Toric topology

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1. Simplicial complexes and face rings.

K: *simplicial complex* on $V = \{v_1, \ldots, v_m\}$.

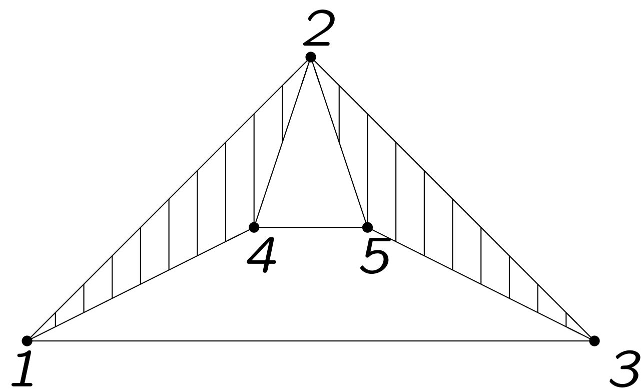
$\sigma \in K$ a *simplex*.

$R[v_1, \ldots, v_m]$: polynomial algebra on V over R, $\deg v_i = 2$. Given $\omega \subseteq V$, set $v_\omega := \prod_{i \in \omega} v_i$. The *Stanley-Reisner algebra* (or *face ring*) of K is

$$R[K] := R[v_1, \ldots, v_m]/(v_\omega \colon \omega \notin K).$$

Ex 1

K:



$R[K] = R[v_1, \ldots, v_5]/(v_1v_5, v_3v_4, v_1v_2v_3, v_2v_4v_5)$.

2. Moment-angle complexes.

$D^2 \subset \mathbb{C}$: the unit disk.

$$B_\omega := \{(z_1, \dots, z_m) \in (D^2)^m \colon |z_i| = 1 \text{ for } v_i \notin \omega\}.$$

The *moment-angle complex*

$$\mathcal{Z}_K := \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^m.$$

Prop 2 *$\mathcal{Z}_K$ is a T^m-space, with quotient* cone K'*:*

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & (D^2)^m \\ \downarrow & & \downarrow \\ \text{cone}\, K' & \longrightarrow & I^m \end{array},$$

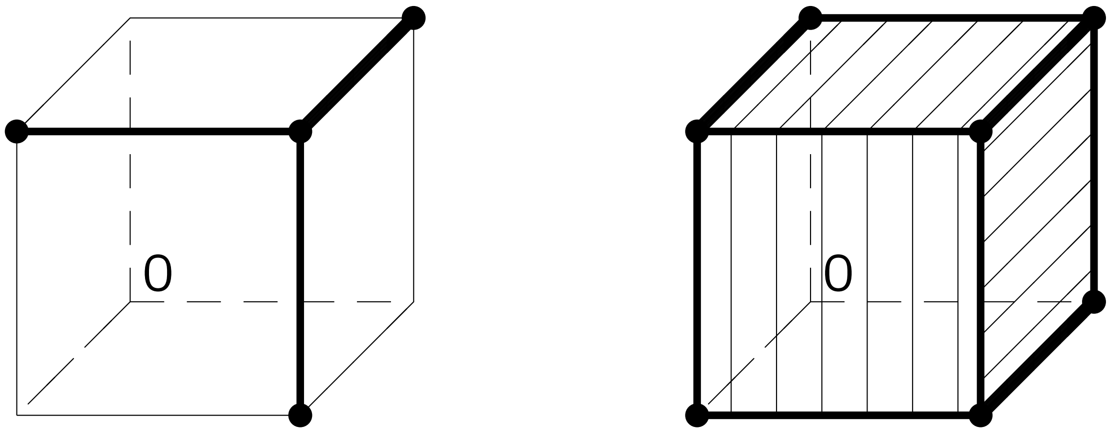
where K' is the barycentric subdivision of K,

$$\sigma = \{v_{i_1}, \dots, v_{i_k}\} \mapsto e_\sigma = (\varepsilon_1, \dots, \varepsilon_m),$$

where $\varepsilon_i = 0$ if $v_i \in \sigma$ and $\varepsilon_i = 1$ if $v_i \notin \sigma$.

$$\varnothing \mapsto e_\varnothing = (1, \dots, 1).$$

Ex 3 Embedding cone $K' \hookrightarrow I^m$:



$K = 3$ points $\qquad\qquad$ $K = \partial\Delta^2$

Prop 4 *(a) Let $|K| \cong S^{n-1}$ (a sphere triangulation on m vertices). Then $\mathcal{Z}_K$ is an $(m+n)$-manifold;*
(b) Let K be a triangulated manifold. Then $\mathcal{Z}_K \backslash \mathcal{Z}_\varnothing$ is an open manifold, where $\mathcal{Z}_\varnothing = T^m$.

