Model categories and homotopy colimits in toric topology

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joint work with Nigel Ray

1. Motivations.

Object of study of "toric topology": torus actions on manifolds or complexes with a rich combinatorial structure in the orbit quotient.

Particular examples:

- Non-singular compact toric varieties M^{2n}
  T^n-action is a part of an algebraic $\mathbb{C}^{*n}$-action with a dense orbit;

- (Quasi)toric manifolds M^{2n} of Davis–Januszkiewicz
  "locally standard" (i.e., locally look like T^n acting on $\mathbb{C}^n$) and M/T combinatorially is a simple polytope;

- Torus manifolds of Hattori–Masuda, "moment-angle complexes", complex coordinate subspace arrangement complements etc.

2. Simplicial complexes and face rings.

K a simplicial complex on $V = \{v_1, \dots, v_m\}$ (e.g., the dual to the boundary of a simplicial polytope).

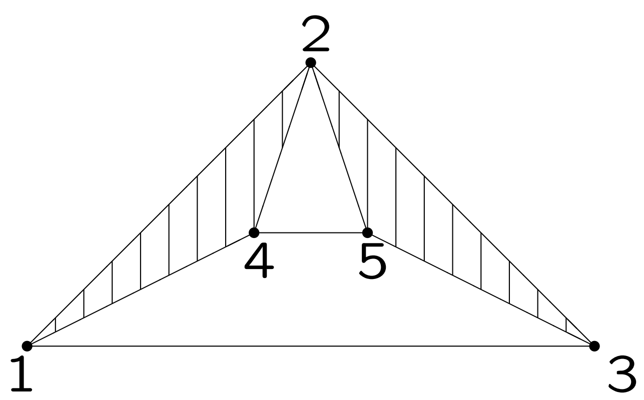
$\sigma \in K$ a simplex.

$R[v_1, \dots, v_m]$ polynomial algebra on V over R, $\deg v_i = 2$. Given $\omega \subseteq V$, set $v_\omega := \prod_{i\in\omega} v_i$. The Stanley-Reisner algebra (or face ring) of K is

$$R[K] := R[v_1, \dots, v_m]/(v_\omega \colon \omega \notin K).$$

Ex 1.

K:



$R[K] = R[v_1, \dots, v_5]/(v_1v_5, v_3v_4, v_1v_2v_3, v_2v_4v_5).$

The Poincaré series of $R[K]$ is given by

$$F(R^*[K];t) = \sum_{i=-1}^{n-1} \frac{f_i t^{2(i+1)}}{(1-t^2)^{i+1}} = \frac{h_0 + h_1 t^2 + \ldots + h_n t^{2n}}{(1-t^2)^n},$$

where $\dim K = n-1$, f_i is the number of i-dimensional simplices in K, $f_{-1} = 1$, and the numbers h_i are defined from the second identity.

A missing face of K is a subset $\omega \subseteq V$ s.t. $\omega \notin K$, but every proper subset of ω is a simplex. K is a flag complex if any of its missing faces has two vertices. In this case

$$R[K] = T(v_1, \ldots, v_m) \Big/ \Big(v_i v_j - v_j v_i = 0 \text{ for } \{i,j\} \in K, \quad v_i v_j = 0 \text{ for } \{i,j\} \notin K \Big),$$

a quadratic algebra.

3. Sample questions.

[g-conjecture] Characterise the f-vectors $(f_0, \ldots, f_{n-1})$ of triangulations of S^{n-1} (done for polytopes).

[Charney–Davis conj] Let K^{2q-1} be flag Gorenstein* (e.g., a sphere triangulation). Then

$$(-1)^q(h_0 - h_1 + h_2 - h_3 + \ldots + h_{2q}) \geqslant 0.$$

Calculate the (co)homology of $R[K]$. When the Ext-cohomology $\mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k}, \mathbf{k})$ has a rational Poincaré series?

