Model categories and homotopy colimits in toric topology

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joint work with Nigel Ray
1. Motivations.

Object of study of “toric topology”: torus actions on manifolds or complexes with a rich combinatorial structure in the orbit quotient.

Particular examples:

- Non-singular compact toric varieties $M^{2n}$
  $T^n$-action is a part of an algebraic $\mathbb{C}^n$-action with a dense orbit;

- (Quasi)toric manifolds $M^{2n}$ of Davis–Januszkiewicz “locally standard” (i.e., locally look like $T^n$ acting on $\mathbb{C}^n$) and $M/T$ combinatorially is a simple polytope;

- Torus manifolds of Hattori–Masuda, “moment-angle complexes”, complex coordinate subspace arrangement complements etc.
2. Simplicial complexes and face rings.

$K$ a simplicial complex on $V = \{v_1, \ldots, v_m\}$ (e.g., the dual to the boundary of a simplicial polytope).

$\sigma \in K$ a simplex.

$R[v_1, \ldots, v_m]$ polynomial algebra on $V$ over $R$, $\deg v_i = 2$. Given $\omega \subseteq V$, set $v_\omega := \prod_{i \in \omega} v_i$. The Stanley-Reisner algebra (or face ring) of $K$ is

$R[K] := R[v_1, \ldots, v_m]/(v_\omega : \omega \notin K)$.

Ex 1.

$R[K] = R[v_1, \ldots, v_5]/(v_1v_5, v_3v_4, v_1v_2v_3, v_2v_4v_5)$. 
The Poincaré series of \( R[K] \) is given by

\[
F(R^*[K]; t) = \sum_{i=-1}^{n-1} \frac{f_it^2(i+1)}{(1-t^2)^{i+1}}
= \frac{h_0 + h_1t^2 + \ldots + h_n t^{2n}}{(1-t^2)^n},
\]

where \( \dim K = n - 1 \), \( f_i \) is the number of \( i \)-dimensional simplices in \( K \), \( f_{-1} = 1 \), and the numbers \( h_i \) are defined from the second identity.

A missing face of \( K \) is a subset \( \omega \subseteq V \) s.t. \( \omega \notin K \), but every proper subset of \( \omega \) is a simplex. \( K \) is a flag complex if any of its missing faces has two vertices. In this case

\[
R[K] = T(v_1, \ldots, v_m) / (v_i v_j - v_j v_i = 0 \text{ for } \{i, j\} \in K, \quad v_i v_j = 0 \text{ for } \{i, j\} \notin K),
\]

a quadratic algebra.
3. Sample questions.

\[ \text{[g-conjecture]} \] Characterise the \( f \)-vectors \((f_0, \ldots, f_{n-1})\) of triangulations of \( S^{n-1} \) (done for polytopes).

\[ \text{[Charney–Davis conj]} \] Let \( K^{2q-1} \) be flag Gorenstein* (e.g., a sphere triangulation). Then

\[ (-1)^q (h_0 - h_1 + h_2 - h_3 + \ldots + h_{2q}) \geq 0. \]

Calculate the (co)homology of \( R[K] \). When the Ext-cohomology \( \text{Ext}_{k[K]}(k, k) \) has a rational Poincaré series?
The Davis–Januszkiewicz space

\[ DJ(K) := \bigcup_{\sigma \in K} BT^\sigma \subseteq BT^m = (\mathbb{C}P^\infty)^m. \]

Let \( M^{2n} \) be a toric variety (or a quasitoric manifold) and \( K^{n-1} \) the underlying simplicial complex of the corresponding fan.

Prop 2. \( DJ(K) \cong ET^n \times_T M^{2n}; \)
\[ H^*(DJ(K); \mathbb{Z}) \cong H^*_T(M; \mathbb{Z}) \cong \mathbb{Z}[K]. \]

Define

\[ Z_K := \text{hofibre}(DJ(K) \hookrightarrow BT^m). \]

The space \( Z_K \) is a finite cell complex acted on by \( T^m \), called the moment-angle complex. There is a principal \( T^{m-n} \)-bundle \( Z_K \to M \). This space also has many other interesting interpretations, e.g. as a complex coordinate subspace arrangement complement or as a level surface for a certain moment map.
4. (Co)homology of face rings and toric spaces.

**Thm 3** (Buchstaber-P). *There is an isomorphism of bigraded algebras*

\[ H^*(\mathbb{Z}_K; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[v_1, \ldots, v_m]}^* (\mathbb{Z}[K], \mathbb{Z}) \]

\[ \cong H\left[ \Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K]; d \right], \]

*where* \( du_i = v_i, \ dv_i = 0 \).

What about \( \text{Ext}_{k[K]}(k, k) \)?

The fibration \( DJ(K) \to BT^m \) with fibre \( \mathbb{Z}_K \) splits after looping: \( \Omega DJ(K) \simeq \Omega \mathbb{Z}_K \times T^m \). This is not an \( H \)-space splitting, and the exact sequence of Pontrjagin homology rings

\[ 0 \to H_*(\Omega \mathbb{Z}_K) \to H_*(\Omega DJ(K)) \to \Lambda[u_1, \ldots, u_m] \to 0 \]

does not split in general.

**Prop 4.** \( H_*(\Omega DJ(K), k) \cong \text{Ext}_{k[K]}(k, k) \)

*Idea of proof:* Use Adams' cobar construction and formality of \( DJ(K) \).
Prop 5. Suppose $K$ is flag. Then
\[ H_*(\Omega DJ(K), k) \cong T_k(u_1, \ldots, u_m)/(u_i^2 = 0, \quad u_iu_j + u_ju_i = 0 \text{ for } \{i, j\} \in K). \]

Idea of proof: Use Koszul duality for algebras.