Ex 5 $\mathcal{Z}_{\partial\Delta^n} \cong S^{2n+1}$. For $n = 1$,

$$S^3 = D^2 \times S^1 \cup S^1 \times D^2 \subset D^2 \times D^2.$$

3. Other important toric spaces.

(a) Original Davis-Januszkiewicz construction:

$$\mathcal{Z}_K \cong T^m \times |\operatorname{cone} K| \big/ \!\sim,$$

where the equivalence relation $\sim$ is defined via the *dual cell decomposition* of $|K|$:

The *facets* of $|K|$ are of the form

$$F_i := \operatorname{star}_{K'} v_i.$$

Given $x \in |\operatorname{cone} K|$, set

$$T(x) := \{(t_1, \ldots, t_m) \in T^m \colon t_i = 1 \text{ if } x \notin F_i\}.$$

Then set $(t, x) \sim (s, x)$ if $t^{-1}s \in T(x)$.

Important particular case: $\operatorname{cone} K = P^n$, a *simple convex polytope*.

(b) Homotopy fibre construction:

The *Davis-Januszkiewicz space* is the Borel construction

$$\mathit{DJ}(K) := ET^m \times_{T^m} \mathcal{Z}_K = ET^m \times \mathcal{Z}_K/\sim,$$

where $(e,z) \sim (et^{-1}, tz)$.

Prop 6 *There is a canonical homotopy eqce*

$$\mathit{DJ}(K) \xrightarrow{\simeq} \bigcup_{\sigma \in K} BT^\sigma \subseteq BT^m = (\mathbb{C}P^\infty)^m,$$

therefore, DJ(K) may be regarded as a canonical cellular subcomplex in the product $(\mathbb{C}P^\infty)^m$.

Cor 7 *(a)* $\mathcal{Z}_K \simeq \text{hofibre}\Big(\bigcup_{\sigma \in K} BT^\sigma \hookrightarrow BT^m\Big)$;

(b) $H^*(\mathit{DJ}(K); R) \cong H^*_{T^m}(\mathcal{Z}_K; R) \cong R[K]$.

(c) Subspace arrangement complements:

A *coordinate subspace* in $\mathbb{C}^m$ is given by

$$L_\omega = \{(z_1,\dots,z_m) \in \mathbb{C}^m \colon z_{i_1} = \dots = z_{i_k} = 0\},$$

where $\omega = \{i_1,\dots,i_k\}$. *Coordinate subspace arrangements* are parametrised by simplicial complexes K on m vertices. Set

$$U(K) = \mathbb{C}^m \setminus \bigcup_{\omega \notin K} L_\omega,$$

a *coordinate subspace arrangement complement*.

Prop 8 *There is a T^m-equivariant deformation retraction*

$$U(K) \xrightarrow{\simeq} \mathcal{Z}_K.$$

Pf Write

$$U(K) = \bigcup_{\sigma \in K} U_\sigma, \quad \mathcal{Z}_K = \bigcup_{\sigma \in K} B_\sigma,$$

where

$$U_\sigma := \{(z_1,\dots,z_m) \in \mathbb{C}^m \colon z_i \neq 0 \text{ for } i \notin \sigma\}.$$

Then there are obvious homotopy equivalences

$$\mathbb{C}^\sigma \times (\mathbb{C} \setminus 0)^{V \setminus \sigma} \cong U_\sigma \xrightarrow{\simeq} B_\sigma \cong (D^2)^\sigma \times (S^1)^{V \setminus \sigma}.$$

Ex 9 1. Let $K = \partial\Delta^{m-1}$. Then $U(K) = \mathbb{C}^m \setminus 0$.

