*Proceedings of the Steklov Institute of Mathematics, Vol. 247, 2004, pp. 1–17.*
*Translated from Trudy Matematicheskogo Instituta imeni V.A. Steklova, Vol. 247, 2004, pp. 41–58.*


# Combinatorics of Simplicial Cell Complexes and Torus Actions

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Received April 2004

**Abstract**—Simplicial cell complexes are special cellular decompositions also known as virtual or ideal triangulations; in combinatorics, appropriate analogues are given by simplicial partially ordered sets. In this paper, combinatorial and topological properties of simplicial cell complexes are studied. Namely, the properties of $f$-vectors and face rings of simplicial cell complexes are analyzed and described, and a number of well-known results on the combinatorics of simplicial partitions are generalized. In particular, we give an explicit expression for the operator on $f$- and $h$-vectors that is defined by a barycentric subdivision, derive analogues of the Dehn–Sommerville relations for simplicial cellular decompositions of spheres and manifolds, and obtain a generalization of the well-known Stanley criterion for the existence of regular sequences in the face rings of simplicial cell complexes. As an application, a class of manifolds with a torus action is constructed, and generalizations of some of our previous results on the moment–angle complexes corresponding to triangulations are proved.

## 1. INTRODUCTION

This paper focuses on spaces with a special cellular decomposition, simplicial cell complexes. In combinatorics, these complexes correspond to the so-called simplicial partially ordered sets.

In topology, simplicial cell complexes have been studied virtually since the emergence of triangulations and combinatorial methods. However, in combinatorics, the systematic study of simplicial partially ordered sets started only around the mid-1980s (see [13]). Many fundamental constructions of commutative and homological algebra that were developed for studying triangulations are naturally carried over to simplicial cell complexes. For instance, in [14], an important concept of a *face ring* (or a *Stanley–Reisner ring*) of a simplicial partially ordered set is introduced and the main algebraic properties of these rings are described.

In [9], a *group of projectivities* of a triangulation is defined. The study of this group has led to the construction of an interesting class of simplicial cell complexes, called the *unfoldings* of triangulations. In the same work, on the basis of this construction, an explicit combinatorial description was obtained for the Hilden–Montesinos branched covering of an arbitrary closed oriented manifold over the 3-sphere.

In the combinatorics of simplicial complexes, the so-called *bistellar subdivisions* are used, which play an important role in applications to torus actions (see [2, Ch. 7]). In this way, one naturally encounters simplicial cellular decompositions because the application of a bistellar subdivision to a triangulation yields, in general, only a simplicial cellular decomposition.

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Simplicial cell complexes have found important applications in the study of *torus manifolds* (see [11]). A torus manifold is a $2n$-dimensional smooth compact closed manifold $M$ on which an action of the $n$-dimensional compact torus $T$ is defined which is effective and has at least one fixed point. Torus manifolds were introduced in [8] as a far-reaching generalization of algebraic nonsingular compact *toric varieties*. The fixed points of the effective action of a half-dimensional torus lead to the rich combinatorial structure of the orbit space $Q = M/T$, which often allows one to describe the torus manifold $M$ itself. Under the additional condition that the action is *locally standard*, the orbit space $Q$ is a *manifold with corners* (a particular case of manifolds with corners is given by simple polyhedra). The set of faces of a manifold with corners forms a simplicial partially ordered set with the order given by reverse inclusion. The study of manifolds with corners (or, equivalently, simplicial partially ordered sets) as orbit spaces of torus manifolds allows one to interpret many important topological invariants of these manifolds in combinatorial terms. For example, as is shown in [11], equivariant cohomology of a torus manifold $M$ is isomorphic to the face ring $\mathbb{Z}[Q]$ of the orbit space $Q$ in a number of natural cases (for instance, when all faces in $Q$ are acyclic). According to another result of [11], the cohomology of a torus manifold vanishes in odd dimensions if and only if all faces of the orbit space $Q$ are acyclic.

Let us briefly describe the contents of the paper. In Section 2, we introduce the basic concepts, simplicial and pseudosimplicial cell complexes, and describe the relations between them and their properties. These cell complexes provide useful approximations to classical triangulations (simplicial complexes); in particular, the concept of barycentric subdivision is introduced for these complexes. The barycentric subdivision of a pseudosimplicial cell complex is a simplicial cell complex, while the barycentric subdivision of a simplicial cell complex is a simplicial complex (Proposition 2.4).

In Section 3, we consider $f$- and $h$-vectors of cell complexes. In the case of simplicial complexes, a transition to the barycentric subdivision yields a linear operator on the spaces of $f$- and $h$-vectors. We present an explicit expression for the matrix of this operator (Lemmas 3.1 and 3.2). The operator constructed can formally be applied to the $f$-vector of any cell complex. The application of this operator to the $f$-vector of a pseudosimplicial cell complex yields an $f$-vector of a simplicial cell complex, whereas its application to the $f$-vector of a simplicial cell complex yields an $f$-vector of a simplicial complex. The main result of this section is the derivation of analogues of the Dehn–Sommerville relations for pseudosimplicial and simplicial cell complex of manifolds (Theorems 3.4 and 3.5 and Corollary 3.6).

In Section 4, we introduce the concept of branched combinatorial covering and prove that a space $X$ is a simplicial cell complex if and only if there exists a branched combinatorial covering $X \to K$ for a certain simplicial complex $K$ (Theorem 4.1).

In Section 5, we study the properties of the face rings of simplicial cell complexes introduced by Stanley [14]. Here, we obtain results on the functoriality of such rings with respect to simplicial mappings (Propositions 5.3, 5.9, and 5.10) and present conditions under which these rings contain linear systems of parameters (Theorem 5.4 and Lemma 5.5); this is important for applications to torus actions.

