MOMENT-ANGLE COMPLEXES AND COMBINATORICS OF SIMPLICIAL MANIFOLDS

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Let $\rho : (D^2)^m \to I^m$ be the orbit map for the diagonal action of torus T^m on the unit poly-disk $(D^2)^m \subset \mathbb{C}^m$. Each face of the cube $I^m = [0,1]^m$ (viewed as a cubical complex) has the form

$$F_{I\subset J} = \{(y_1, \dots, y_m) \in I^m \ : \ y_i = 0 \text{ if } i \in I,\ y_j = 1 \text{ if } j \notin J\},$$

where $I \subset J$ are two subsets of the index set $[m] = \{1, \dots, m\}$. For each face $F_{I\subset J}$ put $B_{I\subset J} := \rho^{-1}(F_{I\subset J})$. If $\#I = i$, $\#J = j$, then $B_{I\subset J} \cong (D^2)^{j-i} \times T^{m-j}$.

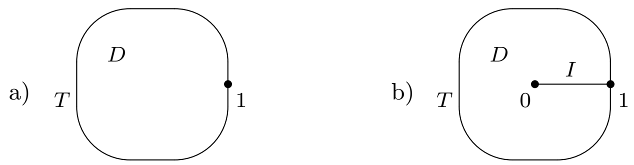
Definition 1. Let C be a cubical subcomplex of I^m. The *moment-angle complex* $\mathrm{ma}(C)$ is the T^m-invariant decomposition of the subset $\rho^{-1}(|C|) \subset (D^2)^m$ into blocks $B_{I\subset J}$ corresponding to the faces $F_{I\subset J}$ of complex C.

Many combinatorial problems concerning cubical complexes may be treated by studying the equivariant topology of moment-angle complexes. In the present paper we realize this approach in the case of cubical complexes determined by simplicial complexes. Let K^{n-1} be an $(n-1)$-dimensional simplicial complex with m vertices, and $|K|$ the corresponding polyhedron. If $I = \{i_1, \dots, i_k\} \subset [m]$ is a simplex of K, then we would write $I \in K$. Define the following two cubical subcomplexes of I^m:

$$\mathrm{cub}(K) = \{F_{I\subset J} \ : \ J \in K, I \neq \varnothing\}, \quad \mathrm{cc}(K) = \{F_{I\subset J} \ : \ J \in K\}.$$

Lemma 2. *As a topological space, the complex* $\mathrm{cub}(K)$ *is homeomorphic to* $|K|$*, while* $\mathrm{cc}(K)$ *is homeomorphic to the cone* $|\operatorname{cone}(K)|$.

The cubical complex $\mathrm{cc}(K)$ was introduced in [1] and then studied in [2]. The cubical complex $\mathrm{cub}(K)$ appeared in [3].



Denote the moment-angle complexes corresponding to $\mathrm{cub}(K)$ and $\mathrm{cc}(K)$ by $\mathcal{W}_K$ and $\mathcal{Z}_K$ respectively. Consider the cellular decomposition of the poly-disk $(D^2)^m$ that is obtained by subdividing each factor D^2 into 0-dimensional cell 1, 1-dimensional cell T, and 2-dimensional cell D, see Fig. a). Each cell of $(D^2)^m$ is a product of cells D_i, T_i, 1_i, $i = 1, \dots, m$, i.e., can be written as $D_I T_J 1_{[m]\setminus I\cup J}$, where I, J are disjoint subsets of $[m]$. Set $D_I T_J := D_I T_J 1_{[m]\setminus I\cup J}$. Now it can be easily seen that $\mathcal{Z}_K$ is cellular subcomplex of $(D^2)^m$ consisting of all cells $D_I T_J$ such that $I \in K$.

Lemma 3. *The embedding* $T^m = \rho^{-1}(1, \dots, 1) \hookrightarrow \mathcal{Z}_K$ *is a cellular map homotopic to the map to a point.*

As it was shown in [2], for any field $\mathbf{k}$ there is the following isomorphism of algebras:

$$(1) \qquad H^*(\mathcal{Z}_K) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\ \mathbf{k}(K), \mathbf{k}\ = H^*\ \mathbf{k}(K) \otimes \Lambda[u_1, \dots, u_m], d\ ,$$

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where $\mathbf{k}(K)$ is the Stanley–Raisner ring of complex K, and the differential d is defined by $d(v_i) = 0$, $d(u_i) = v_i$, $i = 1, \ldots, m$. The Tor-algebra from (1) is naturally a *bigraded* algebra with $\operatorname{bideg}(v_i) = (0, 2)$, $\operatorname{bideg}(u_i) = (-1, 2)$. The calculation of the ring $H^*(\mathcal{Z}_K)$ allowed to describe the multiplicative structure in the cohomology of the complement of a coordinate subspace arrangement in $\mathbb{C}^m$ [2].