2. Let $K = \{v_1, \ldots, v_m\}$ (m points). Then

$$U(K) = \mathbb{C}^m \setminus \bigcup_{1\leqslant i<j\leqslant m} \{z_i = z_j = 0\},$$

the complement to the set of all codim 2 coordinate planes.

3. More generally, if K is the i-skeleton of Δ^{m-1}, then $U(K)$ is the complement to the set of all coordinate planes of codimension $(i + 2)$.

(d) (Quasi)toric manifolds:

Define $s = s(K)$ as the maximal dimension for which there exists a subgroup

$$T^s \subset T^m$$

acting freely on $\mathcal{Z}_K$.

The number $s(K)$ is a combinatorial invariant of K. We have

$$1 \leqslant s(K) \leqslant m - n.$$

Let K be a simplicial m-vertex subdivision of S^{n-1} (e.g., a boundary of a *simplicial polytope*) and assume $s(K) = m - n$. Then the quotient

$$M^{2n} := \mathcal{Z}_K / T^{m-n}$$

is called a *quasitoric manifold*.

All compact nonsingular toric varieties are quasitoric manifolds. Quotients of $\mathcal{Z}_K$ by *almost free* torus actions produce *toric orbifolds*.

4. From combinatorial to toric topology.

Let K_1, K_2 be simplicial complexes on vertex sets

$$V = \{v_1, \ldots, v_{m_1}\} \text{ and } W = \{w_1, \ldots, w_{m_2}\}$$

respectively. A *simplicial map* $\varphi\colon K_1 \to K_2$ is induced by a vertices map $\varphi\colon V \to W$ such that

$$\varphi(\sigma) \in K_2 \text{ for all } \sigma \in K_1.$$

Such φ induces a map

$$\psi\colon (D^2)^{m_1} \to (D^2)^{m_2},$$
$$(z_1, \ldots, z_{m_1}) \mapsto (y_1, \ldots, y_{m_2})$$

where

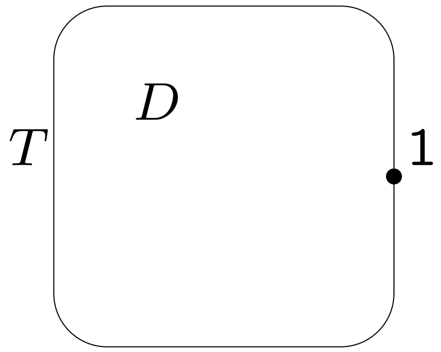
$$y_j = \begin{cases} 1 & \text{if } \varphi^{-1}(w_j) = \varnothing, \\ \prod_{v_i \in \varphi^{-1}(w_j)} z_i & \text{otherwise.} \end{cases}$$

It restricts to a map

$$\varphi\colon \mathcal{Z}_{K_1} \to \mathcal{Z}_{K_2}.$$

Thus, the correspondence $K \mapsto \mathcal{Z}_K$ gives rise to a functor from simplicial complexes and maps to spaces with torus actions and equivariant maps.

Bigraded cell decomposition of D^2:

cells 

$[1]$, $[T]$, $[D]$ — cellular chain generators

Set the *bidegree* (bdg) of the generators by

$$\text{bdg}[1] = (0,0), \quad \text{bdg}[T] = (-1,2), \quad \text{bdg}[D] = (0,2);$$
$$\partial[1] = 0, \qquad \partial[T] = 0, \qquad \partial[D] = [T].$$

Then

$$C_*((D^2)^m;\partial) = \bigotimes_{i=1}^{m} C_*(D^2;\partial),$$

and $\mathcal{Z}_K \subset (D^2)^m$ is a cellular subcomplex!

Thus, the cellular chains $C_*(\mathcal{Z}_K)$ are defined.

The functor $K \mapsto \mathcal{Z}_K$ induces a homomorphism between the standard simplicial chain complex of K and the bigraded cellular chain complex of $\mathcal{Z}_K$.

