Spaces of polytopes, and cobordisms of toric manifolds

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joint work with Victor Buchstaber and Nigel Ray

1. Simple polytopes.

An arrangement of half-spaces is a collection $\mathcal{H}$ of subsets

$$S_i = \{x \in V : (a_i, x) - b_i \geqslant 0\}, \quad 1 \leqslant i \leqslant m,$$

where $a_i \in V \cong \mathbb{R}^n$ and $b_i \in \mathbb{R}$. When the intersection of the S_i is bounded, it forms a convex polytope P; otherwise, it is a polyhedron.

We may specify P by the matrix inequality $A_P x \geqslant b$.

Assume $\dim P = n$, and no redundant half-spaces. So P has m facets F_i, defined by its intersection with the bounding hyperplanes

$$H_i = \{x \in V\colon (a_i, x) = b_i\}.$$

When the bounding hyperplanes are in general position, every vertex is the intersection of precisely n facets, and P is simple.

The positive cone

$$\mathbb{R}^m_{\geqslant} = \{h \in \mathbb{R}^m\colon h_i \geqslant 0 \text{ for } i = 1, \ldots, m\}.$$

2. Spaces of polytopes.

Following Khovanskii, we fix A_P and identify the m-dimensional vector b with the arrangement $\mathcal{H}$, and hence with the polytope P. The coordinates b_i describe the signed distances of the hyperplanes H_i from the origin 0 in V, so long as the normal vectors a_i have length 1; otherwise, the distances have to be scaled accordingly. The sign is positive or negative as 0 lies in the interior or exterior of S_i respectively.

Every vector in $\mathbb{R}^m$ may then be identified with an analogous arrangement of halfspaces, obtained from $\mathcal{H}$ by parallel displacement of the S_i. Some such arrangements define polytopes, and others, dubbed virtual polytopes by Khovanskii, do not; in either case, we also describe the corresponding intersections as analogous. The zero vector is therefore identified with the virtual polytope $\{0\}$. An n-parameter family of examples is given by the translations of V, for which the corresponding polytopes are congruent to P.

In this context, we denote the m-dimensional vector space of polytopes analogous to P by $\mathbb{R}(P)$, and interpret the identification as an isomorphism $k\colon \mathbb{R}^m \to \mathbb{R}(P)$.

We may interpret the matrix A_P as a linear transformation $V \to \mathbb{R}^m$. Since the points of P are specified by the constraint $A_P x \geqslant b$, it follows that the intersection of the affine subspace $A_P(V) - b$ with the positive cone $\mathbb{R}^m_{\geqslant}$ is a copy of P in $\mathbb{R}^m$. Denote

$$i_P\colon V \to \mathbb{R}^m; \quad V \mapsto A_P(v) - b.$$

So $\chi = k \circ i_P$ restricts to an affine embedding $P \to \mathbb{R}(P)$, for which $\chi(x)$ is the polytope congruent to P with origin at x, for all $x \in P$. In particular, $\chi(P)$ is a submanifold of the positive cone $\mathbb{R}(P)_{\geqslant}$.

3. Toric manifolds.

A toric manifold (cf. Davis and Januszkiewicz) $M = M^{2n}$ over a simple polytope $P = P^n$ has

- an action of an n-dimensional torus $\mathbf{T}$ that locally looks like the standard $\mathbf{T}$-action on $\mathbb{C}^n$;

- the orbit map $\pi\colon M \to P$ sending every set of orbits with the same isotropy group onto the interior of a face of P.

The notion of a toric manifolds is a purely topological approximation to smooth compact toric varieties from algebraic geometry.

M has m codimension-two characteristic submanifolds M_i, $i = 1, \ldots, m$, which can be defined either as the inverse images $\pi^{-1}(F_i)$ of the facets F_i of P, or as connected submanifolds fixed pointwise by a certain circle subgroup of $\mathbf{T}$.

We denote by ρ_i the canonical orientable 2-dimensional real bundle over M determined by M_i; it restricts to the normal bundle $\nu(M_i \hookrightarrow M)$. An omniorientation of M consists of a choice of orientation for each ρ_i; we also always assume here that M itself is oriented.

A choice of orientation of ρ_i identifies it as a complex line bundle. The one-dimensional isotropy subgroup $\mathbf{T}_{M_i}$ of M_i acts in the fibres of the normal bundle $\nu(M_i \hookrightarrow M)$. We orient the circle $\mathbf{T}_{M_i}$ in such way that this action preserves the orientation determined by the complex structure in ρ_i. Thereby we obtain a map

$$\lambda\colon T^m \to \mathbf{T}, \quad T^{F_i} \mapsto \mathbf{T}_{M_i},$$

called the characteristic map of M. Due to a nonsingularity condition the kernel $K(\lambda)$ of λ is isomorphic to a $(m-n)$-dimensional torus.

Thm 1 (Davis–Januszkiewicz). *There is an isomorphism of real $2m$-plane bundles:*

$$\tau(M) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \ldots \oplus \rho_m.$$

Idea of proof. Consider the pullback diagram

$$\begin{array}{ccc}
\mathcal{Z}_P = T^m \times P/\sim & \xrightarrow{i_Z} & T^m \times \mathbb{R}^m_{\geqslant}/\sim = \mathbb{C}^m \\
\big\downarrow & & \big\downarrow \rho \\
P & \xrightarrow{i_P} & \mathbb{R}^m.
\end{array},$$

where $\rho(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. It determines a canonically framed manifold $\mathcal{Z}_P$.

