

# Spaces of polytopes, and cobordisms of toric manifolds

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## 1. Simple polytopes.

An **arrangement** of half-spaces is a collection  $\mathcal{H}$  of subsets

$$S_i = \{x \in V : (a_i, x) - b_i \geq 0\}, \quad 1 \leq i \leq m,$$

where  $a_i \in V \cong \mathbb{R}^n$  and  $b_i \in \mathbb{R}$ . When the intersection of the  $S_i$  is bounded, it forms a **convex polytope**  $P$ ; otherwise, it is a **polyhedron**.

We may specify  $P$  by the matrix inequality  $A_P x \geq b$ .

Assume  $\dim P = n$ , and no redundant half-spaces. So  $P$  has  $m$  facets  $F_i$ , defined by its intersection with the **bounding hyperplanes**

$$H_i = \{x \in V : (a_i, x) = b_i\}.$$

When the bounding hyperplanes are in general position, every vertex is the intersection of precisely  $n$  facets, and  $P$  is **simple**.

The **positive cone**

$$\mathbb{R}_{\geq}^m = \{h \in \mathbb{R}^m : h_i \geq 0 \text{ for } i = 1, \dots, m\}.$$

## 2. Spaces of polytopes.

Following Khovanskii, we fix  $A_P$  and identify the  $m$ -dimensional vector  $b$  with the arrangement  $\mathcal{H}$ , and hence with the polytope  $P$ . The coordinates  $b_i$  describe the signed distances of the hyperplanes  $H_i$  from the origin  $0$  in  $V$ , so long as the normal vectors  $a_i$  have length 1; otherwise, the distances have to be scaled accordingly. The sign is positive or negative as  $0$  lies in the interior or exterior of  $S_i$  respectively.

Every vector in  $\mathbb{R}^m$  may then be identified with an analogous arrangement of halfspaces, obtained from  $\mathcal{H}$  by parallel displacement of the  $S_i$ . Some such arrangements define polytopes, and others, dubbed virtual polytopes by Khovanskii, do not; in either case, we also describe the corresponding intersections as analogous. The zero vector

is therefore identified with the virtual polytope  $\{0\}$ . An  $n$ -parameter family of examples is given by the translations of  $V$ , for which the corresponding polytopes are **congruent** to  $P$ .

In this context, we denote the  $m$ -dimensional vector space of polytopes analogous to  $P$  by  $\mathbb{R}(P)$ , and interpret the identification as an isomorphism  $k: \mathbb{R}^m \rightarrow \mathbb{R}(P)$ .

We may interpret the matrix  $A_P$  as a linear transformation  $V \rightarrow \mathbb{R}^m$ . Since the points of  $P$  are specified by the constraint  $A_P x \geq b$ , it follows that the intersection of the affine subspace  $A_P(V) - b$  with the positive cone  $\mathbb{R}_{\geq}^m$  is a copy of  $P$  in  $\mathbb{R}^m$ . Denote

$$i_P: V \rightarrow \mathbb{R}^m; \quad V \mapsto A_P(v) - b.$$

So  $\chi = k \circ i_P$  restricts to an affine embedding  $P \rightarrow \mathbb{R}(P)$ , for which  $\chi(x)$  is the polytope congruent to  $P$  with origin at  $x$ , for all  $x \in P$ . In particular,  $\chi(P)$  is a submanifold of the positive cone  $\mathbb{R}(P)_{\geq}$ .

### 3. (Quasi)toric manifolds.

A (quasi)toric manifold (cf. Davis–Januszkiewicz)  $M = M^{2n}$  over a simple polytope  $P = P^n$  has

- an action of an  $n$ -dimensional torus  $\mathbf{T}$  that locally looks like the standard  $\mathbf{T}$ -action on  $\mathbb{C}^n$ ;
- the orbit map  $\pi: M \rightarrow P$  sending every set of orbits with the same isotropy group onto the interior of a face of  $P$ .

In comparison with the smooth compact toric varieties from algebraic geometry, (quasi)toric manifolds enjoy much larger flexibility for both

topological and combinatorial applications, while retaining most of the important properties of their algebraic counterpart.



## Codimension-two **characteristic submanifolds**

$M_i \subset M$ ,  $i = 1, \dots, m$ :

- $M_i = \pi^{-1}(F_i)$ , where  $F_i$  is a facet of  $P$ ;

or, equivalently,

- $M_i$  is a connected submanifold fixed pointwise by a circle subgroup of  $\mathbf{T}$ .