The Davis-Januszkiewicz space

$$\mathit{DJ}(K) := \bigcup_{\sigma \in K} BT^\sigma \subseteq BT^m = (\mathbb{C}P^\infty)^m.$$

Let M^{2n} be a toric variety (or a quasitoric manifold) and K^{n-1} the underlying simplicial complex of the corresponding fan.

Prop 2. $\mathit{DJ}(K) \simeq ET^n \times_{T^n} M^{2n}$;
$H^*(\mathit{DJ}(K);\mathbb{Z}) \cong H^*_{T^n}(M;\mathbb{Z}) \cong \mathbb{Z}[K]$.

Define

$$\mathcal{Z}_K := \text{hofibre}\Big(\mathit{DJ}(K) \hookrightarrow BT^m\Big).$$

The space $\mathcal{Z}_K$ is a finite cell complex acted on by T^m, called the moment-angle complex. There is a principal T^{m-n}-bundle $\mathcal{Z}_K \to M$. This space also has many other interesting interpretations, e.g. as a complex coordinate subspace arrangement complement or as a level surface for a certain moment map.

4. (Co)homology of face rings and toric spaces.

Thm 3 (Buchstaber-P). *There is an isomorphism of bigraded algebras*

$$
\begin{aligned}
H^*(\mathcal{Z}_K;\mathbb{Z}) &\cong \mathrm{Tor}^{*,*}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K],\mathbb{Z})\\
&\cong H\Big[\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[K];d\Big],
\end{aligned}
$$

where $du_i = v_i$, $dv_i = 0$.

What about $\mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k})$?

The fibration $DJ(K)\to BT^m$ with fibre $\mathcal{Z}_K$ splits after looping: $\Omega\,DJ(K)\simeq\Omega\mathcal{Z}_K\times T^m$. This is not an H-space splitting, and the exact sequence of Pontrjagin homology rings

$$0\to H_*(\Omega\mathcal{Z}_K)\to H_*(\Omega\,DJ(K))\to\Lambda[u_1,\ldots,u_m]\to 0$$

does not split in general.

Prop 4. $H_*(\Omega\,DJ(K),\mathbf{k})\cong\mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k})$

Idea of proof: Use Adams' cobar construction and formality of $DJ(K)$.

Prop 5. *Suppose K is flag. Then*

$$H_*(\Omega\, DJ(K), \mathbf{k}) \cong T_{\mathbf{k}}(u_1, \dots, u_m)\Big/\Big(u_i^2 = 0,\ u_iu_j + u_ju_i = 0 \text{ for } \{i,j\} \in K\Big).$$

Idea of proof: Use Koszul duality for algebras.

Cor 6. *If K is flag then*

$$\pi_*(\Omega\, DJ(K)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong FL(u_1, \dots, u_m)\Big/\Big([u_i, u_i] = 0,\ [u_i, u_j] = 0 \text{ for } \{i,j\} \in K\Big),$$

where FL( ) is a free Lie algebra and $\deg u_i = 1$.

Cor 7. *If K is flag, then the rational homology Poincaré series of $\Omega\, DJ(K)$ is given by*

$$F\Big(H_*(\Omega\, DJ(K)); t\Big) = \frac{(1+t)^n}{1 - h_1t + \ldots + (-1)^n h_n t^n}.$$

5. Categories and colimits.

cat(K): face category of K (simplices and incl);
mc: a model category (e.g., top, tgp or dga);

$X \in$ mc
X^K: cat$(K) \to$ mc exponential diagram; its value on $\sigma \subseteq \tau$ is the inclusion $X^\sigma \subseteq X^\tau$; $X^\varnothing = pt$.

Many previous constructions are colimits, e.g.,

$$DJ(K) = \operatorname{colim}^{\text{top}} BT^K,$$

$R_*[K]$ = dual coalgebra of $R[K] = \operatorname{colim}^{\text{dgc}} C(v)^K$,

where $C(v)$ is the symmetric coalgebra on v, $\deg v = 2$.