5. Cellular cochain algebras.

The map $\widetilde{\Delta}\colon D^2 \to D^2 \times D^2$ given by

$$\rho e^{i\varphi} \mapsto \begin{cases} \bigl(1+\rho(e^{2i\varphi}-1),1\bigr) & \text{for } \varphi \in [0,\pi], \\ \bigl(1,1+\rho(e^{2i\varphi}-1)\bigr) & \text{for } \varphi \in [\pi,2\pi), \end{cases}$$

is a cellular map sending ∂D^2 to $\partial D^2 \times \partial D^2$ and homotopic to the diagonal $\Delta\colon D^2 \to D^2 \times D^2$ in the class of such maps. Therefore, it gives rise to a canonical *cellular diagonal approximation*

$$\widetilde{\Delta}\colon \mathcal{Z}_K \to \mathcal{Z}_K \times \mathcal{Z}_K.$$

Thm 10 *The bigraded cellular cochain algebra* $C^*(\mathcal{Z}_K;R)$ *is given by*

$$C^*(\mathcal{Z}_K;R) = \Lambda[u_1,\ldots,u_m] \otimes R[K]\big/(u_iv_i = v_i^2 = 0),$$

where $u_i = [T_i]^*$, $v_i = [D_i]^*$ *are the dual cochain generators of bideg* $(-1,2)$ *and* $(0,2)$ *resp.*

6. Cohomology of toric spaces.

Thm 11 *There is an isomorphism of bigraded algebras*

$$\begin{aligned} H^*(\mathcal{Z}_K;\mathbb{Z}) &\cong \mathrm{Tor}^{*,*}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K],\mathbb{Z}) \\ &\cong H\Big[\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[K];d\Big], \end{aligned}$$

where $du_i = v_i$, $dv_i = 0$. In particular,

$$H^p(\mathcal{Z}_K) \cong \sum_{-i+2j=p} \mathrm{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K],\mathbb{Z}).$$

Two ways of proving:

(a) Use the Eilenberg–Moore spectral sequence of the fibration

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & ET^m \\ \big\downarrow & & \big\downarrow \\ DJ(K) & \longrightarrow & BT^m \end{array}.$$

This gives the Tor part of the answer.

(b) Use the above calculations with cellular cochains (note: $(u_iv_i, v_i^2; i=1,\ldots,m)$ is an acylic ideal).

(a) and (b) are related by the Koszul complex!

Ex 12 1. $K = \partial\Delta^{m-1}$. Then

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \dots, v_m]/(v_1 \cdots v_m).$$

The fundamental class of $\mathcal{Z}_K = S^{2m-1}$ is represented by the bideg $(-1, 2m)$ cocycle

$$u_1 v_2 v_3 \cdots v_m \in \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K].$$

2. Let K be the boundary of a 5-gon. Then

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \dots, v_5]/(v_1v_3, v_2v_4, v_3v_5, v_4v_1, v_5v_2).$$

$H^3(\mathcal{Z}_K) = H^{-1,4}(\mathcal{Z}_K)$ has 5 generators

$$u_i v_{i+2} \in \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K], \quad i = 1, \dots, 5.$$

$H^4(\mathcal{Z}_K) = H^{-2,6}(\mathcal{Z}_K)$ has 5 generators

$$u_i u_{i+1} v_{i+3}, \quad i = 1, \dots, 5.$$

$H^7(\mathcal{Z}_K) = H^{-3,10}(\mathcal{Z}_K)$ is generated by $u_1u_2u_3v_4v_5$. Thus, $\mathcal{Z}_K$ is a 7-manifold with Betti vector

$$(1, 0, 0, 5, 5, 0, 0, 1).$$

Similarly, for K the boundary of an m-gon

$$\dim H^*(\mathcal{Z}_K) = (m-4)2^{m-2} + 4.$$

3. Let $K = \{v_1, \ldots, v_m\}$ (m points). Then

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_i v_j,\ i \neq j).$$

Cocycles in $\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K]$:

$$v_{i_1} u_{i_2} u_{i_3} \cdots u_{i_k}, \quad k \geqslant 2 \text{ and } i_p \neq i_q \text{ for } p \neq q.$$

Coboundaries:

$$d(u_{i_1} \cdots u_{i_k}).$$

Therefore,

$$\begin{aligned}
&\dim H^0\Big(U(K)\Big) = 1,\\
&\dim H^1\Big(U(K)\Big) = H^2\Big(U(K)\Big) = 0,\\
&\dim H^{k+1}\Big(U(K)\Big) = m\tbinom{m-1}{k-1} - \tbinom{m}{k} = (k-1)\tbinom{m}{k},\\
&\hfill 2 \leqslant k \leqslant m,
\end{aligned}$$

and the multiplication in the cohomology is trivial.