We denote by  $\rho_i$  the canonical orientable 2-dimensional real bundle over  $M$  determined by  $M_i$ ; it restricts to the normal bundle  $\nu(M_i \hookrightarrow M)$ .

An **omniorientation** of  $M$  consists of a choice of orientation for  $M$  and each  $\rho_i$ .

A choice of orientation of  $\rho_i$  identifies it as a complex line bundle. The one-dimensional isotropy subgroup  $\mathbf{T}_{M_i}$  of  $M_i$  acts in the fibres of the normal bundle  $\nu(M_i \hookrightarrow M)$ . We orient the circle  $\mathbf{T}_{M_i}$  in such way that this action preserves the orientation determined by the complex structure in  $\rho_i$ . Thereby we obtain a map

$$\lambda: T^m \rightarrow \mathbf{T}, \quad T^{F_i} \mapsto \mathbf{T}_{M_i},$$

called the **characteristic map** of  $M$ . Due to a non-singularity condition the kernel  $K(\lambda)$  of  $\lambda$  is isomorphic to a  $(m - n)$ -dimensional torus.

**Thm 1 (Davis–Januszkiewicz).** *There is an isomorphism of real  $2m$ -plane bundles:*

$$\tau(M) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \dots \oplus \rho_m.$$

*Idea of proof.* Consider the pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P = T^m \times P / \sim & \xrightarrow{i_Z} & T^m \times \mathbb{R}_{\geq}^m / \sim = \mathbb{C}^m \\ \downarrow & & \downarrow \rho \\ P & \xrightarrow{i_P} & \mathbb{R}^m. \end{array},$$

where  $\rho(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$ . It determines a canonically framed manifold  $\mathcal{Z}_P$ .

The map  $\lambda: T^m \rightarrow \mathbf{T}$  induces a principal  $K(\lambda)$ -fibration  $p: \mathcal{Z}_P = T^m \times P/\sim \rightarrow \mathbf{T} \times P/\approx = M$ . The tangent bundle to  $\mathcal{Z}_P$  decomposes as

$$\tau(\mathcal{Z}_P) = p^*\tau(M) \oplus \xi(p)$$

where  $\xi(p)$  is the tangent bundle along the fibres of  $p$ . The required bundle isomorphism comes from [Szczarba's](#) identification

$$\tau(M) \oplus (\xi(p)/K(\lambda)) \oplus (\nu(i_Z)/K(\lambda)) \cong \rho_1 \oplus \dots \oplus \rho_m$$

by noticing that both  $\xi(p)/K(\lambda)$  and  $\nu(i_Z)/K(\lambda)$  are trivial real  $(m-n)$ -plane bundles over  $M$ . □

## 4. Stably almost complex structures.

**Thm 2.** *A choice of omniorientation of  $M$ , ordering of facets, and initial vertex of  $P$  gives rise to a canonical framing of the real  $2(m-n)$ -bundle  $\nu(i_Z)/K(\lambda) \oplus \xi(p)/K(\lambda)$  over  $M$ , thereby determining a canonical  $\mathbf{T}$ -invariant stably complex structure for  $M$ .*

**Prop 3.** *The equivalence class of the stably complex structure on  $M$  defined in Thm 2, and therefore the corresponding complex cobordism class, depends only on a choice of orientations for  $M$  and for each normal bundle  $\nu(M_i \hookrightarrow M)$ .*

## 5. Complex cobordisms.

**Thm 4** (Buchstaber–Ray’01, corrected by Buchstaber–Ray–P.). *In dimensions  $> 2$ , every complex cobordism class contains a toric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

