Rational aspects of toric topology

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1. Motivations and definitions.

Object of study: T^n-manifolds M^{2n} and their orbit quotients.

Notation: $T := T^n$, $M := M^{2n}$, $Q := M/T$.

Particular examples:

- Non-singular compact toric varieties
  T-action is a part of algebraic $\mathbb{C}^{*n}$-action with a dense orbit;

- (Quasi)toric manifolds of Davis-Januszkiewicz
  "locally standard" (i.e., locally look like T^n acting on $\mathbb{C}^n$) and Q combinatorially is a simple polytope;

- Torus manifolds of Hattori-Masuda.

K a simplicial complex on $V = \{v_1, \dots, v_m\}$ (e.g., K is the dual to the boundary of Q).

$S(V)$ a symmetric algebra on V over a ring R, $\deg v_i = 2$. Given $\omega \subseteq V$, set $v_\omega := \prod_{i\in\omega} v_i$. The *Stanley-Reisner algebra* (or *face ring*) of K is

$$R^*[K] := S(V)/(v_\omega \colon \omega \notin K).$$

The *Davis-Januszkiewicz space*

$$\mathit{DJ}(K) := \bigcup_{\sigma\in K} BT^\sigma \subseteq BT^m = (\mathbb{C}P^\infty)^m.$$

Properties:

- $\mathit{DJ}(K) \simeq ET \times_T M$ for $K = (\partial Q)^*$;

- $H^*(\mathit{DJ}(K); R) \cong H_T^*(M; R) \cong R^*[K]$.

Define

$$\mathcal{Z}_K := \mathrm{hofibre}\Big(\mathit{DJ}(K) \hookrightarrow BT^m\Big).$$

Thus, we have two homotopy pullback diagrams

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & ET^m \\ \downarrow & & \downarrow \\ DJ(K) & \longrightarrow & BT^m \end{array} \quad \text{and} \quad \begin{array}{ccc} M^{2n} & \longrightarrow & ET^n \\ \downarrow & & \downarrow \\ DJ(K) & \longrightarrow & BT^n \end{array}$$

The map $DJ(K) \to BT^n$ is determined by a choice of a regular sequence in the Cohen-Macaulay algebra $\mathbb{Z}[K] = H^*(DJ(K))$.

Overal aim: Relate

- Topology of M, $\mathcal{Z}_K$, $DJ(K)$ and their loop spaces;

- Combinatorics of Q, K;

- Commutative and homological algebra of $\mathbb{Q}[K]$

through rational homotopy theory

2. Rational homotopy theory.

Sullivan's framework: *Piecewise polynomial differential forms functor* A^*: top $\to$ cdga together with a natural isomorphism

$$H(A^*(X)) \xrightarrow{\cong} H^*(X, \mathbb{Q})$$

for any $X \in$ top. The algebra $A^*(X)$ may be thought of as a commutative replacement for the singular $\mathbb{Q}$-cochains.

A space X is *formal* if $A^*(X)$ is a formal dga, i.e. if there is a weak equivalence $A^*(X) \simeq H^*(X)$. In particular, X is formal if there is a multiplicative "choice of a representative" map $H^*(X, \mathbb{Q}) \to A^*(X)$.

Quillen's framework: Quillen's approach is dual (in the Eckmann-Hilton sense) to Sullivan's. The rational homotopy groups $\pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$ of a space X form a graded Lie algebra, called the *rational homotopy Lie algebra of* X, with respect to the *Samelson bracket*. There is a Quillen functor

$$Q_* \colon \mathsf{top}_1 \to \mathsf{dgl}_0$$

from pointed simply connected spaces to connected differential graded Lie algebras, with a natural isomorphism

$$H(Q_*(X)) \xrightarrow{\cong} \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for any $X \in \mathsf{top}_1$.

A space X is called *coformal* if $Q_*(X)$ is a coformal differential graded Lie algebra. In particular, X is coformal if there is a weak equivalence $Q_*(X) \to \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

3. Cohomology of $DJ(K)$, $\mathcal{Z}_K$ and M.

According to a result of Notbohm-Ray, there is a commutative diagram

$$\begin{array}{ccc} H^*(BT^m) = S(V) & \longrightarrow & A^*(BT^m) \\ \downarrow & & \downarrow \\ H^*(DJ(K)) = \mathbb{Q}[K] & \longrightarrow & A^*(DJ(K)) \end{array}$$

in which the horizontal arrows are weak equivalences. In particular, $DJ(K)$ is formal.

Applying $A^*(\ )$ to the pullback square defining $\mathcal{Z}_K$, we get

$$\begin{array}{ccc} A^*(\mathcal{Z}_K) & \longleftarrow & A^*(ET^m) \\ \uparrow & & \uparrow \\ A^*(DJ(K)) & \longleftarrow & A^*(BT^m) \end{array} .$$

This may fail to be a pushout square in cdga.