In Section 6, we introduce and study a functor that assigns a space $\mathcal{Z}_{\mathcal{S}}$ with an action of the torus $T^m$ to any simplicial cell complex $\mathcal{S}$ of dimension $n-1$ with $m$ vertices; the space $\mathcal{Z}_{\mathcal{S}}$ generalizes the concept of a moment–angle complex $\mathcal{Z}_K$ (see [6] and [2, Ch. 7]). Like for simplicial decompositions, $\mathcal{Z}_{\mathcal{S}}$ is a manifold for piecewise linear simplicial cellular decompositions of spheres (Theorem 6.3). The *toral rank* of a space $X$ is the maximal number $k$ such that there exists an almost free $T^k$-action on $X$. We prove that, for any simplicial cell complex $\mathcal{S}$ of dimension $n-1$ with $m$ vertices, the toral rank of the space $\mathcal{Z}_{\mathcal{S}}$ is no less than $m-n$ (Theorem 6.4). This result leads to an interesting relation between the well-known toral rank conjecture of Halperin [7] and the combinatorics of triangulations (Corollary 6.5).

## 2. SIMPLICIAL AND PSEUDOSIMPLICIAL CELL COMPLEXES

An abstract *simplicial complex* on a set $\mathcal{M}$ is a collection $K = \{\sigma\}$ of subsets in $\mathcal{M}$ such that, for every $\sigma \in K$, all subsets in $\sigma$ (including $\varnothing$) also belong to $K$. A subset $\sigma \in K$ is called an (abstract) *simplex* of the complex $K$. One-element subsets are called *vertices*. The *dimension* of a simplex $\sigma \in K$ is the number of its elements minus one: $\dim \sigma = |\sigma| - 1$. The dimension of an abstract simplicial complex is the maximal dimension of its simplices.

Along with abstract simplices, we consider *geometric simplices*, which are convex hulls of sets of affinely independent points in $\mathbb{R}^n$. A *geometric simplicial complex* (or a *polyhedron*) is a set $\mathcal{P}$ of geometric simplices of arbitrary dimensions lying in a certain $\mathbb{R}^n$, such that every face of a simplex from $\mathcal{P}$ lies in $\mathcal{P}$ and the intersection of any two simplices from $\mathcal{P}$ is a face of each of them.

A polyhedron $\mathcal{P}$ is called a *geometric realization* of an abstract simplicial complex $K$ if there exists a one-to-one correspondence between the vertex sets of the complex $K$ and the polyhedron $\mathcal{P}$ under which simplices of the complex $K$ are mapped into the vertex sets of the simplices of the polyhedron $\mathcal{P}$. For every simplicial complex $K$, there exists a unique, up to a simplicial isomorphism, geometric realization, which is denoted by $|K|$. In what follows, we will not distinguish between abstract simplicial complexes and their geometric realizations.

Let $\mathcal{S}$ be an arbitrary partially ordered set. Its *order complex* $\operatorname{ord}(\mathcal{S})$ is the set of all chains $x_1 < x_2 < \ldots < x_k,\ x_i \in \mathcal{S}$. It is obvious that $\operatorname{ord}(\mathcal{S})$ is a simplicial complex.

The *barycentric subdivision* $K'$ of a simplicial complex $K$ is defined as the order complex $\operatorname{ord}(K \setminus \varnothing)$ of the partially ordered (with respect to inclusion) set of nonempty simplices of the complex $K$.

A partially ordered set $\mathcal{S}$ is called *simplicial* if it contains the least element $\widehat{0}$ and, for any $\sigma \in \mathcal{S}$, the lower segment

$$\bigl[\widehat{0}, \sigma\bigr] = \bigl\{\tau \in \mathcal{S} \colon \widehat{0} \le \tau \le \sigma\bigr\}$$

is the partially ordered (with respect to inclusion) set of faces of a certain simplex. Introduce the rank of elements of the set $\mathcal{S}$ putting $\operatorname{rk} \widehat{0} = 0$ and $\operatorname{rk} \sigma = k$ if $\bigl[\widehat{0}, \sigma\bigr]$ is identified with the set of faces of a $(k-1)$-simplex. We also put $\dim \mathcal{S} = \max_{\sigma \in \mathcal{S}} \operatorname{rk} \sigma - 1$. We call elements of rank 1 the *vertices* of the set $\mathcal{S}$. Then, any element of rank $k$ contains exactly $k$ vertices.

It is obvious that the set of all simplices of a certain simplicial complex is a simplicial partially ordered set with respect to inclusion. However, not all simplicial partially ordered sets are obtained in this manner.

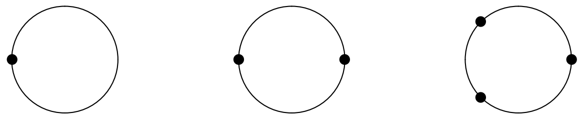
Recall that a *cell complex* is a Hausdorff topological space $X$ represented as the union $\bigcup e_i^q$ of pairwise disjoint sets $e_i^q$ (called *cells*) such that, for every cell $e_i^q$, a unique mapping of a $q$-dimensional closed ball $D^q$ into $X$ is fixed (the *characteristic mapping*) whose restriction to the interior of the ball $D^q$ is a homeomorphism onto $e_i^q$. (Here, as usual, we assume that the interior of a point is the point itself.) Moreover, it is assumed that the following axioms hold:

(C) The boundary of a cell $e_i^q$ is contained in the union of a finite number of cells $e_j^r$ of dimension $r < q$.

(W) A subset $Y \subset X$ is closed if and only if, for any cell $e_i^q$, the intersection $Y \cap \overline{e}_i^q$ is closed.

In what follows, we will always assume that partially ordered sets and simplicial and cell complexes are finite.