The map $\lambda\colon T^m \to \mathbf{T}$ induces a principal $K(\lambda)$-fibration $p\colon \mathcal{Z}_P = T^m \times P/\sim \longrightarrow \mathbf{T} \times P/\approx = M$. The tangent bundle to $\mathcal{Z}_P$ decomposes as

$$\tau(\mathcal{Z}_P) = p^*\tau(M) \oplus \xi(p)$$

where $\xi(p)$ is the tangent bundle along the fibres of p. The required bundle isomorphism comes from Szczarba's identification

$$\tau(M) \oplus (\xi(p)/K(\lambda)) \oplus (\nu(i_Z)/K(\lambda)) \cong \rho_1 \oplus \ldots \oplus \rho_m$$

by noticing that both $\xi(p)/K(\lambda)$ and $\nu(i_Z)/K(\lambda)$ are trivial real $(m-n)$-plane bundles over M. $\square$

4. Stably almost complex structures.

Thm 2. *A choice of omniorientation of M, ordering of facets, and initial vertex of P gives rise to a canonical framing of the real $2(m-n)$-bundle $\nu(i_Z)/K(\lambda) \oplus \xi(p)/K(\lambda)$ over M, thereby determining a canonical stably complex structure for M.*

Prop 3. *The equivalence class of the stably complex structure on M defined in Theorem 2, and therefore the corresponding complex cobordism class, depends on only on a choice of orientations for M and for each normal bundle $\nu(M_i \hookrightarrow M)$.*

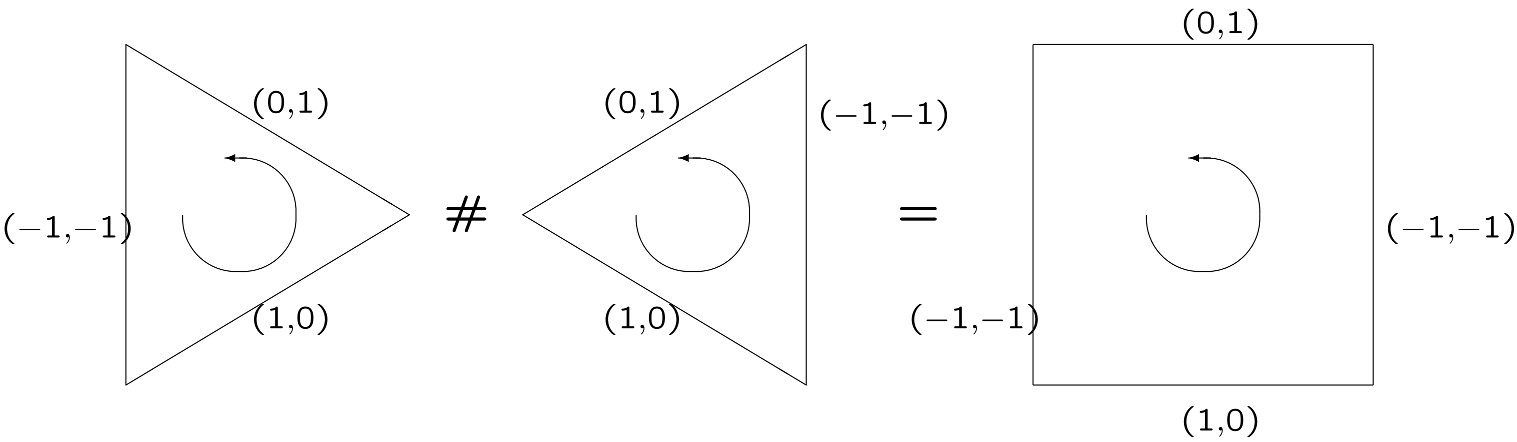
5. Complex cobordisms.

Thm 4 (Buchstaber–Ray'01, corrected by Buchstaber–Ray–P.). *In dimensions > 2, every complex cobordism class contains a toric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

Idea of proof. Start with the additive basis in the complex cobordism ring consisting of toric manifolds, constructed by Buchstaber and Ray in 1999. Then one needs to replace the disjoint union (representing the cobordism sum) by something connected.

Given two cobordism classes represented by $2n$-dimensional omnioriented toric manifolds M_1 and M_2, with quotient polytopes P_1 and P_2 respectively, we need to construct a third such manifold M, with quotient polytope P, representing the sum of the cobordism classes of M_1 and M_2. This is done using the connected sum construction. $\square$

Ex 5. Connected sum of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$. The resulting stably complex structure on the manifold $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ bounds, that is, represents the zero cobordism class.



However, it is not possible to take the connected sum of two copies of $\mathbb{C}P^2$ with the standard omniorientation by a procedure like this. A modification is needed here.

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[3] Victor M Buchstaber and Nigel Ray. Tangential structures on toric manifolds, and connected sums of polytopes. *Internat. Math. Res. Notices,* 4:193–219, 2001.

[4] Michael W Davis and Tadeusz Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.,* 62(2):417–451, 1991.

[5] Alexandr V Pukhlikov and Askold G Khovanskii. Finitely additive measures of virtual polyhedra (Russian). *Algebra i Analiz,* 4(2): 161–185, 1992.