# HIRZEBRUCH GENERA OF MANIFOLDS WITH TORUS ACTION

TARAS E. PANOV

ABSTRACT. A quasitoric manifold is a smooth $2n$-manifold $M^{2n}$ with an action of the compact torus $T^n$ such that the action is locally isomorphic to the standard action of $T^n$ on $\mathbb{C}^n$ and the orbit space is diffeomorphic, as manifold with corners, to a simple polytope $P^n$. The name refers to the fact that topological and combinatorial properties of quasitoric manifolds are similar to that of non-singular algebraic toric varieties (or toric manifolds). Unlike toric varieties, quasitoric manifolds may fail to be complex; however, they always admit a stably (or weakly almost) complex structure, and their cobordism classes generate the complex cobordism ring. As it have been recently shown by Buchstaber and Ray, a stably complex structure on a quasitoric manifold is defined in purely combinatorial terms, namely, by an orientation of the polytope and a function from the set of codimension-one faces of the polytope to primitive vectors of an integer lattice. We calculate the $\chi_y$-genus of a quasitoric manifold with fixed stably complex structure in terms of the corresponding combinatorial data. In particular, this gives explicit formulae for the classical Todd genus and signature. We also relate our results with well-known facts in the theory of toric varieties.

## INTRODUCTION

Manifolds with torus action arise in different areas of topology, algebraic and differential geometry, and mathematical physics. Specific properties of torus action or additional structures on manifolds usually allow to solve corresponding problems by geometrical or combinatorial methods. The most well known examples here are Hamiltonian torus actions in symplectic geometry and smooth toric varieties in algebraic geometry. Both cases allow a natural topological generalization, namely *quasitoric manifolds*, introduced by Davis and Januszkiewicz in [DJ]. (Davis and Januszkiewicz used the term "toric manifolds"; the term "quasitoric manifolds" firstly appeared in [BP2] and [BR2] because of the reasons discussed below.) A quasitoric manifold is a manifold with a torus action that satisfies two natural conditions. The first one is that the action locally looks like the standard torus action on a complex space by diagonal matrices. If this condition is satisfied, then the orbit space is a manifold with corners; the second condition is that this manifold with corners is diffeomorphic to a convex simple polytope. These two properties are well known for smooth algebraic toric varieties [Da], [Fu]. Though quasitoric manifolds retain most topological and combinatorial properties of smooth toric varieties, they may fail to admit a complex structure. Like toric varieties, quasitoric manifolds can be defined in purely combinatorial terms. Namely, any quasitoric manifold is defined by combinatorial data: the lattice of faces of a simple polytope

1991 *Mathematics Subject Classification.* 57R20, 57S25 (Primary) 14M25, 58G10 (Secondary).

This work was partly supported by the Russian Foundation for Fundamental Research (grant no. 99-01-00090).

and a *characteristic function* that assigns an integer primitive vector defined up to sign to each facet. Despite their simple and specific definition, quasitoric manifolds in many cases may serve as model examples (for instance, as it was shown in [BR1], each complex cobordism class contains a quasitoric manifold). All these facts allow to use quasitoric manifolds for solving topological problems by combinatorial methods and vice versa. A number of such relations was firstly discovered in the theory of toric varieties. Some applications were obtained in [BP1], [BP2], where quasitoric manifolds are studied in the general context of "manifolds defined by simple polytopes". Another example of interplay between topology and combinatorics is the calculation of the $KO$-theory of quasitoric manifolds held in [BB].

Besides, it is important to mention that the term "quasitoric manifold" is not common. Some authors starting from Davis and Januszkiewicz call the object of our study just "toric manifolds". However, we prefer to call those manifolds "quasitoric", reserving the term "toric manifold" for smooth toric varieties, as commonly used in algebraic geometry (see, for instance, [Ba]).

In the present paper we calculate well-known cobordism invariant, the $\chi_y$-genus, for quasitoric manifolds in terms of the corresponding combinatorial data. The most important particular cases here are the signature and the Todd genus, which correspond to the values $y = 1$ and $y = 0$ respectively. The signature is an oriented cobordism invariant and is defined for any oriented manifold (it equals zero in dimensions other than $4k$). At the same time the Todd genus, as well as the general $\chi_y$-genus, requires the Chern classes of manifold to be defined. As it is mentioned above, a quasitoric manifold is not necessarily complex, however as it was recently shown by Buchstaber and Ray [BR2], it always admits a stably (or weakly almost) complex structure. Moreover, stably complex structures (i.e. complex structures in the stable tangent bundle) on quasitoric manifolds are also defined combinatorially, namely, by specifying an orientation of the simple polytope and choosing signs for vectors corresponding to facets. This in turn is equivalent to a choice of orientations for the manifold and all submanifolds corresponding to facets. A quasitoric manifold with such additional structure was called in [BR2] *multioriented.* Hence, a multioriented quasitoric manifold determines a complex cobordism class, for which one can define characteristic numbers and complex Hirzebruch genera. We calculate the $\chi_y$-genus of a multioriented (i.e. with fixed stably complex structure) quasitoric manifold in terms of its combinatorial data. To do this, we construct a circle action with only isolated fixed points and then apply Atiyah and Bott's generalized Lefschetz fixed point theorem [AB]. The obtained formula allows to calculate the $\chi_y$-genus as a sum of contributions corresponding to the vertices of polytope. These contributions depend only on the "local combinatorics" near the vertex. Our formula contains one external parameter (an integer primitive vector $\nu$), which does not affect the answer. In the particular case of smooth toric varieties the Todd genus (or the arithmetic genus) is always equal to 1 [Fu]. However, this fact fails to be true for general quasitoric manifolds, since the Todd genus is a complex cobordism invariant and each cobordism class contains a quasitoric representative [BR1].

## 1. Quasitoric manifolds and polytopes, facet and edge vectors, and stably complex structures

Here we briefly discuss the definition of quasitoric manifolds and introduce stably complex structures on them, following [DJ] and [BR2]. We also describe some new relations between the combinatorial data associated to a quasitoric manifold.