Let $\mathcal{S}$ be a simplicial partially ordered set. To each element $\sigma \in \mathcal{S} \setminus \widehat{0}$, we assign a simplex with the set of faces given by $\bigl[\widehat{0}, \sigma\bigr]$ and glue together all these geometric simplices according to the order relation in $\mathcal{S}$. Then, we obtain a cell complex in which the closure of each cell is identified with a simplex, the structure of faces being preserved; moreover, all characteristic mappings are embeddings. This complex is called a *simplicial cell complex* and is denoted by $|\mathcal{S}|$. When $\mathcal{S}$ is a set of simplices of a certain simplicial complex $K$, the space $|\mathcal{S}|$ coincides with the geometric realization



(a) (b) (c)

**Fig. 1.** Pseudosimplicial cellular decompositions of a circle

of $|K|$. Henceforth, we will not distinguish between simplicial partially ordered sets and simplicial cell complexes.

**Example 2.1.** A cell complex obtained by identifying two $(n-1)$-dimensional simplices along their boundaries (with the structure of faces preserved) is a simplicial cell complex. The corresponding simplicial partially ordered set is not a set of faces of any simplicial complex for $n > 1$.

This example is a particular case of the general construction of a union of simplicial cell complexes along a common subcomplex. Note that simplicial cell complexes represent a minimal extension of the class of simplicial complexes that is closed with respect to the operation of taking a union along a common subcomplex.

Recall that a continuous mapping of cell complexes is called *cellular* if the image of any cell lies in a union of cells of the same or lower dimension. A cellular mapping $\phi\colon \mathcal{S}_1 \to \mathcal{S}_2$ of simplicial cell complexes is called *simplicial* if the image of every simplex from $\mathcal{S}_1$ is a simplex in $\mathcal{S}_2$. It is clear that a simplicial mapping is a mapping of the corresponding partially ordered sets (i.e., preserves order). A simplicial mapping is called a *simplicial isomorphism* if there exists a simplicial inverse for it. In geometrical terms, one may assume that a simplicial mapping of simplicial complexes (polyhedra) is linear on simplices.

By analogy with simplicial complexes, we define the *barycentric subdivision* $\mathcal{S}'$ of a simplicial cell complex $\mathcal{S}$ as the order complex $\operatorname{ord}\big(\mathcal{S} \setminus \widehat{0}\big)$. It follows directly from the definition that the barycentric subdivision $\mathcal{S}'$ is a simplicial complex. Thus, a simplicial mapping between arbitrary simplicial cell complexes can be assumed to be linear on the simplices of the barycentric subdivision.

**Remark.** The identity mapping of a simplicial cell complex defines a cellular, but not simplicial, mapping $\mathcal{S} \to \mathcal{S}'$ such that its inverse is not a cellular mapping.

A cell complex $X$ is called *pseudosimplicial* if the characteristic mapping of any cell is a mapping of the simplex $\Delta^q$ into $X$ whose restriction onto each face is the characteristic mapping for a certain other cell.

**Example 2.2.** Figure 1 shows three pseudosimplicial cellular decompositions of a circle. The first of them (a) is not a simplicial cellular decomposition. The second (b) is a simplicial cellular decomposition but is not a simplicial complex. The third (c) is a simplicial complex.

**Lemma 2.3.** *A pseudosimplicial cell complex $X$ is a simplicial cell complex if and only of the characteristic mapping of any of its cells is an embedding.*

**Proof.** For a given pseudosimplicial cell complex $X$, we introduce a partially ordered set $\mathcal{S}$ whose elements are closures of cells and the order relation corresponds to embedding. Let all the characteristic mappings be embeddings. Then, it is obvious that, for every cell $e_i^q$, the corresponding characteristic mapping is a homeomorphism of $\Delta^q$ onto $\overline{e}_i^q$, and the characteristic mapping of any cell $e_j^p \subset \overline{e}_i^q$ is a restriction of this homeomorphism onto a certain $p$-face of the simplex $\Delta^q$. Therefore, $\mathcal{S}$ is a simplicial partially ordered set and $X$ is a simplicial cell complex $|\mathcal{S}|$. The converse assertion is obvious. $\square$

Pseudosimplicial cellular decompositions of manifolds play an important role in various constructions of the low-dimensional topology, where they are referred to as *ideal* or *singular triangulations* (see, e.g., [12]).

Let $X$ be a pseudosimplicial cell complex. Its *barycentric subdivision* is a cell complex $X'$ whose cells are images of open simplices of the standard barycentric subdivision of simplices $\Delta^q$ under all possible characteristic mappings $\Delta^q \to X$ of the cells of the complex $X$. The correctness of this definition follows immediately from the definition of a pseudosimplicial cell complex.

**Remark.** The above "geometrical" definition of a barycentric subdivision is consistent with the "combinatorial" definitions, introduced above, of the barycentric subdivision of a simplicial complex and a simplicial partially ordered set. However, simple examples show that, in the case of an arbitrary pseudosimplicial cell complex $X$, the complex $X'$ differs from the order complex of the partially ordered set of the closures of the cells of the complex $X$.

It follows directly from the definition that $X'$ is a pseudosimplicial cell complex. Indeed, the following more general proposition holds (which is a part of mathematical folklore; see, e.g., [12]).