7. Combinatorial homological algebra.

Let $\omega = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V$, and K_ω the *full subcomplex* of K. Then there is a canonical retraction

$$\mathcal{Z}_{K_\omega} \stackrel{i}{\hookrightarrow} \mathcal{Z}_K \stackrel{p}{\longrightarrow} \mathcal{Z}_{K_\omega},$$

where i is induced by $K_\omega \subseteq K$ and p is the projection induced by the contraction $\operatorname{cone} K \to \operatorname{cone} K_\omega$ sending the extra vertices to the vertex of the cone.

Moreover, given a subcomplex $L \subseteq K$, the subcomplex $\mathcal{Z}_L$ retracts off $\mathcal{Z}_K$ if and only if L is full.

Put $C^{*,*}(\mathcal{Z}_K) = \bigoplus_{\omega \subseteq V} C^{*,2\omega}(\mathcal{Z}_K)$ (multigrading), where $C^{*,2\omega}(\mathcal{Z}_K)$ is generated by $u_{\omega \setminus \sigma} v_\sigma$, $\sigma \subseteq \omega$, $\sigma \in K$. Then

$$H^{-i,2j}(\mathcal{Z}_K) = \bigoplus_{\omega \subseteq V \colon |\omega| = j} H^{-i,2\omega}(\mathcal{Z}_K),$$

where $H^{-i,2\omega}(\mathcal{Z}_K) = H[C^{-i,2\omega}(\mathcal{Z}_K)]$.

Given complexes K_1 on V and K_2 on W, define their *join* as the following complex on $V \sqcup W$:

$K_1 * K_2 = \{\omega \subseteq V \sqcup W : \omega = \sigma_1 \cup \sigma_2,\ \sigma_1 \in K_1,\ \sigma_2 \in K_2\}$.

We introduce a multiplication on

$$\bigoplus_{\substack{p \geqslant -1 \\ \omega \subseteq V}} \widetilde{H}^p(K_\omega)$$

(here $\widetilde{H}^{-1}(\varnothing) = \mathbb{Z}$).

Take $a \in \widetilde{H}^p(K_{\omega_1}),\ b \in \widetilde{H}^q(K_{\omega_2})$.

Assume $\omega_1 \cap \omega_2 = \varnothing$. Then we get

$$i\colon \quad K_{\omega_1 \sqcup \omega_2} = K_{\omega_1} \sqcup K_{\omega_2} \hookrightarrow K_{\omega_1} * K_{\omega_2},$$

$$f\colon \quad \widetilde{C}^p(K_{\omega_1}) \otimes \widetilde{C}^q(K_{\omega_2}) \xrightarrow{\cong} \widetilde{C}^{p+q+1}(K_{\omega_1} * K_{\omega_2}).$$

Now define

$$a \cdot b = \begin{cases} 0, & \text{if } \ \omega_1 \cap \omega_2 \neq \varnothing, \\ i^* f(a \otimes b) \in \widetilde{H}^{p+q+1}(K_{\omega_1 \sqcup \omega_2}), & \omega_1 \cap \omega_2 = \varnothing. \end{cases}$$

Thm 13 (Baskakov, 2003) *For all p and $\omega \subseteq V$ there are isomorphisms*

$$\widetilde{H}^p(K_\omega) \xrightarrow{\cong} H^{p+1-|\omega|,2\omega}(\mathcal{Z}_K),$$

inducing a ring isomorphism

$$\gamma\colon \bigoplus_{\substack{p\geqslant -1\\ \omega\subseteq V}} \widetilde{H}^p(K_\omega) \xrightarrow{\cong} H^{*,*}(\mathcal{Z}_K).$$

Cor 14

$$H^{-i,2j}(\mathcal{Z}_K) = \bigoplus_{\omega\subseteq V\colon |\omega|=j} \widetilde{H}^{j-i-1}(K_\omega).$$

Cor 15 (Hochster, 1975)

$$\mathrm{Tor}^{-i,*}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k}) \cong \bigoplus_{\omega\subseteq V} \widetilde{H}^{|\omega|-i-1}(K_\omega;\mathbf{k}).$$

(additively and with field coefs).