A *convex polytope* $P^n$ of dimension $n$ is a bounded set in $\mathbb{R}^n$ that is obtained as the intersection of a finite number of half-spaces:

$$P^n = \{x \in \mathbb{R}^n : \langle l_i, x\rangle \geq -a_i,\ i = 1, \dots, m\} \tag{1}$$

for some $l_i \in (\mathbb{R}^n)^*$, $a_i \in \mathbb{R}$. A convex polytope $P^n$ is called *simple* if its bounding hyperplanes are in general position at each vertex, i.e. there exactly $n$ codimension-one faces (or *facets*) meet at each vertex. It follows that any point of a simple polytope lies in a neighbourhood that is affinely isomorphic to an open set of the positive cone

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \dots, x_n \geq 0\}.$$

This means exactly that a simple polytope $P^n$ is an $n$-dimensional *manifold with corners*. The faces of $P^n$ of all dimensions form a partially ordered set with respect to inclusion, which is called the *lattice of faces* of $P^n$. We say that two polytopes are *combinatorially equivalent* if they have same lattices of faces. Two polytopes are combinatorially equivalent if and only if they are diffeomorphic as manifolds with corners.

Let $M^{2n}$ be a compact $2n$-dimensional manifold with an action of the compact torus $T^n$. One can view $T^n$ as a subgroup of the complex torus $(\mathbb{C}^*)^n$ in the standard way:

$$T^n = \{\left(e^{2\pi i\varphi_1}, \dots, e^{2\pi i\varphi_n}\right) \in \mathbb{C}^n\},$$

where $(\varphi_1, \dots, \varphi_n)$ runs over $\mathbb{R}^n$. We say that a $T^n$-action is *locally isomorphic to the standard* diagonal action of $T^n$ on $\mathbb{C}^n$ if every point $x \in M^{2n}$ lies in some $T^n$-invariant neighbourhood $U(x)$ for which there exists an equivariant homeomorphism $f : U(x) \to W$ with some ($T^n$-stable) open subset $W \subset \mathbb{C}^n$. The last statement means that there is an automorphism $\theta : T^n \to T^n$ such that $f(t \cdot y) = \theta(t)f(y)$ for all $t \in T^n$, $y \in U(x)$. The orbit space for such an action of $T^n$ is an $n$-dimensional manifold with corners; we refer to $M^{2n}$ as a *quasitoric manifold* if the orbit space is diffeomorphic, as manifold with corners, to a simple polytope $P^n$. Thus, the orbit space of a quasitoric manifold is decomposed into faces in such a way that points from the relative interior of each $k$-face correspond to orbits with same isotropy subgroup of codimension $k$. In particular, the action of $T^n$ is free over the interior of $P^n$, while the vertices of $P^n$ correspond to $T^n$-fixed points of $M^{2n}$.

Now let $M^{2n}$ be a quasitoric manifold with orbit space $P^n$, and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be the set of codimension-one faces (facets) of the polytope $P^n$, $m = \sharp\mathcal{F}$. The interior of each facet $F_i$ consists of orbits with the same one-dimensional isotropy subgroup $G_{F_i}$. This one-parameter subgroup of $T^n$ is determined by an integer primitive vector $\lambda_i = (\lambda_{1i}, \dots, \lambda_{ni})^\top$ in the corresponding lattice $L \simeq \mathbb{Z}^n$:

$$G_{F_i} = \{\left(e^{2\pi i\lambda_{1i}\varphi}, \dots, e^{2\pi i\lambda_{ni}\varphi}\right) \in T^n\}, \tag{2}$$

where $\varphi \in \mathbb{R}$. Of course, the vector $\lambda_i$ is defined only up to sign. In this way one defines the *characteristic function* $\lambda : \mathcal{F} \to \mathbb{Z}^n$ that takes a facet to the corresponding primitive vector. By the definition of quasitoric manifolds, the characteristic function satisfies the following condition: if $n$ facets $F_{i_1}, \ldots, F_{i_n}$ meet at same vertex $p$, i.e. $p = F_{i_1} \cap \cdots \cap F_{i_n}$, then the integer vectors $\lambda(F_{i_1}), \ldots, \lambda(F_{i_n})$ constitute an integer basis of the lattice $L \simeq \mathbb{Z}^n$. As it was shown in [DJ], the characteristic pair $(P^n, \lambda : \mathcal{F} \to \mathbb{Z}^n)$ satisfying the above condition defines a quasitoric manifold $M^{2n}$ uniquely up to an equivariant diffeomorphism. Once the numeration of facets of $P^n$ is fixed, the characteristic function can be viewed as the integer $(n \times m)$-matrix $\Lambda$ whose $i$-th column is the vector $\lambda(F_i)$. Each vertex $p$ of $P^n$ can be represented as the intersection of $n$ facets: $p = F_{i_1} \cap \cdots \cap F_{i_n}$; we denote by $\Lambda_{(p)} = \Lambda_{(i_1, \ldots, i_n)}$ the minor matrix of $\Lambda$ formed by the columns $i_1, \ldots, i_n$. It follows from the above that

$$\det \Lambda_{(p)} = \det(\lambda_{i_1}, \ldots, \lambda_{i_n}) = \pm 1. \tag{3}$$

In the sequel we refer to the vectors $\lambda_i = \lambda(F_i)$ as *facet vectors*. Note again that for now they are defined only up to sign.

The next things we need in order to define genera of a quasitoric manifold is an orientation and stably complex structures. We fix an orientation of $T^n$ and specify an orientation of the polytope $P^n$ by orienting the ambient space $\mathbb{R}^n$. These orientation data define an orientation of the quasitoric manifold $M^{2n}$. The above construction of the characteristic function $\lambda$ involves arbitrary choice of signs for the vectors $\lambda(F_i)$. The inverse image $\pi^{-1}(F_i)$ of the facet $F_i$ under the orbit map $\pi : M^{2n} \to P^n$ is a quasitoric submanifold $M_i \subset M^{2n}$ of codimension 2. This facial submanifold $M_i$ belongs to the fixed point set of the circle subgroup $G_{F_i} \subset T^n$ (see (2)), which therefore acts on the normal bundle $\nu_i := \nu(M_i \subset M^{2n})$. Hence, one can invest $\nu_i$ with a complex structure (and orientation) by specifying a sign of the primitive vector $\lambda(F_i)$. Given an orientation of $M^{2n}$ (i.e. an orientation of $P^n$) and an orientation of $\nu_i$ (i.e. a sign of $\lambda_i$), one can orient the facial submanifold $M_i$ (or, equivalently, the facet $F_i \subset P^n$). Now, the oriented submanifold $M_i \subset M^{2n}$ of codimension 2 gives rise, by the standard topological construction, to a complex line bundle $\sigma_i$ over $M^{2n}$ that restricts to $\nu_i$ over $M_i$. The following theorem was proved by Buchstaber and Ray in [BR2]:

