ for all } j\}.$$

Given $\sigma \in \Sigma$, define

$$\mathcal{Z}(\sigma) = \{z \in (\mathsf{D}^2)^m \colon |z_j| = 1 \text{ if } j \notin g(\sigma)\},$$

and

$$\mathcal{Z}(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathcal{Z}(\sigma).$$

The subset $\mathcal{Z}(\Sigma) \subseteq (D^2)^m$ is invariant with respect to the T^m-action.

Prop 6. *Assume that Σ is a complete simplicial fan. Then $\mathcal{Z}(\Sigma)$ is a compact T^m-manifold of dimension $m + n$.*

Note that $\mathcal{Z}(\sigma) \subset U(\sigma)$, and therefore, $\mathcal{Z}(\Sigma) \subset U(\Sigma)$.

Thm 7. *Assume that Σ is a simplicial fan.*

a) *The composition $\mathcal{Z}(\Sigma) \hookrightarrow U(\Sigma) \to U(\Sigma)/\mathrm{G}$ is proper and induces a homeomorphism $\mathcal{Z}(\Sigma)/\mathrm{K} \to U(\Sigma)/\mathrm{G}$.*

b) *There is a T^m-equivariant deformation retraction of $U(\Sigma)$ to $\mathcal{Z}(\Sigma)$.*

Proof. b): there are obvious equivariant deformation retractions $U(\sigma) \to \mathcal{Z}(\sigma)$ for all $\sigma \in \Sigma$, which patch together to get the necessary map $U(\Sigma) \to \mathcal{Z}(\Sigma)$. □

By comparing this result with Theorem 2, we see that $\mathcal{Z}(\Sigma)$ has the same properties with respect to the G-action on $U(\Sigma)$ as the set KN_S with respect to a G-action on an affine variety S. We therefore refer to $\mathcal{Z}(\Sigma)$ as the Kempf–Ness set of $U(\Sigma)$.

3. Normal fans.

The next step in our study of the Kempf–Ness set for torus actions on quasiaffine varieties like $U(\Sigma)$ would be to obtain an explicit description like the one given by (1) in the affine case. Although we do not now of such a description in general, it does exist in the particular case when Σ is the normal fan of a simple polytope.

$M_\mathbb{R} = (N_\mathbb{R})^*$ the dual vector space. Given primitive vectors $a_1, \ldots, a_m \in N$ and $b_1, \ldots, b_m \in \mathbb{Z}$, consider

$$P = \{x \in M_\mathbb{R} \colon \langle a_i, x\rangle + b_i \geqslant 0, \quad i = 1, \ldots, m\}.$$

Assume that P is bounded, the affine hull of P is the whole $M_\mathbb{R}$, and there are no redundant inequalities. Then P is a convex polytope with exactly m facets F_i and normal vectors a_i, $i = 1, \ldots, m$.

If G is an l-dimensional face, then the set of all its normal vectors $\{a_{i_1}, \ldots, a_{i_k}\}$ spans an $(n-l)$-dimensional normal cone σ_G.

$\Sigma_P = \{\sigma_G \colon G \text{ a face of } P\}$ normal fan of P (a complete fan).

From now on, assume: Σ_P is simplicial $\Leftrightarrow$ P is simple.
In this case, $\{a_{i_1}, \ldots, a_{i_k}\}$ spans a cone of Σ_P $\Leftrightarrow$ $F_{i_1} \cap \ldots \cap F_{i_k} \neq \varnothing$.

Kempf–Ness sets $\mathcal{Z}(\Sigma_P)$ admit a very transparent interpretation as complete intersections of real algebraic quadrics.

We may specify P by a matrix inequality $A_P x + b_P \geqslant 0$, where A_P is the $m \times n$ matrix of row vectors a_i, and b_P is the column vector of scalars b_i.

The linear transformation $A_P\colon M_{\mathbb{R}} \to \mathbb{R}^m$ is exactly the one obtained from the map $T^m \to \mathsf{T}$ from (3) by applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, S^1) \otimes_{\mathbb{Z}} \mathbb{R}$.

The formula $i_P(x) = A_P x + b_P$ defines an affine injection

$$i_P\colon M_{\mathbb{R}} \longrightarrow \mathbb{R}^m,$$

which embeds P into the positive cone $\mathbb{R}^m_{\geqslant}$.

Now define the space $\mathcal{Z}_P$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \varrho_P \Big\downarrow & & \Big\downarrow \varrho \\ P & \xrightarrow{i_P} & \mathbb{R}^m \end{array}$$

where $\varrho(z_1, \ldots, z_m)$ is given by $(|z_1|^2, \ldots, |z_m|^2)$.

Prop 8. a) *We have* $\mathcal{Z}_P \subset U(\Sigma_P)$.

b) *There is a* T^m*-equivariant homeomorphism* $\mathcal{Z}_P \cong \mathcal{Z}(\Sigma_P)$.

Applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,,\mathsf{S}^1)\otimes_{\mathbb{Z}}\mathbb{R}$ to

$$1 \longrightarrow \mathsf{K} \longrightarrow T^m \longrightarrow \mathsf{T} \longrightarrow 1$$

we get

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \longrightarrow 0,$$

where $C = (c_{jk})$ is an $(m-n)\times m$-matrix.

The map i_Z embeds $\mathcal{Z}_P$ in $\mathbb{C}^m$ as the space of solutions of the $m-n$ real quadratic equations

$$\sum_{k=1}^{m} c_{j,k}\left(|z_k|^2 - b_k\right) = 0, \quad \text{for } 1 \leq j \leq m-n. \tag{9}$$

This intersection of real quadrics is non-degenerate, and therefore, $\mathcal{Z}_P \subset \mathbb{R}^{2m}$ is a smooth submanifold with trivial normal bundle.

4. Projective toric varieties and moment maps.

Let $f_v = (dF_v)_e \colon \mathfrak{g} \to \mathbb{R}, \quad \gamma \in \mathfrak{g} \mapsto \langle \gamma v, v \rangle$. Think $f_v \in \mathfrak{g}^*$.

G is reductive $\Rightarrow$ $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$.

K-action is norm preserving $\Rightarrow$ f_v vanishes on $\mathfrak{k}$,
so we consider f_v as an element of $i\mathfrak{k}^* \cong \mathfrak{k}^*$.

Varying $v \in V$ we get the moment map $\mu \colon V \to \mathfrak{k}^*$, which sends $v \in V$, $\kappa \in \mathfrak{k}$ to $\langle i\kappa v, v \rangle$. The Kempf–Ness set is the set of zeroes of μ:

$$KN = \mu^{-1}(0).$$

This description does not directly apply to algebraic torus actions on $U(\Sigma)$: the set

$$\mu^{-1}(0) = \{z \in \mathbb{C}^m \colon \langle \kappa z, z \rangle = 0 \text{ for all } \kappa \in \mathfrak{k}\}$$

consist only of the origin in this case.

Assume now that Σ_P is a regular fan. Then X_{Σ_P} is a smooth projective variety. This implies that X_{Σ_P} is Kähler, and therefore, a symplectic manifold. In this case there is a symplectic version of the previous constructions.

(W, ω) symplectic manifold with symplectic K-action.

Given $\kappa \in \mathfrak{k}$, denote by ξ_κ the corresponding K-invariant vector field on W.

The K-action is Hamiltonian if the 1-form $\omega(\,\cdot\,, \xi_\kappa)$ is exact for every $\kappa \in \mathfrak{k}$, that is, there is a function H_κ on W such that

$$\omega(\xi, \xi_\kappa) = dH_\kappa(\xi) = \xi(H_\kappa)$$

for every vector field ξ on W. Under this assumption, the moment map

$$\mu\colon W \to \mathfrak{k}^*, \qquad (x, \kappa) \mapsto H_\kappa(x)$$

is defined.

Ex 10. 1. $W = \mathbb{C}^m$, $\omega = 2\sum_{k=1}^m dx_k \wedge dy_k$, $z_k = x_k + iy_k$. The coordinatewise action of $\mathsf{K} = T^m$ is Hamiltonian with moment map

$$\mu\colon \mathbb{C}^m \to \mathbb{R}^m, \qquad (z_1, \dots, z_m) \mapsto (|z_1|^2, \dots, |z_m|^2).$$

2. Σ a simplicial fan, $1 \to \mathsf{K} \to T^m \to \mathsf{T} \to 1$.
We can restrict the previous example to the K-action on the invariant subvariety $U(\Sigma) \subset \mathbb{C}^m$. The moment map

$$\mu_\Sigma\colon \mathbb{C}^m \longrightarrow \mathbb{R}^m \longrightarrow \mathfrak{k}^*.$$

A choice of an isomorphism $\mathfrak{k} \cong \mathbb{R}^{m-n}$ allows to identify the map $\mathbb{R}^m \to \mathfrak{k}^*$ with the linear transformation given by matrix C in

$$0 \longrightarrow M_\mathbb{R} \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \longrightarrow 0.$$

Unlike the affine case, the Kempf–Ness set $\mathcal{Z}_P$ for the G-action on $U(\Sigma_P)$ *does not* coincide with the level set

$$\mu_\Sigma^{-1}(0) = \{z \in \mathbb{C}^m \colon \langle \kappa z, z\rangle = 0 \text{ for all } \kappa \in \mathfrak{k}\}.$$

The latter is given by the equations $\sum_{k=1}^m c_{j,k}|z_k|^2 = 0,\ 1 \le j \le m-n$, which have only zero solution.

The right statement is as follows.

Prop 11. *Then the Kempf–Ness set $\mathcal{Z}(\Sigma_P)$ is given by*

$$\mathcal{Z}(\Sigma_P) \cong \mu_{\Sigma_P}^{-1}(Cb_P).$$

In other words, the difference with the affine situation is that we have to take Cb_P instead of 0 as the value of the moment map. The reason is that Cb_P is a regular value of μ, unlike 0.