Nevertheless, an Eilenberg-Moore type result implies that the induced map

$$B\Big(A^*(DJ(K)), A^*(BT^m), A^*(ET^m)\Big) \to A^*(\mathcal{Z}_K)$$

is a quism. Consider the free extension diagram

$$\begin{array}{ccc} \Lambda(U)\otimes\mathbb{Q}[K] & \longleftarrow & \Lambda(U)\otimes S(V) \\ \Big\uparrow & & \Big\uparrow \\ \mathbb{Q}[K] & \longleftarrow & S(V) \end{array},$$

where $\Lambda(U) = \Lambda[u_1, \ldots, u_m]$, $\deg u_i = 1$, and the dga structure in $\Lambda(U)\otimes\mathbb{Q}[K]$ and $\Lambda(U)\otimes S(V)$ is defined by $dv_i = 0$, $du_i = v_i$.

Thm 1 *The free extension $\Lambda(U)\otimes\mathbb{Q}[K]$ of the Stanley-Reisner ring $\mathbb{Q}[K]$ is weakly equivalent to $A^*(\mathcal{Z}_K)$. Hence, there are isomorphisms of (bi)graded algebras*

$$H^*(\mathcal{Z}_K;\mathbb{Q}) \cong \mathrm{Tor}_{S(V)}(\mathbb{Q}[K],\mathbb{Q}) \cong H(\Lambda(U)\otimes\mathbb{Q}[K]).$$

The statement is also true with $\mathbb{Z}$ coefficients, although the proof uses different techniques. It has also been proven by M. Franz in his thesis in a slightly different context of non-compact non-singular toric varieties.

The following is a generalisation of an argument due to Bousfield-Gugenheim.

Prop 2 *Let* B *be a simply connected space and* $t_1, \ldots, t_n$ *a sequence of elements in* $H^2(B;\mathbb{Z})$. *Then we have a pullback diagram of fibre bundles*

$$\begin{array}{ccc} E & \longrightarrow & ET^n \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & BT^n \end{array}.$$

Assume that B *is formal and* $H^*(B;\mathbb{Q})$ *is free as a* $S(t_1, \ldots, t_n)$*-module. Then* E *is also formal and*

$$H^*(E;\mathbb{Q}) \cong H^*(B;\mathbb{Q})/(t_1, \ldots, t_n).$$

Cor 3 *All toric manifolds are formal.*

The same argument works also in a more general case of *torus manifolds* over *homology polytopes*. Even more generally, torus manifolds M with $H^{odd}(M) = 0$ are also formal. In this case the face ring $\mathbb{Z}[K]$ has to be replaced by the face ring $\mathbb{Z}[\mathcal{S}]$ of an appropriate *simplicial poset* $\mathcal{S}$ determined by M. This face ring still admits an lsop $t_1, \ldots t_n$ and is free as a $\mathbb{Z}[t_1, \ldots, t_n]$-module.

4. Flag complexes and loop spaces

A *missing face* of K is a subset $\omega \subseteq V$ s.t. $\omega \notin K$, but every proper subset of ω is a simplex. K is a *flag complex* if any of its missing faces has two vertices. In this case

$$R[K] = T(V)\Big/\Big(v_iv_j - v_jv_i = 0 \text{ for } \{i,j\} \in K, \\ v_iv_j = 0 \text{ for } \{i,j\} \notin K\Big),$$

a *quadratic* algebra.

The fibration $DJ(K) \to BT^m$ with fibre $\mathcal{Z}_K$ splits after looping: $\Omega\, DJ(K) \simeq \Omega\mathcal{Z}_K \times T^m$. This is not an H-space, and the exact sequence of Pontrjagin homology rings

$$0 \to H_*(\Omega\mathcal{Z}_K) \to H_*(\Omega\, DJ(K)) \to \Lambda(U) \to 0$$

does not split in general.

Thm 4 *Suppose K is flag and $\mathbf{k}$ a field. Then*

$$H_*(\Omega\, DJ(K)) \cong T_{\mathbf{k}}(U)\Big/\Big(u_i^2 = 0,\\ u_iu_j + u_ju_i = 0 \text{ for } \{i,j\} \in K\Big).$$

Idea of proof. Use Adams' cobar construction on the S-R coalgebra $\mathbf{k}_*[K]$:

$$H_*(\Omega\, DJ(K)) \cong H(\Omega_* C_*(DJ(K))) \cong \mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k})$$

Then use a result of Fröberg calculating the latter Ext.

Cor 5 *We have an isomorphism of graded Lie algebras:*

$$\pi_*(\Omega\, DJ(K)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong FL(u_1,\ldots,u_m)\Big/\Big([u_i,u_i] = 0,\\ [u_i,u_j] = 0 \text{ for } \{i,j\} \in K\Big),$$

where $FL(\ )$ is a free Lie algebra and $\deg u_i = 1$.