**Theorem 1.1.** *The following isomorphism of real oriented $2m$-bundles holds for any quasitoric manifold $M^{2n}$:*

$$\tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)} \simeq \sigma_1 \oplus \cdots \oplus \sigma_m.$$

*Here $\tau(M^{2n})$ is the tangent bundle and $\mathbb{R}^{2(m-n)}$ denotes the trivial $2(m-n)$-bundle over $M^{2n}$.*

An oriented quasitoric manifold $M^{2n}$ together with fixed orientations of facial submanifolds $M_i = \pi^{-1}(F_i)$ was called in [BR2] *multioriented*. It follows that signs for the facet vectors $\{\lambda_i\}$ of a multioriented quasitoric manifold are defined unambiguously. Theorem 1.1 shows that a multioriented quasitoric manifold can be invested with a canonical stably complex structure. Thus, an oriented simple polytope $P^n$ with characteristic matrix $\Lambda$ not only define a (multioriented) quasitoric manifold, but also specify a cobordism class in the complex cobordism ring $\Omega_U$.

Given an edge (1-dimensional face) $E_j$ of $P^n$, we see that points from its relative interior in $P^n$ correspond to orbits with the same $(n-1)$-dimensional isotropy subgroup $G_{E_j}$. This subgroup can be written as

$$G_{E_j} = \{(e^{2\pi i\varphi_1}, \ldots, e^{2\pi i\varphi_n}) \in T^n : \mu_{1j}\varphi_1 + \ldots + \mu_{nj}\varphi_n = 0\}, \tag{4}$$

i.e. it is determined by a primitive (co)vector $\mu_j = (\mu_{1j}, \dots, \mu_{nj})^\top$ in the dual (or weight) lattice $W = L^*$. We refer to this $\mu_j$ as *edge vector*; again it is defined only up to sign. The edge vectors satisfy the condition similar to that for the facet vectors: if $E_{j_1}, \dots, E_{j_n}$ are edges that meet at same vertex $p$, then

$$\det \mathrm{M}_{(p)} = \det(\mu_{j_1}, \dots, \mu_{j_n}) = \pm 1. \tag{5}$$

Here $\mathrm{M}_{(p)}$ is the square matrix with columns $\mu_{j_1}, \dots, \mu_{j_n}$. The above equality means that the vectors $\mu_{j_1}, \dots, \mu_{j_n}$ constitute a basis of $W \simeq \mathbb{Z}^n$.

The following lemma enables to choose signs of edge vectors for a multioriented quasitoric manifold unambiguously "locally" at each vertex.

**Lemma 1.2.** *Given a vertex $p$ of $P^n$, one can choose signs for edge vectors $\mu_{j_1}, \dots, \mu_{j_n}$ meeting at $p$ in such a way that*

$$\mathrm{M}_{(p)}^\top \cdot \Lambda_{(p)} = E,$$

*where $E$ denotes the identity matrix, and the matrices $\Lambda_{(p)}$, $\mathrm{M}_{(p)}$ are those from* (3), (5).

*Proof.* Since we are interested only in facet and edge vectors meeting at the vertex $p = F_{i_1} \cap \dots \cap F_{i_n}$, we may renumerate the edge vectors $\mu_{j_1}, \dots, \mu_{j_n}$ at $p$ by the index set $\{i_1, \dots, i_n\}$ of the facet vectors at $p$. To do this we just set $j_k = i_k$ if $E_{j_k} \not\subset F_{i_k}$ (i.e. a facet vector and an edge vector have same index if the corresponding facet and edge of $P^n$ span the whole $\mathbb{R}^n$). Then for $k \neq l$ one has $E_{i_k} \subset F_{i_l}$. Hence, $G_{F_{i_l}} \subset G_{E_{i_k}}$ and

$$\langle \mu_{i_k}, \lambda_{i_l} \rangle = 0, \quad k \neq l, \tag{6}$$

(see (2) and (4)). Now, since $\mu_{i_k}$ is a primitive vector, it follows from (6) that $\langle \mu_{i_k}, \lambda_{i_k} \rangle = \pm 1$. Changing the sign of $\mu_{i_k}$ if necessary, we obtain

$$\langle \mu_{i_k}, \lambda_{i_k} \rangle = 1, \tag{7}$$

which together with (6) gives $\mathrm{M}_{(p)}^\top \cdot \Lambda_{(p)} = E$, as needed. □

Throughout the rest of this paper we assume that once a stably complex structure of quasitoric manifold is fixed (i.e. signs for columns of the characteristic $(n \times m)$-matrix $\Lambda$ are specified), signs of edge vectors for each vertex are chosen as in Lemma 1.2.

We fix an orientation of the torus $T^n$ once and forever; then a choice of orientation for $M^{2n}$ is equivalent to a choice of orientation for the polytope $P^n \subset \mathbb{R}^n$. Hence, edges of $P^n$ meeting at the same vertex $p$ can be ordered canonically in such a way that the ordered set of vectors along the edges pointing out of $p$ constitute a positively oriented basis of $\mathbb{R}^n$ (i.e. an orientation of $P^n$ defines an ordering of edges at each vertex). In the sequel we assume that once an orientation of $M^{2n}$ is fixed, the edge vectors $\mu_{i_1}, \dots, \mu_{i_n}$ meeting at $p$ are ordered in accordance with the above ordering of edges at $p$. In this situation the edge vectors $\mu_{i_1}, \dots, \mu_{i_n}$ themselves may constitute a positively or negatively oriented basis of $\mathbb{R}^n$. (We assume that signs for $\mu_{i_1}, \dots, \mu_{i_n}$ are determined by Lemma 1.2.) Thus, we come to the following

**Definition 1.3.** Given a multioriented quasitoric manifold $M^{2n}$ with orbit space $P^n$, the *sign* of a vertex $p \in P^n$ is

$$\sigma(p) = \det \mathrm{M}_{(p)} = \det(\mu_{i_1}, \dots, \mu_{i_n}),$$

where $\mu_{i_1}, \dots, \mu_{i_n}$ are canonically ordered edge vectors meeting at $p$.

Obviously, Lemma 1.2 shows that

$$\sigma(p) = \det \Lambda_{(p)} = \det(\lambda_{i_1}, \dots, \lambda_{i_n}),$$

where $\lambda_{i_1}, \dots, \lambda_{i_n}$ are the facet vectors at $p$. We mention that the above definition of the sign of vertex already appeared in [Do] while studying characteristic functions and quasitoric manifolds over given simple polytope.