8. Massey products and formality.

Given $\sigma \in K$, the *stellar subdivision* of K at σ is

$$\widehat{K} = \zeta_\sigma(K) = \Big(K \setminus \mathrm{star}_K\,\sigma\Big) \cup \Big(\mathrm{cone}\,\partial\,\mathrm{star}_K\,\sigma\Big).$$

Let K_i be a triangulation of S^{n_i-1} with $|V_i| = m_i$ vertices, $i = 1, 2, 3$.

Put $m = m_1 + m_2 + m_3$, $n = n_1 + n_2 + n_3$,

$$\begin{aligned} K^{n-1} &= K_1^{n_1-1} * K_2^{n_2-1} * K_3^{n_3-1}, \\ \mathcal{Z}_K &= \mathcal{Z}_{K_1} \times \mathcal{Z}_{K_2} \times \mathcal{Z}_{K_3}. \end{aligned}$$

Choose maximal simplices

$$\sigma_1 \in K_1, \quad \sigma_2', \sigma_2'' \in K_2,\ \sigma_2' \cap \sigma_2'' = \varnothing, \quad \sigma_3 \in K_3.$$

Set $\widehat{\widehat{K}} = \zeta_{\sigma_1 \cup \sigma_2'}\Big(\zeta_{\sigma_2'' \cup \sigma_3}(K)\Big)$. Then $\widehat{\widehat{K}}$ is a triangulation of S^{n-1} with $m+2$ vertices.

Consider the generators

$$a_i \in \widetilde{H}^{n_i-1}(\widehat{\widehat{K}}_{V_i}),$$

and set

$$b_i = \gamma(a_i) \in H^{n_i-m_i, 2m_i}(\mathcal{Z}_{\widehat{\widehat{K}}}) \subset H^{m_i+n_i}(\mathcal{Z}_{\widehat{\widehat{K}}}).$$

Then

$$\begin{aligned} a_1 a_2 &\in \widetilde{H}^{n_1+n_2-1}\Big(\widehat{\widehat{K}}_{V_1 \sqcup V_2}\Big) \\ &\qquad\qquad \cong \widetilde{H}^{n_1+n_2-1}(S^{n_1+n_2-1} \setminus \mathrm{pt}) = 0, \\ b_1 b_2 &= \gamma(a_1 a_2) = 0. \end{aligned}$$

Similarly, $b_2 b_3 = 0$. Therefore the Massey product $\langle b_1, b_2, b_3 \rangle \in H^{m+n-1}(\mathcal{Z}_{\widehat{\widehat{K}}})$ is defined.

Thm 16 *The cohomology of the $(m+n+2)$-manifold $\mathcal{Z}_{\widehat{\widehat{K}}}$ has non-trivial Massey products (e.g., $\langle b_1, b_2, b_3 \rangle$).*

Cor 17 *We get a family of non-formal 2-connected manifolds.*

The dual homology class $D\langle b_1, b_2, b_3 \rangle \in H_3(\mathcal{Z}_{\widehat{\widehat{K}}})$ is represented by the embedding $S^3 \subset \mathcal{Z}_{\widehat{\widehat{K}}}$ corresponding to the pair of vertices added to $K = K_1 * K_2 * K_3$ under the stellar subdivisions.

9. Toral rank conjecture.

A T^k-action on X is *almost free* if all isotropy subgroups are finite. The *toral rank* of X, denoted $\mathrm{trk}(X)$, is the largest k for which there exist a almost free T^k-action on X.

Prop 18 *If K is an $(n-1)$-dim complex on m vertices, then* $\mathrm{trk}\,\mathcal{Z}_K \geqslant m-n$.

In 1985 S. Halperin conjectured that

$$\dim H^*(X;\mathbb{Q}) \geqslant 2^{\mathrm{trk}(X)}$$

for any finite dimensional space X.

Cor 19 *Assuming that the toral rank conjecture is true, we come to the following inequality:*

$$\dim \bigoplus_{\omega\subseteq[m]} \widetilde{H}^*(K_\omega;\mathbb{Q}) \geqslant 2^{m-n}.$$

Ex 20 *K the boundary of an m-gon. Then*

$$\dim H^*(\mathcal{Z}_K) = (m-4)2^{m-2}+4 \geqslant 2^{m-2}.$$

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