**Proposition 2.4.** *The barycentric subdivision $X'$ of a pseudosimplicial cell complex $X$ is a simplicial cell complex, while the second barycentric subdivision $X''$ is a simplicial complex.*

**Proof.** By Lemma 2.3, to prove the first assertion, it suffices to verify that all characteristic mappings of the cells of the complex $X'$ are embeddings. Suppose the contrary, i.e., some two points $x$ and $y$ go to a single point under the characteristic mapping of a certain $q$-dimensional cell $e^q$ of the complex $X'$. One may assume that $q$ is the minimal dimension of such cells. By the definition of $X'$, the characteristic mapping of its cell $e^q$ is a restriction of the characteristic mapping $\Delta^q \to X$ of a certain cell of the complex $X$ onto a certain $q$-dimensional simplex of the barycentric subdivision of the standard simplex $\Delta^q$. Since $q$ is minimal, at least one of the points $x$ or $y$ is contained in the interior of the simplex $\Delta^q$. Now, using the fact that the restriction of a characteristic mapping onto the interior of a cell is one-to-one, we arrive at a contradiction. This proves the first assertion, which implies the second. $\square$

## 3. $f$-VECTORS AND THE DEHN–SOMMERVILLE RELATIONS

Let $X$ be a cell complex of dimension $n-1$. Denote by $f_i$ the number of its $i$-dimensional cells. An integer vector $\boldsymbol{f}(X) = (f_0, \dots, f_{n-1})$ is called the *$f$-vector* of the complex $X$. It is convenient to assume that $f_{-1} = 1$. Introduce the *$h$-vector* of the complex $X$ as an integer vector $(h_0, h_1, \dots, h_n)$ determined from the equation

$$h_0 t^n + \ldots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \ldots + f_{n-1}. \tag{3.1}$$

Note that the $f$-vector and the $h$-vector carry the same information about the cell complex and are expressed in terms of each other by linear relations, namely,

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{n-k} f_{i-1}, \qquad f_{n-1-k} = \sum_{q=k}^{n} \binom{q}{k} h_{n-q}, \qquad k = 0, \dots, n. \tag{3.2}$$

In particular, $h_0 = 1$ and $h_n = (-1)^n (1 - f_0 + f_1 + \ldots + (-1)^n f_{n-1}) = (-1)^n (1 - \chi(X))$, where $\chi(X)$ is the Euler characteristic of the complex $X$.

First, we describe some properties of $f$-vectors in the case when $X = \mathcal{S}$ is a simplicial cell complex (i.e., the closures of cells form a simplicial partially ordered set).

The *join* of simplicial partially ordered sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is a set $\mathcal{S}_1 * \mathcal{S}_2$ consisting of elements $\sigma_1 * \sigma_2$, where $\sigma_1 \in \mathcal{S}_1$ and $\sigma_2 \in \mathcal{S}_2$. A partial order relation is introduced as follows: $\sigma_1 * \sigma_2 \le \tau_1 * \tau_2$ if $\sigma_1 \le \tau_1$ and $\sigma_2 \le \tau_2$. It is clear that $\mathcal{S}_1 * \mathcal{S}_2$ is a simplicial partially ordered set. The space (cell complex) $|\mathcal{S}_1 * \mathcal{S}_2|$ is known in topology as the *join* of the spaces $|\mathcal{S}_1|$ and $|\mathcal{S}_2|$.

A simplicial cell complex $\mathcal{S}$ is called *pure* if all of its maximal simplices have the same dimension. Consider two pure complexes $\mathcal{S}_1$ and $\mathcal{S}_2$ of the same dimension, select maximal simplices $\sigma_1 \in \mathcal{S}_1$

and $\sigma_2 \in \mathcal{S}_2$, and fix a certain identification for them (this is done, for example, by choosing the order of vertices in $\sigma_1$ and $\sigma_2$). A simplicial cell complex $\mathcal{S}_1 \#_{\sigma_1,\sigma_2} \mathcal{S}_2$ obtained by gluing together $\mathcal{S}_1$ and $\mathcal{S}_2$ along $\sigma_1$ and $\sigma_2$ followed by the removal of the simplex $\sigma = \sigma_1 = \sigma_2$ is called the *connected sum* of the complexes $\mathcal{S}_1$ and $\mathcal{S}_2$. When the result is independent of the choice of the simplices $\sigma_1$ and $\sigma_2$ and the method of their identification, we will use an abbreviated notation $\mathcal{S}_1 \# \mathcal{S}_2$.

Let us express the $f$-vector and the $h$-vector of the connected sum $\mathcal{S}_1 \# \mathcal{S}_2$ in terms of the $f$-vectors and $h$-vectors of the complexes $\mathcal{S}_1$ and $\mathcal{S}_2$. Let $\dim \mathcal{S}_1 = \dim \mathcal{S}_2 = n-1$; then, we have

$$\begin{aligned} f_i(\mathcal{S}_1 \# \mathcal{S}_2) &= f_i(\mathcal{S}_1) + f_i(\mathcal{S}_2) - \binom{n}{i+1}, \qquad i = 0, 1, \dots, n-2, \\ f_{n-1}(\mathcal{S}_1 \# \mathcal{S}_2) &= f_{n-1}(\mathcal{S}_1) + f_{n-1}(\mathcal{S}_2) - 2. \end{aligned}$$

Then, (3.2) implies that

$$\begin{aligned} &h_0(\mathcal{S}_1 \# \mathcal{S}_2) = 1, \\ &h_i(\mathcal{S}_1 \# \mathcal{S}_2) = h_i(\mathcal{S}_1) + h_i(\mathcal{S}_2), \qquad i = 1, 2, \dots, n-1, \\ &h_n(\mathcal{S}_1 \# \mathcal{S}_2) = h_n(\mathcal{S}_1) + h_n(\mathcal{S}_2) - 1. \end{aligned} \tag{3.3}$$

Now, let $\dim \mathcal{S}_1 = n_1 - 1$ and $\dim \mathcal{S}_2 = n_2 - 1$; then, we have the following equation for the join $\mathcal{S}_1 * \mathcal{S}_2$:

$$f_k(\mathcal{S}_1 * \mathcal{S}_2) = \sum_{i=-1}^{n_1-1} f_i(\mathcal{S}_1) f_{k-i-1}(\mathcal{S}_2), \qquad k = -1, 0, \dots, n_1 + n_2 - 1.$$