Cor 6. If $K$ is flag then
\[ \pi_*(\Omega DJ(K)) \otimes \mathbb{Q} \cong FL(u_1, \ldots, u_m)/(\,[u_i, u_i] = 0, \quad [u_i, u_j] = 0 \text{ for } \{i, j\} \in K), \]
where $FL(\ )$ is a free Lie algebra and $\deg u_i = 1$.

Cor 7. If $K$ is flag, then the rational homology Poincaré series of $\Omega DJ(K)$ is given by
\[ F(H_*(\Omega DJ(K)); t) = \frac{(1 + t)^n}{1 - h_1t + \ldots + (-1)^nh_nt^n}. \]
5. Categories and colimits.

cat($K$): face category of $K$ (simplices and incl);
mc: a model category (e.g., top, tgp or dga);

$X \in \text{mc}$

$X^K: \text{cat}(K) \to \text{mc}$ exponential diagram; its value on $\sigma \subseteq \tau$ is the inclusion $X^\sigma \subseteq X^\tau$; $X^\emptyset = \text{pt}$.

Many previous constructions are colimits, e.g.,

$$DJ(K) = \text{colim}^{\text{top}} BT^K,$$

$$R_*[K] = \text{dual coalgebra of } R[K] = \text{colim}^{\text{dgc}} C(v)^K,$$

where $C(v)$ is the symmetric coalgebra on $v$, $\deg v = 2$.

**Cor 8. Assume $K$ is flag. Then**

$$\Omega DJ(K) \cong \text{colim}^{\text{tgp}} T^K;$$

$$H_*(\Omega DJ(K), \mathbb{Q}) \cong \text{colim}^{\text{ga}} \Lambda[u]^K;$$

$$\pi_*(\Omega DJ(K)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \text{colim}^{\text{gl}} CL(u)^K,$$

where $CL(u)$ is the commutative Lie algebra, $\deg u = 1$.

In general colimit models do not work! (Look at $K = \partial \Delta^2$, in which case $DJ(K)$ is not coformal.)
6. Homotopy colimit models.

Appropriate notions of homotopy colimits exist in the model categories tgp, tmon, dga, dgc and dgl.

**Thm 9.** (P.-Ray-Vogt) The loop space functor \( \Omega: \text{top} \to \text{tmon} \) commutes with the homotopy colimit, i.e., there is a weak equivalence

\[
\Omega \operatorname{hocolim}^{\text{top}} D \to \operatorname{hocolim}^{\text{tmon}} \Omega D
\]

for every diagram \( D: c \to \text{top} \).

For diagrams over \( \text{cat}(K) \) we get

**Thm 10.** (P.-Ray-Vogt) There is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega \operatorname{hocolim}^{\text{top}} BT^K & \xrightarrow{\overline{h}_K} & \operatorname{hocolim}^{\text{tgp}} T^K \\
\downarrow \Omega p_K & & \downarrow \\
\Omega DJ(K) & \xrightarrow{h_K} & \operatorname{colim}^{\text{tgp}} T^K
\end{array}
\]

in which \( \Omega p_K \) and \( \overline{h}_K \) are weak equivalences, while \( h_K \) is a weak equivalence only if \( K \) is flag.
There is a similar result in algebraic mc. The algebraic analogue of the loop functor is the cobar construction $\Omega_* : \mathsf{dgc} \rightarrow \mathsf{dga}$.

**Thm 11.** There is a htpy commutative diagram

\[
\begin{array}{ccc}
\Omega_* \mathsf{hocolim}^{\mathsf{dgc}} C(v)K & \xrightarrow{\bar{\eta}K} & \mathsf{hocolim}^{\mathsf{dga}} \Lambda[u]K \\
\downarrow \Omega_*\rho_K & & \downarrow \\
\Omega_* (\mathbb{Q}_*[K]) & \xrightarrow{\eta_K} & \mathsf{colim}^{\mathsf{dga}} \Lambda[u]K
\end{array}
\]

in which $\Omega_*\rho_K$ and $\bar{\eta}_K$ are weak equivalences, while $\eta_K$ is a weak equivalence only if $K$ is flag.

**Cor 12.**

\[
\begin{align*}
H_*(\Omega DJ(K); \mathbb{Q}) & \cong H\left(\mathsf{hocolim}^{\mathsf{dga}} \Lambda[u]K\right) \\
\pi_*(\Omega DJ(K)) \otimes_{\mathbb{Z}} \mathbb{Q} & \cong H\left(\mathsf{hocolim}^{dgl} CL(u)K\right)
\end{align*}
\]
Ex 13. Let $K$ be the 1-skeleton of a 3-simplex. A calculation using the previous results gives

$$H_*(\Omega DJ(K)) \cong T(u_1, u_2, u_3, u_4, w_{123}, w_{124}, w_{134}, w_{123})$$

(relations)

where $\deg w_{ijk} = 4$ and there are 3 types of relations:

(a) exterior algebra relations for $u_1, u_2, u_3, u_4$;

(b) $[u_i, w_{jkl}] = 0$ for $i \in \{j, k, l\}$;

(c) $[u_1, w_{234}] + [u_2, w_{134}] + [u_3, w_{124}] + [u_4, w_{123}] = 0$.

$w_{ijk}$ is the higher commutator (Hurewicz image of the higher Samelson product) of $u_i, u_j$ and $u_k$, so the last equation is a higher Jacobian identity.

