Homology and homotopy of certain loop spaces

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Plan

1. Prerequisites: SR ring and DJ space
2. History and motivations
3. Homology of DJ(K) and Koszul duality
4. Higher Whitehead and Samelson prods

Builds on

T. Panov, N. Ray, R. Vogt
"Colimits, Stanley-Reisner algebras, and loop spaces",
http://arXiv.org/abs/math.AT/0202081

1. Prerequisites

k a commutative ring with unit

V a finite set, usually $V=[m]=\{1,\dots,m\}$

$k[V]$ polynomial algebra, generators $v_1,\dots,v_m$

$\Lambda_k[V]$ exterior algebra, generators $u_1,\dots,u_m$

$T_k[V]$ free (tensor) algebra, generators $x_1,\dots,x_m$

$W\subset V \Rightarrow v_W = \prod_{i\in W} v_i$ a square-free monomial

Gradings: $k=\mathbb{Z}$ or char $k\neq 2 \Rightarrow$ deg $u_i=1$ for $\Lambda_k[V]$
deg $=2$ otherwise

char $k=2 \Rightarrow$ deg $=1$ for all 3 cases

Given a simplicial complex K on V,

Def: the Stanley-Reisner ring is the quotient

$$SR_k(K) = k[V]/(v_W : W\notin K)$$

M. Davis & Januskiewicz intoduced homotopy types $DJ_{\mathbb{R}}(K)$ & $DJ_{\mathbb{C}}(K)$ such that

$$H^*(DJ_{\mathbb{R}}(K)) \cong SR_{\mathbb{Z}/2}(K), \quad H^*(DJ_{\mathbb{C}}(K)) \cong SR_{\mathbb{Z}}(K)$$

D&J models can be identified with colimit spaces

$$\underbrace{\operatorname{colim}_{\sigma\in K} B\mathbb{Z}/2^{\sigma} = \bigcup_{\sigma\in K} B\mathbb{Z}/2^{\sigma}}_{\text{is } DJ_{\mathbb{R}}(K)}, \quad \underbrace{\operatorname{colim}_{\sigma\in K} BT^{\sigma} = \bigcup_{\sigma\in K} BT^{\sigma} \subset BT}_{\text{is } DJ_{\mathbb{C}}(K)}$$

(or simply $DJ(K)$)

2. History and motivations

Have an obvious map:

$$\underset{\nwarrow \text{homotopy fibre}}{\mathcal{Z}_K} \longrightarrow DJ(K) \longrightarrow (\mathbb{C}P^\infty)^m \qquad (*)$$

$\mathcal{Z}_K$ has many interesting properties, e.g.

(1) $\mathcal{Z}_K$ is a manifold if $|K| \cong S^n$

(2) $\mathcal{Z}_K$ is homotopy equivalent to the coordinate subspace arrangement complement (def. by K)

(3) $H^*(\mathcal{Z}_K, k) \cong \operatorname{Tor}_{k[v_1,\dots,v_m]}(SR_k(K), k)$

... (Buchstaber - P.)

$(*)$ splits after looping:

$$\Omega DJ(K) \simeq \Omega \mathcal{Z}_K \times T^m,$$

but not multiplicatively!

$\Omega \mathcal{Z}_K$ is the "commutant" of $\Omega DJ(K)$, i.e. $\ker(\Omega DJ(K) \xrightarrow{Ab} T^m)$

<u>Def.</u>: K is a flag <u>complex</u> if any set of vertices which are pairwise connected spans a simplex.

<u>Prop.</u> (P.-Ray-Vogt): K flag $\Longrightarrow$

$$DJ(K) = \operatorname{colim}_K BT^\sigma \simeq B\operatorname{colim}_K^{TGRP} T^\sigma$$

Also holds for $DJ_{\mathbb{R}}(K) = \operatorname{colim}_K B\mathbb{Z}/2^\sigma$, $DJ_\Lambda(K) = \operatorname{colim}_K B\mathbb{Z}^\sigma$

<u>Corollary</u>: $\exists$ homotopy homomorphism $\operatorname{colim}_K^{TGRP} T^\sigma \xrightarrow{\simeq} \Omega DJ(K)$

<u>Ex.</u>: $K = :$ $\quad \underset{\nwarrow \text{free prod.}}{T * T} \xrightarrow{\simeq} \Omega(BT \vee BT)$

In general,

Thm (P-R-V):

$$\begin{tikzcd} \operatorname{hocolim}_K BT^\sigma \arrow[r, "\simeq"] \arrow[d] & B\operatorname{hocolim}_K^{TGRP} T^\sigma \arrow[d] \\ \operatorname{colim}_K BT^\sigma \arrow[r] & B\operatorname{colim}_K^{TGRP} T^\sigma \end{tikzcd}$$

in TGRP!