Put

$$h(\mathcal{S}; t) = h_0 + h_1 t + \ldots + h_n t^n.$$

Then, the preceding formula and (3.1) imply

$$h(\mathcal{S}_1 * \mathcal{S}_2; t) = h(\mathcal{S}_1; t) h(\mathcal{S}_2; t). \tag{3.4}$$

Next, we will need transformation formulas for the $f$- and $h$-vectors of simplicial cell complexes under barycentric subdivisions. Introduce the matrix

$$B = (b_{ij}), \quad 0 \le i, j \le n-1, \qquad b_{ij} = \sum_{k=0}^{i} (-1)^k \binom{i+1}{k} (i-k+1)^{j+1}.$$

One can verify that $b_{ij} = 0$ for $i > j$ (i.e., $B$ is an upper triangular matrix) and $b_{ii} = (i+1)!$. Thus, the matrix $B$ is invertible.

**Lemma 3.1.** *Let $\mathcal{S}'$ be the barycentric subdivision of an $(n-1)$-dimensional simplicial cell complex $\mathcal{S}$. Then, the $f$-vectors of the complexes $\mathcal{S}$ and $\mathcal{S}'$ are related by the formula*

$$f_i(\mathcal{S}') = \sum_{j=i}^{n-1} b_{ij} f_j(\mathcal{S}), \qquad i = 0, \dots, n-1,$$

*i.e.,*

$$\boldsymbol{f}(\mathcal{S}') = B \boldsymbol{f}(\mathcal{S}).$$

**Proof.** Consider the barycentric subdivision of a $j$-dimensional simplex $\Delta^j$ and suppose that $b'_{ij}$ is the number of $i$-simplices in $(\Delta^j)'$ that do not lie in $\partial\Delta^j$. Then, we have $f_i(K') = \sum_{j=i}^{n-1} b'_{ij} f_j(K)$. Let us prove that $b_{ij} = b'_{ij}$. Indeed, it is obvious that the number $b'_{ij}$ satisfies the following recurrent relation:

$$b'_{ij} = (j+1)b'_{i-1,j-1} + \binom{j+1}{2} b'_{i-1,j-2} + \ldots + \binom{j+1}{j-i+1} b'_{i-1,i-1}.$$

Hence, one can easily derive by induction that $b'_{ij}$ is defined by the same formula as $b_{ij}$. □

Now, let us introduce the matrix

$$D = (d_{pq}), \quad 0 \le p, q \le n, \qquad d_{pq} = \sum_{k=0}^{p} (-1)^k \binom{n+1}{k} (p-k)^q (p-k+1)^{n-q}$$

(here, we assume that $0^0 = 1$).

**Lemma 3.2.** *The $h$-vectors of the complexes $\mathcal{S}$ and $\mathcal{S}'$ are related by*

$$h_p(\mathcal{S}') = \sum_{q=0}^{n} d_{pq} h_q(\mathcal{S}), \qquad p = 0, \ldots, n,$$

*i.e.,*

$$\boldsymbol{h}(\mathcal{S}') = D\boldsymbol{h}(\mathcal{S}).$$

*Moreover, the matrix $D$ is invertible.*

**Proof.** The lemma is proved by a routine verification with the use of Lemma 3.1, formulas (3.1), and a number of identities for binomial coefficients, which can be found, for example, in [3]. If we add the component $f_{-1} = 1$ to the $f$-vector and appropriately change the matrix $B$, then we obtain the relation $D = C^{-1}BC$, where $C$ is the transition matrix from the $h$-vector to the $f$-vector (the explicit form of this matrix can easily be obtained from (3.1)). This implies the invertibility of the matrix $D$. □

Thus, the barycentric subdivision induces invertible linear operators $B$ and $D$ on the $f$- and $h$-vectors of simplicial cell complexes.

The following fact is a classical result of combinatorial geometry: if a simplicial complex $K$ is a triangulation of an $(n-1)$-dimensional sphere, i.e., $|K| \cong S^{n-1}$, then its $h$-vector is symmetric:

$$h_i = h_{n-i}, \qquad i = 0, \ldots, n. \tag{3.5}$$

These equations (as well as their different expressions in terms of $f$-vectors) are known as the *Dehn–Sommerville relations*. Note that $h_0 = 1$ for any simplicial complex and the first relation $h_0 = h_n$ is equivalent to the formula for the Euler characteristic (see (3.1)). Various proofs of the Dehn–Sommerville relations, as well as an account of the history of the question, can be found in [2].

We will need the following result, which is due to Klee [10].

**Proposition 3.3.** *The Dehn–Sommerville relations are the most general linear equations that hold true for the $f$-vectors of all triangulations of spheres.*

**Proof.** In [10], this proposition was proved in terms of $f$-vectors. However, the use of $h$-vectors significantly simplifies the proof. It is sufficient to show that the affine hull of the $h$-vectors $(h_0, h_1, \ldots, h_n)$ of the triangulations of spheres is an $\left[\frac{n}{2}\right]$-dimensional plane (recall that $h_0 = 1$ always). This can be done, for example, by indicating $\left[\frac{n}{2}\right] + 1$ triangulations of spheres with affinely independent $h$-vectors. Put $K_j := \partial\Delta^j * \partial\Delta^{n-j}$, $j = 0, 1, \ldots, \left[\frac{n}{2}\right]$, where $\partial\Delta^j$ stands for the boundary of a $j$-dimensional simplex. Since $\boldsymbol{h}(\partial\Delta^j) = 1 + t + \ldots + t^j$, it follows from (3.4) that

$$\boldsymbol{h}(K_j) = \frac{1-t^{j+1}}{1-t} \, \frac{1-t^{n-j+1}}{1-t}.$$

Thus, $\boldsymbol{h}(K_{j+1}) - \boldsymbol{h}(K_j) = t^{j+1} +$ higher order terms, $j = 0, 1, \ldots, \left[\frac{n}{2}\right] - 1$. Hence, the vectors $\boldsymbol{h}(K_j)$, $j = 0, 1, \ldots, \left[\frac{n}{2}\right]$, are affinely independent. □

**Theorem 3.4.** *If $\mathcal{S}$ is a simplicial cellular decomposition of an $(n-1)$-dimensional sphere, then the h-vector $\boldsymbol{h}(\mathcal{S})$ satisfies the Dehn–Sommerville relations* (3.5).