As it is mentioned in the introduction, quasitoric manifolds can be viewed as a topological generalization of algebraic smooth *toric varieties*. Toric variety [Da] is a normal algebraic variety on which the complex torus $(\mathbb{C}^*)^n$ acts with a dense orbit. Though any smooth toric variety (also called *toric manifold*) is quasitoric manifold, we illustrate our above constructions in the more restricted case of smooth projective toric varieties arising from polytopes. Any such toric variety is obtained via the standard procedure (see [Fu]) from a simple polytope (1) with vertices in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (we refer to such polytope as *integral*). Normal covectors $l_i$ of facets of an integral polytope $P^n$ can be chosen integer and primitive. The toric variety $M_P$ obtained from an integral simple polytope $P^n$ is necessarily projective and has complex dimension $n$. The compact torus $T^n \subset (\mathbb{C}^*)^n$ acts on $M_P$ with orbit space $P^n$; moreover, there is a map from $M_P$ to $\mathbb{R}^n$ which is constant on $T^n$-orbits and has image $P^n$ (the *moment map*). The variety $M_P$ is smooth whenever for each vertex $p \in P^n$ the normal covectors $l_{i_k}$, $k = 1, \dots, n$, of facets containing $p$ constitute an integer basis of the dual lattice. This means exactly that the map $F_i \to l_i$ defines a characteristic function. It can be easily seen that it is exactly the characteristic function of $M_P$ regarded as a quasitoric manifold (or, conversely, the quasitoric manifold corresponding to the above characteristic function is equivariantly diffeomorphic to $M_P$). Hence, facet vectors for $M_P$ are just normal (co)vectors for the corresponding polytope $P^n$. Edge vectors are primitive integer vectors along the edges of $P^n$. In short, this means that in the case of toric $M_P$ "facet vectors" are "vectors normal to facets" and "edge vectors" are "vectors along edges". Lemma 1.2 in this case says that if an edge is contained in a facet, then the vector along the edge is orthogonal to the normal vector of the facet, while if an edge is not contained in a facet, then signs of the corresponding vectors can be chosen in such a way that their scalar product is equal to 1. It can be shown that the canonical complex structure on a toric variety $M_P$, regarded as a stably complex structure on a quasitoric manifold, corresponds to orienting the facet vectors $l_i$ in such a way that all of them are "pointing inside the polytope $P^n$". As it follows from Lemma 1.2, edge vectors locally should be oriented in such a way that all of them are "pointing out of the vertex". Note that globally Lemma 1.2 provides two signs for an edge, one for each of its ends. These signs are always different in the case of toric varieties, however this is not true in general. Note also that since an edge vector for a toric variety is a vector along the edge pointing out of the vertex, one has $\sigma(p) = 1$ (see Definition 1.3) for any vertex $p$.

## 2. Circle action with isolated fixed points on a quasitoric manifold

In this section we show that there exists a circle subgroup of $T^n$ that acts on a quasitoric manifold with only isolated fixed points corresponding to vertices of the underlying simple polytope. This action will be used in the next section for calculating the $\chi_y$-genus via contributions of fixed points.

So, we start with a multioriented quasitoric manifold $M^{2n}$ with orbit space $P^n$ and edge vectors $\{\mu_i\}$. Hence, $M^{2n}$ is endowed with a stably complex structure, as described in the previous section.

**Theorem 2.1.** *Suppose that $\nu \in \mathbb{Z}^n$ is an integer primitive vector such that $\langle \mu_i, \nu \rangle \neq 0$ for all edge vectors $\mu_i$. Then the circle subgroup $S^1 \subset T^n$ defined by $\nu$ acts on $M^{2n}$ with isolated fixed points corresponding to vertices of $P^n$. In the tangent space $T_pM^{2n}$ at fixed point corresponding to the vertex $p = F_{i_1} \cap \dots \cap F_{i_n}$ this action induces a representation of $S^1$ with weights $\langle \mu_{i_1}, \nu \rangle, \dots, \langle \mu_{i_n}, \nu \rangle$.*

*Proof.* Let us consider the action of $T^n$ near fixed point $p \in M^{2n}$ corresponding to a vertex $p = F_{i_1} \cap \dots \cap F_{i_n}$ of $P^n$. Let $\mu_{i_1}, \dots, \mu_{i_n}$ be the edge vectors at $p$. The $T^n$-action induces a unitary $T^n$-representation in the tangent space $T_pM^{2n}$. We choose complex coordinates $(x_1, \dots, x_n)$ in $T_pM^{2n}$ such that the tangent space of two-dimensional submanifold $\pi^{-1}(E_{i_k}) \subset M^{2n}$ at $p$ is given by equations $x_1 = \dots = \widehat{x_k} = \dots = x_n = 0$ ($x_k$ is dropped). Then, the corresponding isotropy subgroup $G_{E_{i_k}}$ is given by the equation $\langle \mu_{i_k}, \varphi \rangle = 0$ in $T^n$ (see (4)). Hence, the weights of the $T^n$-representation are $\mu_{i_1}, \dots, \mu_{i_n}$, i.e. an element $t = \left(e^{2\pi i \varphi_1}, \dots, e^{2\pi i \varphi_n}\right) \in T^n$ acts on $T_pM^{2n}$ as

$$
\begin{aligned}
(8) \qquad t \cdot (x_1, \dots, x_n) &= \left(e^{2\pi i(\mu_{1i_1}\varphi_1 + \dots + \mu_{ni_1}\varphi_n)} x_1, \dots, e^{2\pi i(\mu_{1i_n}\varphi_1 + \dots + \mu_{ni_n}\varphi_n)} x_n\right) \\
&= \left(e^{2\pi i\langle \mu_{i_1}, \Phi\rangle} x_1, \dots, e^{2\pi i \langle \mu_{i_n}, \Phi\rangle} x_n\right),
\end{aligned}
$$

where $(x_1, \dots, x_n) \in T_pM^{2n}$, $\Phi = (\varphi_1, \dots, \varphi_n)$.