3. Homology of $\Omega DJ(K)$

Thm: if K is flag and k a field then

$H_*(\Omega DJ(K)) \cong T_k\langle x_1, \dots, x_m\rangle / (x_i^2 = 0,\ x_i x_j + x_j x_i = 0,\ \{i,j\} \in K)$

$d = \deg x_i = 1$

Remark: the rhs is the colimit in ALG of the diagram assigning $\sigma \longrightarrow \Lambda[x_i, i \in \sigma]$

Tool: Koszul duality

$T = T_k\langle x_1, \dots, x_n\rangle$ + inner product in 2-tensors
$\deg x_i = d$ ($x_i x_j$ form an orthonormal basis)

$A = A(I) = T/I$ is quadratic if I is generated, as an ideal, by 2-tensors (elts of degree $2d$)

$I^!$: ideal of T generated by $I_{2d}^\perp$

$A^! = A(I^!) = T/I^!$ is called the quadratic dual of A

Def: a quadratic algebra $A = A(I)$ is called Koszul if

$\exists \quad \cdots \to R^i \to \cdots \to R^1 \to R^0 \to k \to 0$

such that R^i is generated in degree di.

$E(A) := \mathrm{Ext}_A(k,k)$, $H(W,t) = \sum_n \dim_k W_n t^n$

Lemma: the following statements are equivalent:

(a) A is Koszul;

(b) $A^!$ is Koszul;

(c) $E(A)$ is a quadratic algebra;

(d) $E(A) \cong A^!$

(e) $H(A,t) \cdot H(E(A),-t) = 1$

Ex.: $A = k[V]$, $A^! = \Lambda_k[V]$. Both are Koszul.

Adams's cobar construction gives

Prop.: $H_*(\Omega DJ(K), k) \cong \mathrm{Ext}_{SR_k(K)}(k,k)$

Note: $SR_k(K)$ is quadratic $\Leftrightarrow$ K is flag

$$SR(K) = T_k\langle x_1,\dots,x_m\rangle \Big/ \begin{pmatrix} x_i x_j - x_j x_i = 0 & \text{for } \{i,j\} \in K \\ x_i x_j = 0 & \text{for } \{i,j\} \notin K \end{pmatrix}$$

$$(SR(K))^! = T_k\langle x_1,\dots,x_m\rangle \Big/ \begin{pmatrix} x_i^2 = 0 & \\ x_i x_j + x_j x_i = 0 & \text{for } \{i,j\} \in K \end{pmatrix}$$

Thm (Fröberg): K is flag $\Rightarrow$ $SR_k(K)$ is Koszul.

4. Higher Whitehead and Samelson prods

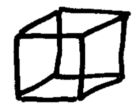
Ex. 1. $K =$ m points, $DJ(K) = (\mathbb{C}P^\infty)^{\vee m}$

$U := \text{hofibre}(DJ(K) \hookrightarrow (\mathbb{C}P^\infty)^m) \simeq \mathbb{C}^m \setminus \bigcup_{1 \le i < j \le m} (z_i = z_j = 0)$

$H_*(\Omega DJ(K)) = T_k\langle x_1, \dots, x_m \rangle / x_i^2 = 0$

2. $DJ_{\mathbb{R}}(K) = (\mathbb{R}P^\infty)^{\vee m}$

hofibre $(DJ_{\mathbb{R}}(K) \hookrightarrow (\mathbb{R}P^\infty)^m) \simeq$ 1-skeleton of I^m

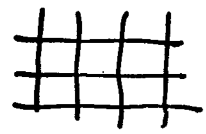


$\simeq$ wedge of $(m-2)2^{m-1}+1$ circles

Its π_1 is the commutant of $(\mathbb{Z}/2)^{*m}$

3. $DJ_\wedge(K) = T^{\vee m}$ $\quad (T = S^1)$

hofibre $(DJ_\wedge(K) \hookrightarrow T^m) \simeq$ 1-grid in $\mathbb{R}^m$



$\simeq$ infinite wedge of circles

Its π_1 is the commutant of $\mathbb{Z}^{*m}$

Prop.: $U \simeq \bigvee_{k=3}^{m+1} \left(\bigvee_{i=1}^{a_k} S^k \right)$; $a_k = (k-2)\binom{m}{k-1}$, $\Sigma a_k = (m-2)2^{m-1}+1$