**Proof.** Consider the barycentric subdivision $\mathcal{S}'$. By Lemma 3.2, $\boldsymbol{h}(\mathcal{S}') = D\boldsymbol{h}(\mathcal{S})$, where the vector $\boldsymbol{h}(\mathcal{S}')$ is symmetric because $\mathcal{S}'$ is a triangulation of a sphere. A routine test with the use of well-known binomial identities shows that the matrix $D$ (and its inverse) transforms symmetric vectors into symmetric vectors (this is equivalent to $d_{pq} = d_{n+1-p,n+1-q}$, i.e., the matrix $D$ is centrally symmetric). However, this can be proved without resort to computations and even without using the explicit form of the matrix $D$ from Lemma 3.2. Indeed, the Dehn–Sommerville relations determine a linear subspace $W$ of dimension $k = \left[\frac{n}{2}\right] + 1$ (or an affine space of dimension $\left[\frac{n}{2}\right]$ if we add the relation $h_0 = 1$) in the space $\mathbb{R}^{n+1}$ (with coordinates $h_0, \dots, h_n$). We have to verify that this subspace is invariant with respect to the invertible linear operator $D$. To this end, it suffices to choose an arbitrary basis $e_1, \dots, e_k$ in $W$ and verify that $De_i \in W$ for all $i$. However, the subspace $W$ admits a basis composed of the $h$-vectors of simplicial spheres (see the proof of Proposition 3.3). Since the barycentric subdivision of a simplicial sphere is again a simplicial sphere, the images $De_i$, $i = 1, \dots, k$, also satisfy the Dehn–Sommerville relations. Hence, the subspace $W$ is $D$-invariant. This implies that the vector $\boldsymbol{h}(\mathcal{S}) = D^{-1}\boldsymbol{h}(\mathcal{S}')$ satisfies the Dehn–Sommerville relations. $\square$

**Remark.** The proof of the Dehn–Sommerville relations for *Eulerian partially ordered sets* (which include simplicial cellular decompositions of spheres as a particular case) was first obtained by Stanley [13, (3.40)].

In [2], a generalization of the Dehn–Sommerville relations for the triangulations of arbitrary topological manifolds was obtained. Namely, the differences between symmetric components of the $h$-vector of the triangulation of an $(n-1)$-dimensional manifold $M$ are expressed in terms of its Euler characteristic:

$$h_{n-i} - h_i = (-1)^i\bigl(\chi(M) - \chi(S^{n-1})\bigr)\binom{n}{i} = (-1)^i(h_n - 1)\binom{n}{i}, \qquad i = 0, 1, \dots, n. \tag{3.6}$$

In particular, if $M = S^{n-1}$ or $n$ is odd, we obtain relations (3.5). Arguments analogous to those used in the proof of Theorem 3.4 allow us to generalize this result to arbitrary simplicial cellular decompositions of manifolds.

**Theorem 3.5.** *Let $\mathcal{S}$ be a simplicial cellular decomposition of an $(n-1)$-dimensional manifold $M$. Then, the h-vector $\boldsymbol{h}(\mathcal{S}) = (h_0, \dots, h_n)$ satisfies relations* (3.6).

**Proof.** Let $\mathcal{S}'$ be the barycentric subdivision of the complex $\mathcal{S}$ and $D$ be the matrix (operator) from Lemma 3.2 such that $\boldsymbol{h}(\mathcal{S}') = D\boldsymbol{h}(\mathcal{S})$. Let, next, $A$ denote an affine subspace of dimension $k = \left[\frac{n}{2}\right]$ in $\mathbb{R}^{n+1}$ (with coordinates $h_0, \dots, h_n$) defined by the relations from the text of the theorem. Since $\mathcal{S}'$ is a simplicial complex, just as in the proof of Theorem 3.4, we have to verify that $A$ is invariant with respect to $D$. To this end, it suffices to choose a basis in $A$ (i.e., a set of $\left[\frac{n}{2}\right]+1$ affinely independent vectors) that consists of the $h$-vectors of simplicial decompositions of the manifold $M$. This can be done as follows. Consider the $h$-vectors of $\left[\frac{n}{2}\right] + 1$ triangulations of an $(n-1)$-sphere of the form $\partial\Delta^j * \partial\Delta^{n-j}$, which constitute a basis in the subspace $W$ specified by the relations $h_i = h_{n-i}$ (see Proposition 3.3). Then, consider the connected sums $K \# (\partial\Delta^j * \partial\Delta^{n-j})$, where $K$ is a certain fixed triangulation of the manifold $M$. Then, the corresponding $h$-vectors form a basis in $A$ (this easily follows from relations (3.3)). $\square$

**Corollary 3.6.** *Analogues of Theorems* 3.4 *and* 3.5 *hold for pseudosimplicial cellular decompositions of spheres and manifolds, respectively.*

**Proof.** The proof follows from Proposition 2.4. $\square$

(a) (b)

**Fig. 2.** Cellular decompositions of the disk $D^2$

**Example 3.7.** In Fig. 2, two cellular decompositions of the disk $D^2$ are shown. The first of them (a) has two 0-dimensional, two 1-dimensional, and one 2-dimensional cells and is pseudosimplicial. This can be seen as follows. Let us realize $D^2$ as a unit disk in $\mathbb{C}$ and the simplex $\Delta^2$ as a quarter of the unit disk that is specified by the conditions $\operatorname{Re} z \geq 0$ and $\operatorname{Im} z \geq 0$. Then, the mapping $z \mapsto z^4$ defines a characteristic mapping for the two-dimensional cell.