In [1] there was introduced the subcomplex $\mathcal{C}^*(K) \subset \mathbf{k}(K) \otimes \Lambda[u_1, \ldots, u_m]$ spanned by monomials u_J and $v_I u_J$ such that $I \cap J = \varnothing$, $I \in K$. It was also shown there that the cohomology of $\mathcal{C}^*(K)$ is also isomorphic to that of $\mathcal{Z}_K$. Denote by $\mathcal{C}_*(\mathcal{Z}_K)$ and $\mathcal{C}^*(\mathcal{Z}_K)$ the chain and the cochain complex for type a) cellular decomposition of $\mathcal{Z}_K$ respectively.

Theorem 4. *Let $(D_I T_J)^* \in \mathcal{C}^*(\mathcal{Z}_K)$ denote the cellular cochain dual to the cell $D_I T_J \in \mathcal{Z}_K$. The correspondence $v_I u_J \mapsto (D_I T_J)^*$ defines a canonical isomorphism of complexes $\mathcal{C}^*(K)$ and $\mathcal{C}^*(\mathcal{Z}_K)$, each of which calculates $H^*(\mathcal{Z}_K)$.*

The pair $(\mathcal{Z}_K, T^m)$ acquires a bigraded cellular structure by setting $\operatorname{bideg}(D_i) = (0, 2)$, $\operatorname{bideg}(T_i) = (-1, 2)$, $\operatorname{bideg}(1_i) = (0, 0)$. Put $b_{-q,2p}(\mathcal{Z}_K, T^m) = \dim H_{-q,2p}[\mathcal{C}_*(\mathcal{Z}_K, T^m)]$. Consider the new cellular decomposition of the poly-disc $(D^2)^m$ that is obtained by subdividing each factor D^2 into 5 cells D, T, I, 1, 0, see Fig. b). This allows to introduce a bigraded cellular structure on $\mathcal{W}_K$ and define the numbers $b_{q,2p}(\mathcal{W}_K) = \dim H_{q,2p}[\mathcal{C}_*(\mathcal{W}_K)]$. Put

$$\chi(\mathcal{Z}_K, T^m; t) = \sum_{p,q} (-1)^q b_{-q,2p}(\mathcal{Z}_K, T^m) t^{2p}, \quad \chi(\mathcal{W}_K; t) = \sum_{p,q} (-1)^q b_{q,2p}(\mathcal{W}_K) t^{2p}.$$

Let f_i be the number of i-simplices of K, and $(h_0, \ldots, h_n)$ the *h-vector* of K determined from the equation $h_0 t^n + \ldots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \ldots + f_{n-1}$.

Theorem 5. *Put $h(t) = h_0 + h_1 t + \cdots + h_n t^n$. Then*

$$\chi(\mathcal{Z}_K, T^m; t) = (1 - t^2)^{m-n} h(t^2) - (1 - t^2)^m,$$
$$\chi(\mathcal{W}_K; t) = (1 - t^2)^{m-n} h(t^2) + (-1)^{n-1} h_n (1 - t^2)^m.$$

Lemma 6. *If $|K| \cong S^{n-1}$ (i.e., K is a simplicial sphere), then $\mathcal{Z}_K$ is a closed manifold.*

Suppose now that K^{n-1} is a simplicial manifold. Then the complex $\mathcal{Z}_K$ generally fails to be a manifold. However, removing from $\mathcal{Z}_K$ a small neighbourhood $U_\varepsilon(T^m)$ of the orbit $\rho^{-1}(1, \ldots, 1) \cong T^m$ we obtain manifold $W_K = \mathcal{Z}_K \setminus U_\varepsilon(T^m)$ with boundary $\partial W_K = |K| \times T^m$.

Theorem 7. *The manifold (with boundary) W_K is equivariantly homotopy equivalent to the complex $\mathcal{W}_K$. There is a canonical homeomorphism of pairs $(W_K, \partial W_K) \to (\mathcal{Z}_K, T^m)$.*

The relative Poincaré duality isomorphisms for W_K imply

$$\chi(\mathcal{W}_K; t) = (-1)^{m-n} t^{2m} \chi(\mathcal{Z}_K, T^m; \tfrac{1}{t}).$$

Taking into account Theorem 5 we obtain

Corollary 8. *Let K^{n-1} be a simplicial manifold. Then*

$$h_{n-i} - h_i = (-1)^i (h_n - 1) \binom{n}{i} = (-1)^i \bigl(\chi(K^{n-1}) - \chi(S^{n-1})\bigr) \binom{n}{i}, \quad i = 0, 1, \ldots, n,$$

where $\chi(\cdot)$ denotes the Euler number.

Rewriting the equation (8) in terms of the f-vector we come to more complicated equations, which were deduced in [4], [5]. For $|K| = S^{n-1}$ Corollary 8 gives the classical Dehn–Sommerville equations. In the particular case of PL-manifolds the topological invariance of numbers $h_{n-i} - h_i$ (which follows directly from Corollary 8) was firstly observed by Pachner in [6, (7.11)].

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