Idea of proof:

1) $a_k = \#$ of linearly independent length $k-1$ iterated commutators on m letters

2) compare Betti numbers

3) consider $\pi_*(U) \longrightarrow \pi_*((\mathbb{C}P^\infty)^{\vee m}) \xrightarrow{Ab_*} \pi_*((\mathbb{C}P^\infty)^m)$

iterated W.p. $\in$ of $[f_i]$ $\qquad [f_i : S^2 \to (\mathbb{C}P^\infty)^{\vee m}] \in$

Construct a map $\bigvee(\bigvee S^k) \longrightarrow U$ unducing iso in H_*

Whitehead prods

$f: S^{i+1} \to X \quad g: S^{j+1} \to X$

$[f] \in \pi_{i+1}(X), \; [g] \in \pi_{j+1}(X)$

$$\begin{tikzcd}
S^{i+j+1} = \partial(D^{i+1} \times D^{j+1}) = D^{i+1} \times S^j \cup S^i \times D^{j+1} \arrow[d, "{[f,g]}"'] \arrow[dr] & \\
X & S^{i+1} \vee S^{j+1} \arrow[l, "f \vee g"']
\end{tikzcd}$$

$[f,g] \in \pi_{i+j+1}(X)$

Samelson prods

$f: S^i \to \Omega X, \; g: S^j \to \Omega X$

$[f] \in \pi_i(\Omega X), \; [g] \in \pi_j(\Omega X)$

$$\begin{tikzcd}
S^i \times S^j \arrow[rr] \arrow[dr] & & \Omega X \\
& S^{i+j} = S^i \times S^j / S^i \vee S^j \arrow[ur, "{[f,g]}"'] &
\end{tikzcd}$$

$(x, y) \to f(x) g(x) f^{-1}(x) g^{-1}(x)$

$[f,g] \in \pi_{i+j}(\Omega X)$

Properties:

0. $[f,g] = 0 \iff f \vee g : S^{i+1} \vee S^{j+1} \to X$ extends to $S^{i+1} \times S^{j+1}$
1. $[f_1 + f_2, g] = [f_1, g] + [f_2, g]$
2. $[f_1, f_2] = -(-1)^{ij} [f_2, f_1]$
3. Graded Jacobi identity

So $\pi_*(\Omega X) \otimes \mathbb{Q}$ is a graded Lie algebra

Its enveloping algebra is $H_*(\Omega X, \mathbb{Q})$

<u>Message</u>: commutators $u_iu_j + u_ju_i \in H_*(\Omega DJ(K))$ are images under Hurewicz homomorphism of Samelson prods $[f_i, f_j] \in \pi_3(DJ(K)) = \pi_2(\Omega DJ(K))$. They vanish whenever $\{i,j\}$ is an edge of K.

For general (non-flag) K Samelson prods are <u>not enough</u>.

<u>Ex.</u> <u>1.</u> $K = \partial\Delta^2$, $DJ(K) = BT_{12} \cup BT_{23} \cup BT_{13} \subset BT^3$ (fat wedge)

$S^5 \longrightarrow DJ(K) \longrightarrow BT^3$
(hofibre)

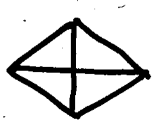
$[f_i, f_j] = 0$, but $[f_1, f_2, f_3] \neq 0$ in $\pi_5(DJ(K))$ — <u>Higher W. prod.</u>

$H_*(\Omega DJ(K)) \cong \mathbb{Z}[v] \otimes \Lambda[u_1, u_2, u_3]$, $\deg u_i = 1$, $\deg v = 4$

v is higher commutator prod of u_1, u_2, u_3.

(Recall that $\Omega DJ(K) \simeq \Omega S^5 \times T^3$ as spaces.)

Similarly for $K = \partial\Delta^n$

<u>2.</u> $K =$  (the 1-skeleton of Δ^3)

$H_*(\Omega DJ(K)) \cong \mathbb{Z}\langle u_1, u_2, u_3, u_4, v_{234}, v_{134}, v_{124}, v_{123}\rangle / \text{rel.}$

$\deg u_i = 1$, $\deg v_{ijk} = 4$

<u>Relations</u>: $u_iu_j + u_ju_i = 0$, $u_i^2 = 0$

$[u_i, v_{jkl}] = 0$ if $\{i,j,k,l\} \neq \{1,2,3,4\}$

and $[u_1, v_{234}] + [u_2, v_{134}] + [u_3, v_{124}] + [u_4, v_{123}] = 0$