A primitive vector $\nu = (\nu_1, \dots, \nu_n)^\top \in \mathbb{Z}^n$ defines a one-parameter circle subgroup $\{\left(e^{2\pi i \nu_1 \varphi}, \dots, e^{2\pi i \nu_n \varphi}\right), \varphi \in \mathbb{R}\} \subset T^n$. It follows from (8) that this circle acts on $T_pM^{2n}$ as

$$
s \cdot (x_1, \dots, x_n) = \left(e^{2\pi i \langle \mu_{i_1}, \nu\rangle \varphi} x_1, \dots, e^{2\pi i \langle \mu_{i_n}, \nu \rangle \varphi} x_n\right),
$$

where $s = e^{2\pi i \varphi} \in S^1$. The fixed point $p$ is isolated if all the weights $\langle \mu_{i_1}, \nu \rangle, \dots, \langle \mu_{i_n}, \nu \rangle$ of the $S^1$-action are non-zero. Thus, if $\langle \mu_i, \nu \rangle \neq 0$ for all edge vectors, then the $S^1$-action on $M^{2n}$ defined by $\nu$ has only isolated fixed points. □

Note that the condition $\langle \mu_i, \nu \rangle \neq 0$ for all $\mu_i$ from the above theorem means that the primitive vector $\nu$ is "of general position". In the case of smooth toric variety constructed from integral simple polytope $P^n$ the above condition is that the vector $\nu$ is not orthogonal to any edge of $P^n$ (or, equivalently, no hyperplane normal to $\nu$ intersects $P^n$ by a face of dimension $> 0$).

In the next section we will need the following

**Definition 2.2.** Given the $S^1$-action on $M^{2n}$ defined by a primitive vector $\nu$, the *index* of the vertex $p = F_{i_1} \cap \dots \cap F_{i_n}$ is

$$
\mathrm{ind}_\nu(p) = \{\sharp k : \langle \mu_{i_k}, \nu \rangle < 0\},
$$

i.e. $\mathrm{ind}_\nu(p)$ equals the number of negative weights at $p$.

The following lemma shows that the index of any vertex $p$ can be defined directly by means of facet vectors at $p$ without calculating edge vectors.

**Lemma 2.3.** *Let $p = F_{i_1} \cap \dots \cap F_{i_n}$ be a vertex of $P^n$. Then the index $\mathrm{ind}_\nu(p)$ equals the number of negative coefficients in the representation of $\nu$ as a linear combination of basis vectors $\lambda_{i_1}, \dots, \lambda_{i_n}$.*

*Proof.* It follows from Lemma 1.2 that for any vertex $p = F_{i_1} \cap \dots \cap F_{i_n}$ one can write

$$\nu = \langle \mu_{i_1}, \nu \rangle \lambda_{i_1} + \ldots + \langle \mu_{i_n}, \nu \rangle \lambda_{i_n}.$$

Then our lemma follows from the definition of $\mathrm{ind}_\nu(p)$. $\square$

## 3. Formulae for $\chi_y$-genus, signature and Todd genus

Here we calculate the $\chi_y$-genus of a multioriented quasitoric manifold in terms of its characteristic pair $(P^n, \Lambda)$. This is done by applying the Atiyah–Hirzebruch formula for the $S^1$-action constructed in the previous section. We also stress upon the most important particular cases of signature and Todd genus.

**Theorem 3.1.** *Let $M^{2n}$ be a multioriented quasitoric manifold, and let $\nu \in \mathbb{Z}^n$ be an integer primitive vector such that $\langle \mu_i, \nu \rangle \neq 0$ for all edge vectors $\mu_i$. Then*

$$\chi_y(M^{2n}) = \sum_{p \in P^n} (-y)^{\mathrm{ind}_\nu(p)} \sigma(p),$$

*where the sum is taken over all vertices of $P^n$, $\sigma(p)$ and $\mathrm{ind}_\nu(p)$ are as in definitions* 1.3 *and* 2.2.

*Proof.* The Atiyah–Hirzebruch formula ([AH]; see also [Kr], where it was deduced within the cobordism theory) states that the $\chi_y$-genus of a stably complex manifold $X$ with a $S^1$-action can be calculated as

$$\chi_y(X) = \sum_i (-y)^{n(F_i)} \chi_y(F_i), \tag{9}$$

where the sum is taken over all $S^1$-fixed submanifolds $F_i \subset X$, and $n(F_i)$ denotes the number of negative weights of the $S^1$-representation in the normal bundle of $F_i \subset X$. In our case all fixed submanifolds are isolated fixed points corresponding to vertices $p \in P^n$. Hence, $\chi_y(F_i) = \chi_y(p) = \pm 1$ depending on whether or not the orientation of $\mathbb{R}^n$ defined by $\mu_{i_1}, \dots, \mu_{i_n}$ coincides with that defined by $P^n$. Thus, for quasitoric $M^{2n}$ we may substitute $\sigma(p)$ for $\chi_y(F_i)$ in (9). Theorem 2.1 shows that the weights of induced $S^1$-representation in $T_pM^{2n}$ equal $\langle \mu_{i_1}, \nu \rangle, \dots, \langle \mu_{i_n}, \nu \rangle$, therefore $n(F_i)$ in (9) is exactly $\mathrm{ind}_\nu(p)$ (see Definition 2.2), and the required formula follows. $\square$

The $\chi_y$-genus $\chi_y(M^{2n})$ at $y = -1$ equals the $n$th Chern number $c_n[M^{2n}]$ for any $2n$-dimensional stably complex manifold $M^{2n}$. Theorem 3.1 gives

$$c_n[M^{2n}] = \sum_{p \in P^n} \sigma(p). \tag{10}$$

If $M^{2n}$ is a complex manifold (e.g., $M^{2n}$ is a smooth toric variety) one has $\sigma(p) = 1$ for all vertices $p \in P^n$ and $c_n[M^{2n}]$ equals the Euler number $e(M^{2n})$. Hence, for complex $M^{2n}$ the Euler number equals the number of vertices of $P^n$ (which is well known for toric varieties). For general quasitoric $M^{2n}$ the Euler number is also equal to the number of vertices of $P^n$ (since the Euler number of any $S^1$-manifold equals the sum of Euler numbers of fixed submanifolds), however this number may be different from $c_n[M^{2n}]$.