The cellular decomposition of a disk shown in Fig. 2b is not pseudosimplicial. Indeed, otherwise, gluing together two such disks along the boundary, we would obtain a pseudosimplicial cellular decomposition of the two-dimensional sphere. However, the $f$-vector of this cellular decomposition is equal to $(1,1,2)$, and therefore the corresponding $h$-vector $(1,-2,2,1)$ does not satisfy the second of the Dehn–Sommerville relations $h_1 = h_2$. The first relation $h_0 = h_3$, which is equivalent to the formula for the Euler characteristic, is obviously valid.

The definition of a pseudosimplicial cellular decomposition implies that this decomposition must contain cells of all dimensions. The decomposition shown in Fig. 2b gives an example of a situation where the converse proposition does not hold.

## 4. BRANCHED COMBINATORIAL COVERINGS

Here, we characterize simplicial cell complexes as a special class of branched coverings over simplicial complexes. In this section, we deal with geometric simplicial complexes (i.e., with polyhedra) $K$. An *open simplex* $\mathring{\tau} \in K$ is defined as the relative interior of a certain simplex $\tau \in K$. When $\tau$ is a vertex, we put $\mathring{\tau} = \tau$.

Let $X$ be a compact Hausdorff topological space and $K$ be a simplicial complex. A continuous mapping $p\colon X \to K$ is called a *branched combinatorial covering* over $K$ if the following two conditions are fulfilled:

(1) for any open simplex $\mathring{\tau} \in K$, the preimage $p^{-1}(\mathring{\tau})$ is a nonempty disjoint union of a finite number of open sets $U_i(\tau)$:

$$p^{-1}(\tau) = \bigsqcup_i U_i(\tau), \qquad i = 1, \dots, I(\tau);$$

(2) the mapping $p\colon U_i(\tau) \to \mathring{\tau}$ is a homeomorphism for any $i$.

It follows immediately from the definition that the open sets $U_i(\tau)$ corresponding to all simplices $\tau \in K$ and all $i$ define a cellular decomposition of the space $X$.

**Theorem 4.1.** *A space $X$ is a simplicial cell complex if and only if there exists a branched combinatorial covering of $X \to K$ for a certain simplicial complex $K$.*

**Proof.** Let $X$ be a simplicial cell complex with vertex set $\mathcal{M}$. Introduce a simplicial complex $K_{\mathcal{S}}$ on the vertex set $\mathcal{M}$ as follows. We say that a subset $\tau \in \mathcal{M}$ is a simplex in $K_{\mathcal{S}}$ if there exists a simplex $\sigma \in X$ with the vertex set $\tau$. Then, the mapping $X \to K_{\mathcal{S}}$, which is the identity mapping on the vertex set $\mathcal{M}$, sends the simplex $\sigma$ to the corresponding simplex $\tau$ and is a branched combinatorial covering by construction.

Now, let $K$ be a simplicial complex and $p\colon X \to K$ be a certain branched combinatorial covering. Then, $X$ is a cell complex with cells of the form $U_i(\tau)$, $\tau \in K$. In order to establish that $X$ is a simplicial cell complex, it is necessary to verify that the set of cells in the closure of each cell $U_i(\tau)$ forms a partially ordered (by inclusion) set of faces of a certain simplex. This is equivalent to the fact that the restriction of the projection $p$ onto the closure of each cell is a homeomorphism (since the corresponding property holds for $K$). Put $\sigma = \overline{U}_i(\tau)$. We have to prove that $p\colon \sigma \to \tau$ is a homeomorphism. Since $\sigma$ and $\tau$ are compact and Hausdorff, it is sufficient to prove that $p\colon \sigma \to \tau$ is one-to-one. Obviously, this mapping is epimorphic; suppose that $p(x_1) = p(x_2) = y$ for some two different points $x_1, x_2 \in \sigma$. Let us connect the points $x_1$ and $x_2$ by a path segment $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$, and $\gamma(s) \in U_i(\tau)$ for $0 < s < 1$. Then, $p \circ \gamma\colon [0,1] \to K$ is a loop with the beginning and end at $y$, whose interior lies in $\mathring{\tau}$. Since $\tau$ is a simplex, there exists a contracting homotopy $F\colon [0,1]\times[0,1] \to K$ such that $F(s,1) = p\circ\gamma(s)$, $F(s,0) = y$, and $F(s,t) \in \mathring{\tau}$ for $0 < s < 1$ and $0 < t < 1$. Put

$$\gamma_i = p^{-1}\Big(F\Big([0,1], \frac{1}{i}\Big)\Big) \cap \sigma$$

(note that $\gamma_1 = \gamma$). Each subset $\gamma_i \subset X$ is connected (as the closure of the connected subset $p^{-1}(F((0,1), \frac{1}{i})))$ and compact as the preimage of a compact subset under a proper mapping. Consider the upper limit

$$\Gamma = \bigcap_N \overline{\bigcup_{i\ge N} \gamma_i}.$$

Since each subset $\gamma_i$ is connected and compact, $\Gamma$ is also connected and compact. However, on the other hand, $p(\Gamma) = y$, so that $\Gamma$ consists of a finite number of points by the definition of a branched combinatorial covering. The contradiction obtained completes the proof. $\square$

## 5. FACE RINGS

The *face ring* of a simplicial complex is an important concept that allows one to translate combinatorial properties into the language of commutative and homological algebra. This ring was introduced by Stanley and Reisner (see [15]) and is often called a *Stanley–Reisner ring.* A generalization of the face ring to arbitrary simplicial partially ordered sets was first introduces in [14], where important algebraic properties of these rings were also described. However, in the present account, which is intended for topological applications, we follow [11] (see also [2, Ch. 4]).