Cor 8. *Assume K is flag. Then*

$$\Omega\, DJ(K) \cong \operatorname{colim}^{\text{tgp}} T^K;$$
$$H_*(\Omega\, DJ(K), \mathbb{Q}) \cong \operatorname{colim}^{\text{ga}} \Lambda[u]^K;$$
$$\pi_*(\Omega\, DJ(K)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \operatorname{colim}^{\text{gl}} CL(u)^K,$$

where $CL(u)$ is the commutative Lie algebra, $\deg u = 1$.

In general colimit models do not work! (Look at $K = \partial\Delta^2$, in which case $DJ(K)$ is not coformal.)

6. Homotopy colimit models.

Appropriate notions of homotopy colimits exist in the model categories tgp, tmon, dga, dgc and dgl.

Thm 9. *(P.-Ray-Vogt) The loop space functor* $\Omega\colon \mathsf{top} \to \mathsf{tmon}$ *commutes with the homotopy colimit, i.e., there is a weak equivalence*

$$\Omega\,\mathsf{hocolim}^{\mathsf{top}} D \to \mathsf{hocolim}^{\mathsf{tmon}}\,\Omega D$$

for every diagram $D\colon \mathsf{c} \to \mathsf{top}$*.*

For diagrams over $\mathsf{cat}(K)$ we get

Thm 10. *(P.-Ray-Vogt) There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega\,\mathsf{hocolim}^{\mathsf{top}} BT^K & \xrightarrow{\overline{h}_K} & \mathsf{hocolim}^{\mathsf{tgp}}\,T^K \\ \downarrow{\scriptstyle \Omega p_K} & & \downarrow \\ \Omega\,DJ(K) & \xrightarrow{h_K} & \mathsf{colim}^{\mathsf{tgp}}\,T^K \end{array},$$

in which Ωp_K *and* $\overline{h}_K$ *are weak equivalences, while* h_K *is a weak equivalence only if* K *is flag.*

There is a similar result in algebraic mc. The algebraic analogue of the loop functor is the cobar construction $\Omega_*\colon \mathsf{dgc} \to \mathsf{dga}$.

Thm 11. *There is a htpy commutative diagram*

$$\begin{array}{ccc}
\Omega_* \,\mathrm{hocolim}^{\mathsf{dgc}}\, C(v)^K & \xrightarrow{\overline{\eta}_K} & \mathrm{hocolim}^{\mathsf{dga}}\, \Lambda[u]^K \\
\Big\downarrow{\scriptstyle \Omega_*\rho_K} & & \Big\downarrow \\
\Omega_*(\mathbb{Q}_*[K]) & \xrightarrow{\eta_K} & \mathrm{colim}^{\mathsf{dga}}\, \Lambda[u]^K
\end{array},$$

in which $\Omega_*\rho_K$ and $\overline{\eta}_K$ are weak equivalences, while η_K is a weak equivalence only if K is flag.

Cor 12.

$$H_*(\Omega\, DJ(K); \mathbb{Q}) \cong H\Big(\mathrm{hocolim}^{\mathsf{dga}}\, \Lambda[u]^K\Big)$$
$$\pi_*(\Omega\, DJ(K)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H\Big(\mathrm{hocolim}^{\mathsf{dgl}}\, CL(u)^K\Big).$$

Ex 13. Let K be the 1-skeleton of a 3-simplex. A calculation using the previous results gives

$$H_*(\Omega\, DJ(K)) \cong \frac{T(u_1, u_2, u_3, u_4, w_{123}, w_{124}, w_{134}, w_{123})}{\text{(relations)}},$$

where $\deg w_{ijk} = 4$ and there are 3 types of relations:

(a) exterior algebra relations for u_1, u_2, u_3, u_4;

(b) $[u_i, w_{jkl}] = 0$ for $i \in \{j, k, l\}$;

(c) $[u_1, w_{234}] + [u_2, w_{134}] + [u_3, w_{124}] + [u_4, w_{123}] = 0$.

w_{ijk} is the higher commutator (Hurevicz image of the higher Samelson product) of u_i, u_j and u_k, so the last equation is a higher Jacobian identity.

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