The $\chi_y$-genus at $y = 1$ equals the signature (or the $L$-genus). Theorem 3.1 gives in this case

**Corollary 3.2.** *The signature of a multioriented quasitoric manifold $M^{2n}$ can be calculated as*

$$\operatorname{sign}(M^{2n}) = \sum_{p\in P^n} (-1)^{\operatorname{ind}_\nu(p)}\sigma(p),$$

*where the sum is taken over all vertices of $P^n$.*

Being an invariant of an oriented cobordism class the signature of a quasitoric manifold $M^{2n}$ does not depend on a stably complex structure (i.e. on a choice of signs for facet vectors) and is determined only by an orientation of $M^{2n}$ (or $P^n$). This can be seen directly from our above considerations. Indeed, we have chosen signs of edge vectors locally at each vertex as described in Lemma 1.2. Then Corollary 3.2 states that the signature can be calculated as the sum over vertices of values $(-1)^{\operatorname{ind}_\nu(p)}\sigma(p)$, where $\operatorname{ind}_\nu(p)$ is the number of negative scalar products $\langle\mu_{i_k},\nu\rangle$ and $\sigma(p)=\det(\mu_{i_1},\dots,\mu_{i_n})$ determines the difference between orientations of $T_pM^{2n}$ defined by the $T^n$-representation with weights $\mu_{i_1},\dots,\mu_{i_n}$ and the standard $T^n$-representation. Instead of this, we may choose signs for edge vectors in such a way that *all* scalar products with $\nu$ are positive. Then one obviously has

$$(-1)^{\operatorname{ind}_\nu(p)}\sigma(p) = \det(\tilde\mu_{i_1},\dots,\tilde\mu_{i_n}),$$

where $\tilde\mu_{i_k}=\pm\mu_{i_k}$ and $\langle\tilde\mu_{i_k},\nu\rangle>0$, $k=1,\dots,n$. Now, the right hand side of the above equality is independent of particular choice of signs for facet vectors (i.e. independent of a stably complex structure). Thus, we come to the following

**Corollary 3.3.** *The signature of an oriented quasitoric manifold $M^{2n}$ can be calculated as*

$$\operatorname{sign}(M^{2n}) = \sum_{p\in P^n} \det(\tilde\mu_{i_1},\dots,\tilde\mu_{i_n}),$$

*where $\{\tilde\mu_{i_k}\}$ are edge vectors at $p$ with signs chosen to satisfy $\langle\tilde\mu_{i_k},\nu\rangle>0$, $k=1,\dots,n$.*

In the case of smooth toric variety $M_P$ we have $\sigma(p)=1$ for all vertices $p\in P^n$, and Corollary 3.2 gives

$$\operatorname{sign}(M_P) = \sum_{p\in P^n} (-1)^{\operatorname{ind}_\nu(p)}. \tag{11}$$

The scalar product with vector $\nu$ can be viewed as an analog of Morse height function on the polytope $P^n\subset\mathbb{R}^n$ (see [Br], [Kh], and [DJ]). Since $\nu$ is not orthogonal to any edge of $P^n$, we may orient the edges of $P^n$ in such a way that the scalar product with $\nu$ increases along the edges. Then the index of vertex $p$ is just the number of edges pointing in (i.e. towards $p$). An easy combinatorial argument [Br, p. 115] shows that the number of vertices with exactly $k$ edges pointing in equals $h_k$. Here $(h_0,h_1,\dots,h_n)$ is the so-called *h-vector* of the polytope $P^n$. It is defined from the equation

$$h_0t^n+\ldots+h_{n-1}t+h_n=(t-1)^n+f_0(t-1)^{n-1}+\ldots+f_{n-1},$$

where $f_k$ is the number of faces of $P^n$ of codimension $(k+1)$. It is well known [Fu] that $h_k$ equals $2k$-th Betti number of the toric variety $M_P$ (this is also true for a general quasitoric manifold [DJ], however we do not use this fact here). Now, using formula (11) we obtain that the signature of a smooth toric variety $M_P$ is

$$\operatorname{sign}(M_P)=\sum_{k=1}^n(-1)^kh_k.$$

This formula can be also deduced from the Hodge structure in the cohomology of $M_P$. However, we believe that obtaining this result from our general formula for quasitoric manifolds could be of interest.

The next important particular case of the $\chi_y$-genus is the Todd genus corresponding to the value $y=0$. In this case summands in the formula from Theorem 3.1 are not defined for those vertices having index 0 , so it requires some additional analysis.

**Theorem 3.4.** *The Todd genus of a multioriented quasitoric manifold can be calculated as*

$$\operatorname{td}(M^{2n}) = \sum_{\substack{p\in P^n \\ \operatorname{ind}_\nu(p)=0}} \sigma(p)$$

*(the sum is taken over all vertices of index 0). Here $\nu$ is any primitive vector such that $\langle\mu_i,\nu\rangle\neq 0$ for all $\mu_i$.*

*Proof.* For each vertex $p=F_{i_1}\cap\dots\cap F_{i_n}$ of $P^n$ the stably complex structure on $M^{2n}$ determined by $(P^n,\Lambda)$ defines a complex structure on $T_pM^{2n}$ via the isomorphism

$$T_pM^{2n}\simeq\sigma_{i_1}\oplus\dots\oplus\sigma_{i_n} \tag{12}$$

(see Theorem 1.1). The $S^1$-action on $M^{2n}$ determined by $\nu\in\mathbb{Z}^n$ induces the complex $S^1$-representation on $T_pM^{2n}$ with weights $w_1(p)=\langle\mu_{i_1},\nu\rangle,\dots,w_n(p)=\langle\mu_{i_n},\nu\rangle$ (signs of edge vectors are determined by Lemma 1.2). Atiyah and Bott's generalized Lefschetz fixed point formula ([AB], see also [HBY]) gives the following expression for the *equivariant $\chi_y$-genus* of $M^{2n}$

$$\chi_y(q,M^{2n})=\sum_{p\in P^n}\prod_{i=1}^{k}\frac{1+yq^{w_k(p)}}{1-q^{w_k(p)}}\sigma(p), \tag{13}$$

where $q=e^{2\pi i\varphi}\in S^1$, and $\sigma(p)=\det(\mu_{i_1},\dots,\mu_{i_n})=\pm1$ depending on whether or not the orientation of $T_pM^{2n}$ defined by (12) coincides with that defined by the original orientation of $M^{2n}$. (We note again that this sign $\sigma(p)$ is equal 1 for all vertices if $M^{2n}$ is a true complex manifold, e.g., a smooth toric variety.) Atiyah and Hirzebruch's theorem [AH] states that the above expression for $\chi_y(q,M^{2n})$ is independent of $q$ and equals $\chi_y(M^{2n})$. Taking the limit of the right hand side of (13) as $q\to0$, one obtains the Atiyah–Hirzebruch formula (9) (since the $\lim_{q\to0}$ of each summand in (13) is exactly $(-y)^{\operatorname{ind}(p)}\sigma(p)$). In the case $y=0$ corresponding to the Todd genus the same limit for the summand corresponding to a vertex $p$ equals 0 if there is at least one $w_k(p)<0$ and equals 1 otherwise. This is exactly what is stated in the theorem. □

As for the Todd genus of a smooth toric variety, it is easy to see that there is only one vertex of index 0 in this case. Indeed, if we orient edges of the polytope by means of the scalar product with $\nu$ (see the above considerations concerning the signature of toric varieties), then only one "bottom" vertex will have all edges pointing out, i.e. index 0. Since one has $\sigma(p)=1$ for all vertices $p\in P^n$, Theorem 3.4 gives $\operatorname{td}(M_P)=1$, which is well known (see e.g. [Fu]).