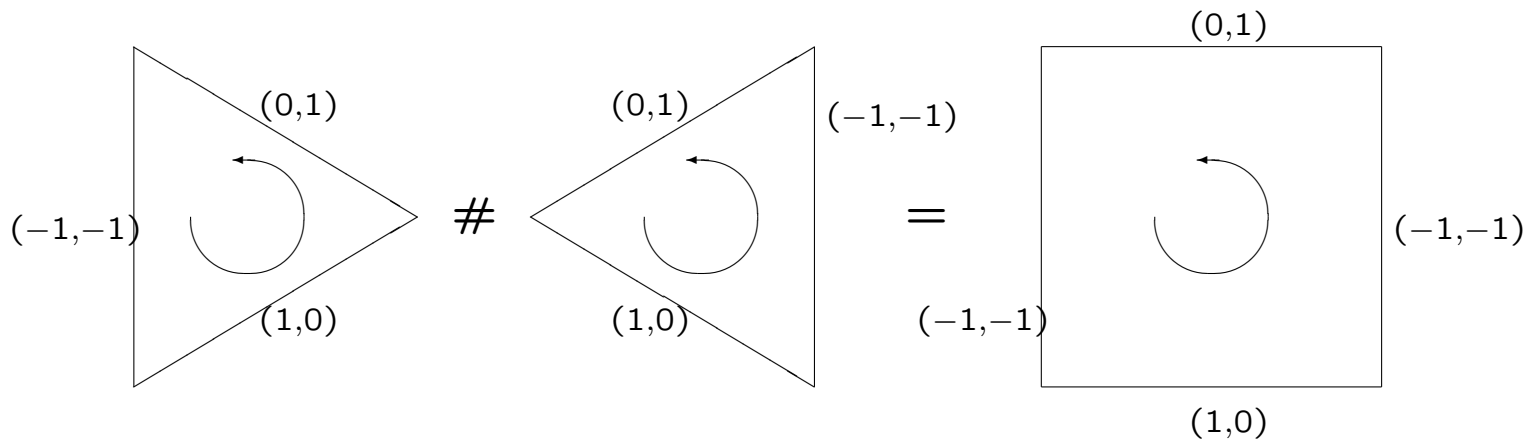
*Idea of proof.* Start with the additive basis in the complex cobordism ring consisting of toric manifolds, constructed by Buchstaber and Ray in 1999. Then one needs to replace the disjoint union (representing the cobordism sum) by something connected.

Given two cobordism classes represented by  $2n$ -dimensional omnioriented toric manifolds  $M_1$  and  $M_2$ , with quotient polytopes  $P_1$  and

$P_2$  respectively, we need to construct a third such manifold  $M$ , with quotient polytope  $P$ , representing the sum of the cobordism classes of  $M_1$  and  $M_2$ . This is done using the connected sum construction.  $\square$

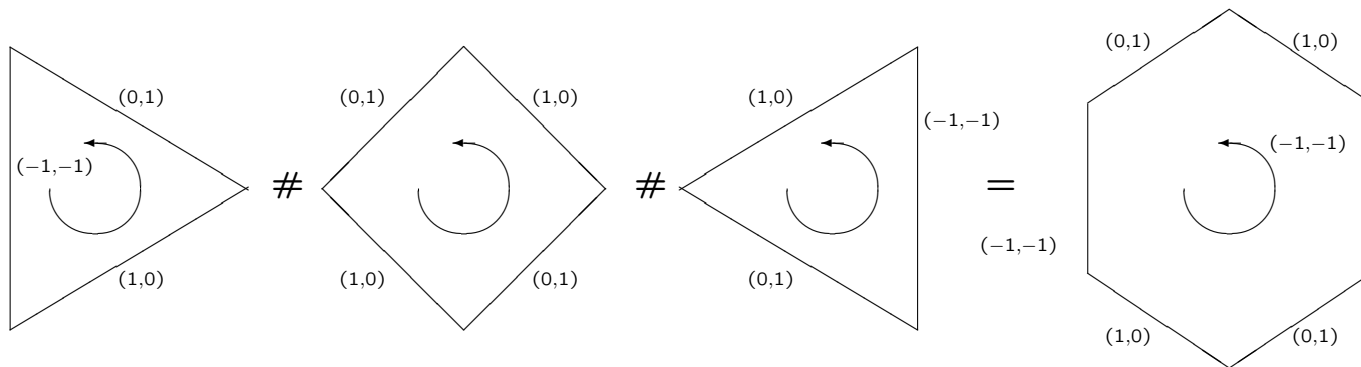


**Ex 5.** Connected sum of  $\mathbb{C}P^2$  and  $\overline{\mathbb{C}P^2}$ . The resulting stably complex structure on the manifold  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$  bounds, that is, represents the zero cobordism class.



However, it is not possible to take the connected sum of two copies of  $\mathbb{C}P^2$  with the standard omniorientation by a procedure like this. A modification is needed here.

**Ex 6.** Connected sum  $\mathbb{C}P^2 \# S \# \mathbb{C}P^2$ .



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