The Poincaré series of $R^*[K]$ is given by

$$F(R^*[K];t) = \sum_{i=-1}^{n-1} \frac{f_i t^{2(i+1)}}{(1-t^2)^{i+1}}$$
$$= \frac{h_0 + h_1 t^2 + \ldots + h_n t^{2n}}{(1-t^2)^n},$$

where $\dim K = n-1$, f_i is the number of i-dimensional simplices in K, $f_{-1} = 1$, and the numbers h_i are defined from the second identity.

Cor 6 *For any flag complex K the rational homology Poincaré series of $\Omega\, DJ(K)$ is given by*

$$F\Big(H_*(\Omega\, DJ(K));t\Big) = \frac{(1+t)^n}{1 - h_1 t + \ldots + (-1)^n h_n t^n}.$$

Proof. Since $H_*(\Omega\, DJ(K);\mathbb{Q})$ is the quadratic dual of the Stanley-Reisner algebra $\mathbb{Q}[K]$ (graded by $\deg v_i = 1$), we have

$$F\Big(\mathbb{Q}[K];-t\Big) \cdot F\Big(H_*(\Omega\, DJ(K));t\Big) = 1.$$

5. Diagrams and homotopy colimits.

cat(K): *category of* K (simplices and inclusions);
mc: a model category (e.g., top or dga);

$X \in$ mc
X^K: cat(K) $\to$ mc *exponential diagram*; its value on $\sigma \subseteq \tau$ is the inclusion $X^\sigma \subseteq X^\tau$; $X^\varnothing = pt$.

Many previous constructions are colimits, e.g.,

$$\mathit{DJ}(K) = \mathrm{colim}^{\mathrm{top}}\, BT^K,$$
$$R_*[K] = \mathrm{colim}^{\mathrm{dgc}}\, C(v)^K, \text{ etc.},$$

where $C(v)$ is the symmetric coalgebra on v, $\deg v = 2$.

Cor 7 *Assume* K *is flag. Then*

$$H_*(\Omega\, \mathit{DJ}(K), \mathbb{Q}) \cong \mathrm{colim}^{\mathrm{ga}}\, \Lambda[u]^K;$$
$$\pi_*(\Omega\, \mathit{DJ}(K)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{colim}^{\mathrm{gl}}\, \mathit{CL}(u)^K,$$

where $\mathit{CL}(u)$ *is the commutative Lie algebra,* $\deg u = 1$.

In general colimit models do not work! (Look at $K = \partial\Delta^2$.)

Thm 8 *DJ*(K) *is coformal iff* K *is flag.*

Proof. For flag K construct a map

$$\Omega_* C_*(DJ(K)) \to H_*(\Omega\, DJ(K); \mathbb{Q})$$

and show that it is a quism using the above homology calculation. For non-flag K *higher Samelson* and *commutator* brackets appear in π_* and H_*, obstructing coformality.

Thm 9 *There is a htpy commutative diagram*

$$\begin{array}{ccc} \Omega_* \,\mathsf{hocolim}^{\mathsf{dgc}}\, C(v)^K & \xrightarrow{\overline{\eta}_K} & \mathsf{hocolim}^{\mathsf{dga}}\, \Lambda[u]^K \\ \big\downarrow \Omega_*\rho_K & & \big\downarrow \\ \Omega_*(\mathbb{Q}_*[K]) & \xrightarrow{\eta_K} & \mathsf{colim}^{\mathsf{dga}}\, \Lambda[u]^K \end{array},$$

in which $\Omega_*\rho_K$ *and* $\overline{\eta}_K$ *are weak equivalences, while* η_K *is a weak equivalence only if* K *is flag.*

Cor 10

$$H_*(\Omega\, DJ(K);\mathbb{Q}) \cong H\Big(\mathsf{hocolim}^{\mathsf{dga}}\, \Lambda[u]^K\Big)$$
$$\pi_*(\Omega\, DJ(K)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H\Big(\mathsf{hocolim}^{\mathsf{dgl}}\, CL(u)^K\Big).$$

Compare:

Thm 11 *(P.-Ray-Vogt) There is a htpy commutative diagram*

$$\begin{array}{ccc} \Omega\, \mathsf{hocolim}^{\mathsf{top}}\, BT^K & \xrightarrow{\overline{h}_K} & \mathsf{hocolim}^{\mathsf{tgrp}}\, T^K \\ \big\downarrow \Omega p_K & & \big\downarrow \\ \Omega\, DJ(K) & \xrightarrow{h_K} & \mathsf{colim}^{\mathsf{tgrp}}\, T^K \end{array},$$

in which Ωp_K *and* $\overline{h}_K$ *are weak equivalences, while* h_K *is a weak equivalence only if* K *is flag.*