At the end we consider some examples illustrating our results. All our examples are multioriented four-dimensional quasitoric manifolds. Thus, all facet vectors are defined unambiguously, while signs for edge vectors are defined "locally" at each

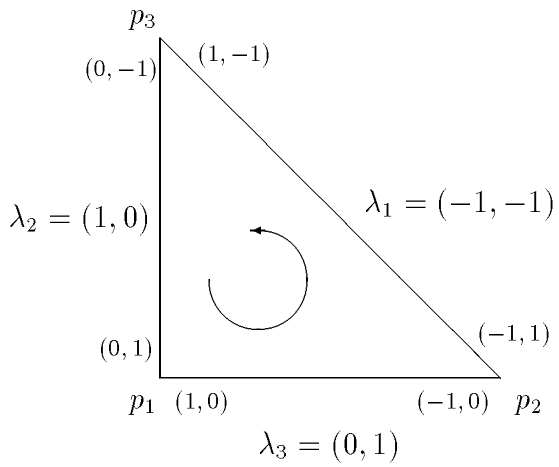


FIGURE 1. $\tau(\mathbb{C}P^2)\oplus\mathbb{C}\simeq\bar\eta\oplus\bar\eta\oplus\bar\eta$

vertex as described in Lemma 1.2. Our pictures contain a polytope (a polygon in our case), facet and edge vectors, and an orientation of the polytope (which is determined in our case by a cyclic order of vertices).

Figure 1 corresponds to $\mathbb{C}P^2$ regarded as a toric variety. Hence, a stably complex structure is determined by the standard complex structure on $\mathbb{C}P^2$, i.e. via the isomorphism of bundles $\tau(\mathbb{C}P^2)\oplus\mathbb{C}\simeq\bar\eta\oplus\bar\eta\oplus\bar\eta$, where $\mathbb{C}$ is the trivial complex line bundle and $\eta$ is the Hopf line bundle over $\mathbb{C}P^2$. The orientation is defined by the complex structure. As we have pointed out above, the toric variety $\mathbb{C}P^2$ arises from an integral polytope (2-dimensional simplex with vertices $(0,0)$, $(1,0)$ and $(0,1)$ in this case). Facet vectors here are primitive normal vectors to facets pointing inside the polytope, and edge vectors are primitive vectors along edges pointing out of a vertex. This can be seen on Figure 1. Let us calculate the Todd genus and the signature by means of Corollary 3.2 and Theorem 3.4. We have $\sigma(p_1)=\sigma(p_2)=\sigma(p_3)=1$. Take $\nu=(1,2)$, then $\mathrm{ind}(p_1)=0$, $\mathrm{ind}(p_2)=1$, $\mathrm{ind}(p_3)=2$ (remember that the index is the number of negative scalar products of edge vectors with $\nu$). Thus, $\mathrm{sign}(\mathbb{C}P^2)=\mathrm{sign}(\mathbb{C}P^2,\bar\eta\oplus\bar\eta\oplus\bar\eta)=1$, $\mathrm{td}(\mathbb{C}P^2)=\mathrm{td}(\mathbb{C}P^2,\bar\eta\oplus\bar\eta\oplus\bar\eta)=1$.

Changing the orientation of polytope on Figure 1 causes changing of signs of the vertices: $\sigma(p_1)=\sigma(p_2)=\sigma(p_3)=-1$, while the indices of vertices remain unchanged. This corresponds to reversing the canonical orientation of $\mathbb{C}P^2$. Denoting the resulting manifold $\overline{\mathbb{C}P}^2$, we obtain $\mathrm{sign}(\overline{\mathbb{C}P}^2)=\mathrm{sign}(\overline{\mathbb{C}P}^2,\bar\eta\oplus\bar\eta\oplus\bar\eta)=-1$, $\mathrm{td}(\overline{\mathbb{C}P}^2)=\mathrm{td}(\overline{\mathbb{C}P}^2,\bar\eta\oplus\bar\eta\oplus\bar\eta)=-1$. The same stably complex structure can be obtained from the initial orientation of the polytope by taking another facet vectors, as it is shown on Figure 2. In this case one has

$$\sigma(p_1)=\begin{vmatrix}-1&0\\0&1\end{vmatrix}=-1,\quad\sigma(p_2)=\begin{vmatrix}1&1\\1&0\end{vmatrix}=-1,\quad\sigma(p_3)=\begin{vmatrix}0&-1\\-1&-1\end{vmatrix}=-1.$$

Again, we can take $\nu=(1,2)$, then $\mathrm{ind}(p_1)=1$, $\mathrm{ind}(p_2)=0$, $\mathrm{ind}(p_3)=2$. Thus, we see again that $\mathrm{sign}(\overline{\mathbb{C}P}^2)=\mathrm{sign}(\overline{\mathbb{C}P}^2,\bar\eta\oplus\bar\eta\oplus\bar\eta)=-1$, $\mathrm{td}(\overline{\mathbb{C}P}^2)=\mathrm{td}(\overline{\mathbb{C}P}^2,\bar\eta\oplus\bar\eta\oplus\bar\eta)=-1$.