In the case of normal fans the following version of our Theorem 7 (a) is known in toric geometry (Audin, Batyrev, Guillemin):

Thm 12. *Assume X_Σ a projective simplicial toric variety and c is in its Kähler cone. Then $\mu_\Sigma^{-1}(c) \subset U(\Sigma)$, and the natural map*

$$\mu_\Sigma^{-1}(c)/\mathsf{K} \to U(\Sigma)/\mathsf{G} = X_\Sigma$$

is a diffeomorphism.

This statement is the essence of the construction of smooth projective toric varieties via symplectic reduction. The submanifold $\mu_\Sigma^{-1}(c) \subset \mathbb{C}^m$ may fail to be symplectic as the restriction of the standard symplectic form ω on $\mathbb{C}^m$ to $\mu_\Sigma^{-1}(c)$ may fail to be non-degenerate. However, the restriction of ω descends to the quotient $\mu_\Sigma^{-1}(c)/\mathsf{K}$ as a symplectic form.

Ex 13. $P = \Delta^n \subset M_{\mathbb{R}}$ standard simplex defined by

$$\langle e_i, x\rangle \geqslant 0,\ \ i = 1, \ldots, n,\ \text{ and } \langle(-1, \ldots, -1), x\rangle + 1 \geqslant 0$$

Cones of Σ_P are generated by proper subsets of the set of vectors $\{e_1, \ldots, e_n, (-1, \ldots, -1)\}$.

$\mathsf{G} \cong \mathbb{C}^*$ and $\mathsf{K} \cong \mathsf{S}^1$ are the diagonal subgroups in $(\mathbb{C}^*)^{n+1}$ and T^{n+1} respectively, while $U(\Sigma) = \mathbb{C}^{n+1} \setminus \{0\}$.

C is a row of units. Moment map is given by

$$\mu_\Sigma(z_1, \ldots, z_{n+1}) = |z_1|^2 + \ldots + |z_{n+1}|^2.$$

Since $Cb_P = 1$, the Kempf–Ness set $\mathcal{Z}_P = \mu_\Sigma^{-1}(1)$ is the unit sphere $\mathsf{S}^{2n+1} \subset \mathbb{C}^{n+1}$, and $X_\Sigma = (\mathbb{C}^{n+1} \setminus \{0\})/\mathsf{G} = \mathsf{S}^{2n+1}/\mathsf{K}$ is the complex projective space $\mathbb{C}P^n$.

Problem 14. *As is known, there are many complete regular fans* Σ *which cannot be realised as normal fans of convex polytopes. The corresponding toric varieties* X_Σ *are not projective (although being non-singular). In this case the Kempf–Ness set* $\mathcal{Z}(\Sigma)$ *is still defined, as well as the moment map. However, the rest of the analysis does not apply here. Can one still describe* $\mathcal{Z}(\Sigma)$ *as a complete intersection of real quadratic (or other) hypersurfaces? And does the moment map* $\mu_\Sigma \colon U(\Sigma) \to \mathfrak{t}^*$ *have any regular values?*

5. Cohomology of Kempf–Ness sets.

Given an abstract simplicial complex K on the set $[m]$, the face ring (or the Stanley–Reisner ring) $\mathbb{Z}[K]$ is defined as the following quotient of the polynomial ring on m generators:

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} \colon \{i_1, \ldots, i_k\} \text{ is not a simplex of } K).$$

Grading $\deg v_i = 2, \quad i = 1, \ldots, m$.

Thm 15. *For every simplicial fan Σ there are algebra isomorphisms*

$$H^*\bigl(\mathcal{Z}(\Sigma); \mathbb{Z}\bigr) \cong \mathrm{Tor}^*_{\mathbb{Z}[v_1,\ldots,v_m]}\bigl(\mathbb{Z}[K_\Sigma], \mathbb{Z}\bigr) \cong H\Bigl[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K_\Sigma], d\Bigr],$$

where the latter denotes the cohomology of a dga with $\deg u_i = 1$, $\deg v_i = 2$, $du_i = v_i$, $dv_i = 0$ *for* $1 \leqslant i \leqslant m$.

Given $I \subseteq [m]$, denote by $K(I)$ the corresponding full subcomplex of K, or the restriction of K to I.

$\widetilde{H}^i(K(I))$ the ith reduced simplicial cohomology group of $K(I)$ with integer coefficients.

A theorem due to Hochster expresses $\mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K],\mathbb{Z})$ in terms of full subcomplexes of K, which leads to the following description of the cohomology of $\mathcal{Z}(\Sigma)$.

Thm 16. *We have*

$$H^k\Big(\mathcal{Z}(\Sigma)\Big) \cong \bigoplus_{I \subseteq [m]} \widetilde{H}^{k-|I|-1}\Big(K_\Sigma(I)\Big).$$

There is also a similar description of the product in $H^*(\mathcal{Z}(\Sigma))$.

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