Our third example (Figure 3) is $\mathbb{C}P^2$ with the standard orientation and stably complex structure determined by the isomorphism $\tau(\mathbb{C}P^2)\oplus\mathbb{C}\simeq\eta\oplus\bar\eta\oplus\bar\eta$ (this is

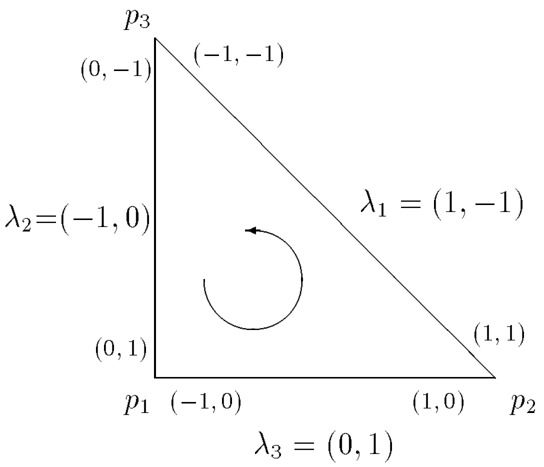


FIGURE 2. $\tau(\overline{\mathbb{C}P^2})\oplus\mathbb{C}\simeq\bar\eta\oplus\bar\eta\oplus\bar\eta$

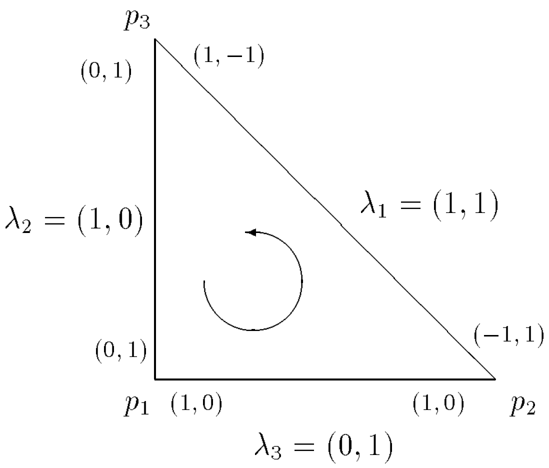


FIGURE 3. $\tau(\mathbb{C}P^2)\oplus\mathbb{C}\simeq\eta\oplus\bar\eta\oplus\bar\eta$

obtained from Figure 1 by changing the sign of first facet vector). Then we have

$$\sigma(p_1)=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1,\quad \sigma(p_2)=\begin{vmatrix}-1&1\\1&0\end{vmatrix}=-1,\quad \sigma(p_3)=\begin{vmatrix}0&1\\1&-1\end{vmatrix}=-1.$$

Taking $\nu=(1,2)$, we find $\operatorname{ind}(p_1)=0,\operatorname{ind}(p_2)=0,\operatorname{ind}(p_3)=1$. Thus, $\operatorname{sign}(\mathbb{C}P^2,\eta\oplus\bar\eta\oplus\bar\eta)=1$, $\operatorname{td}(\mathbb{C}P^2,\eta\oplus\bar\eta\oplus\bar\eta)=0$. Note that in this case formula (10) gives $c_n(\mathbb{C}P^2,\eta\oplus\bar\eta\oplus\bar\eta)[\mathbb{C}P^2]=\sigma(p_1)+\sigma(p_2)+\sigma(p_3)=-1$ (this could be also checked directly), while the Euler number of $\mathbb{C}P^2$ is $c_n(\mathbb{C}P^2,\bar\eta\oplus\bar\eta\oplus\bar\eta)[\mathbb{C}P^2]=3$.

The author is grateful to Prof. V. M. Buchstaber for stimulating discussions and useful recommendations.

## References


[AB] M.F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: I,* Ann. of Math. **86** (1967), no. 2, 374–407.

[AH] M.F. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions,* Essays in Topology and Related Topics, Berlin-Heidelberg-New York, Springer, 1970, pp. 18–28.

[Ba] V.V. Batyrev, *Quantum Cohomology Rings of Toric Manifolds*, Journées de Géometrie Algébrique d'Orsay (Juillet 1992), Astérisque **218**, Sociéte Mathématique de France, Paris, 1993, pp. 9–34; math.AG/9310004.

[BB] A. Bahri and M. Bendersky, *The $KO$-theory of toric manifolds,* math.AT/9904087.

[Br] A. Brønsted, *An introduction to convex polytopes*, Springer-Verlag New-York, 1983.

[BP1] V.M. Bukhshtaber and T.E. Panov, *Algebraic topology of manifolds defined by simple polytopes* (Russian), Uspekhi Mat. Nauk **53** (1998), no. 3, 195–196; English transl. in: Russian Math. Surveys **53** (1998), no. 3, 623–625.

[BP2] V.M. Buchstaber and T.E. Panov, *Torus actions and combinatorics of polytopes* (Russian), Trudy Matematicheskogo Instituta im. Steklova **225** (1999), 96–131; English transl. in: Proceedings of the Steklov Institute of Mathematics **225** (1999), 87–120; math.AT/9909166.

[BR1] V.M. Bukhshtaber and N. Ray, *Toric manifolds and complex cobordisms* (Russian), Uspekhi Mat. Nauk **53** (1998), no. 2, 139–140; English transl. in: Russian Math. Surveys **53** (1998), no. 2, 371–373.

[BR2] V.M. Buchstaber and N. Ray, *Tangential structures on toric manifolds, and connected sums of polytopes*, preprint UMIST, Manchester, 1999.

[Da] V.I. Danilov, *The geometry of toric varieties*, (Russian), Uspekhi Mat. Nauk **33** (1978), no. 2, 85–134; English transl. in: Russian Math. Surveys **33** (1978), 97–154.

[DJ] M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. Journal **62** (1991), no. 2, 417–451.

[Do] N. Dobrinskaya, *The problem of classification of quasitoric manifolds over given simple polytope*, Funct. Anal. Appl., to appear.

[Fu] W. Fulton, *Introduction to Toric Varieties,* Princeton Univ. Press, Princeton, NJ, 1993.

[HBY] F. Hirzebruch, T. Berger, and R. Jung. *Manifolds and Modular Forms,* Second Edition, A Publication from the Max-Planc-Institut für Mathematik, Bonn, 1994.

[Kh] A. G. Khovanskii, *Hyperplane sections of polyhedra,* (Russian), Funk. Anal. i ego Prilozhen. **20**, (1986), no. 1; English transl. in: Funct. Anal. Appl. **20**, (1986), no. 1, 41–50.

[Kr] I.M. Krichever, *Formal groups and Atiyah–Hirzebruch formula*, (Russian), Izvestiya Ross. Akad. Nauk Ser. Mat. **38**, (1974), no. 6, 1289–1304; English transl. in: Math. USSR, Izvestiya **8** (1974).



Department of Mathematics and Mechanics, Moscow State University, Moscow 119899 RUSSIA

*E-mail address*: `tpanov@mech.math.msu.su`