First, suppose that $K$ is a simplicial complex and identify its vertex set $\mathcal{M}$ with the index set $[m] = \{1, \dots, m\}$.

Let $\mathbb{Z}[v_1, \dots, v_m]$ be a graded ring of polynomials with integer coefficients and with $m$ degree-2 generators. The *Stanley–Reisner ring* (or the *face ring*) of a simplicial complex $K$ is the graded quotient ring

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}_K,$$

where $\mathcal{I}_K$ is the homogeneous ideal generated by the monomials $v_{i_1} \cdot \ldots \cdot v_{i_k}$ for which $\{i_1, \dots, i_k\}$ is not a simplex in $K$.

Next, we slightly modify the previous definition by extending both the set of generators of the polynomial ring and the set of relations. In this case, the quotient ring $\mathbb{Z}[K]$ remains unchanged, and the new definition thus obtained can easily be generalized to arbitrary simplicial partially ordered sets.

For any two simplices $\sigma, \tau \in K$, denote by $\sigma \wedge \tau$ their unique greatest lower bound (i.e., the maximal simplex that is contained simultaneously in $\sigma$ and $\tau$). The greatest lower bound always exists but may be an empty simplex $\varnothing \in K$. On the other hand, the least upper bound of the simplices $\sigma$ and $\tau$ (i.e., the minimal simplex that simultaneously contains $\sigma$ and $\tau$) may not exist. If the least upper bound exists, it is unique, and we denote it by $\sigma \vee \tau$.

Consider a polynomial ring $\mathbb{Z}[v_\sigma\colon \sigma \in K \setminus \varnothing]$, which has one generator per each nonempty simplex in $K$. Introduce a grading by putting $\deg v_\sigma = 2|\sigma|$. In addition, identify $v_\varnothing$ with 1. The following proposition gives an alternative "less economic" representation for the face ring $\mathbb{Z}[K]$.

**Proposition 5.1.** *There is a canonical isomorphism of graded rings*

$$\mathbb{Z}[v_\sigma\colon \sigma \in K \setminus \varnothing]/\mathcal{I} \cong \mathbb{Z}[K],$$

*where $\mathcal{I}$ is the ideal generated by all elements of the form*

$$v_\sigma v_\tau - v_{\sigma\wedge\tau} v_{\sigma\vee\tau}.$$

*Here, we assume that $v_{\sigma\vee\tau} = 0$ if the least upper bound $\sigma \vee \tau$ does not exist.*

**Proof.** The isomorphism is established by a mapping that sends $v_\sigma$ into $\prod_{i\in\sigma} v_i$. $\square$

Now, let $\mathcal{S}$ be an arbitrary simplicial cell complex. Then, for any two elements $\sigma, \tau \in \mathcal{S}$, the set $\sigma \vee \tau$ of their least upper bounds may consist of more than one element (but may also be empty). The set $\sigma \wedge \tau$ of greatest lower bounds is always nonempty; moreover, it consists of a single element provided that $\sigma \vee \tau \neq \varnothing$. Introduce a graded polynomial ring $\mathbb{Z}\bigl[v_\sigma\colon \sigma \in \mathcal{S} \setminus \widehat{0}\bigr]$, where $\deg v_\sigma = 2 \operatorname{rk} \sigma$. We also formally set $v_{\widehat{0}} = 1$.

The *face ring* of a simplicial cell complex $\mathcal{S}$ is the quotient ring

$$\mathbb{Z}[\mathcal{S}] := \mathbb{Z}\bigl[v_\sigma\colon \sigma \in \mathcal{S} \setminus \widehat{0}\bigr]/\mathcal{I}_{\mathcal{S}},$$

where $\mathcal{I}_{\mathcal{S}}$ is the ideal generated by all elements of the form

$$v_\sigma v_\tau - v_{\sigma\wedge\tau} \sum_{\rho\in\sigma\vee\tau} v_\rho. \tag{5.1}$$

In particular, if $\sigma \vee \tau = \varnothing$, then $v_\sigma v_\tau = 0$ in $\mathbb{Z}[\mathcal{S}]$.

The relations that generate the ideal $\mathcal{I}_{\mathcal{S}}$ allow one to express the product of the generators $v_\sigma$ and $v_\tau$ that correspond to incomparable elements $\sigma, \tau \in \mathcal{S}$ as a sum of monomials each of which is a product of ordered generators. In the algebraic literature, such relations are called *straightening* relations.

**Example 5.2.** Consider the complex $\mathcal{S}$ described in Example 2.1 for $n = 2$. Thus, $\mathcal{S}$ is obtained by gluing together two segments along their boundaries. We have two elements of rank 1 (two vertices), say, $\sigma_1$ and $\sigma_2$, and two elements of rank 2 (two maximal simplices), say, $\tau_1$ and $\tau_2$. Then, the face ring is expressed as

$$\mathbb{Z}[\mathcal{S}] = \mathbb{Z}[v_{\sigma_1}, v_{\sigma_2}, v_{\tau_1}, v_{\tau_2}]/(v_{\tau_1} + v_{\tau_2} = v_{\sigma_1} v_{\sigma_2},\ v_{\tau_1} v_{\tau_2} = 0),$$

where $\deg v_{\sigma_1} = \deg v_{\sigma_2} = 2$ and $\deg v_{\tau_1} = \deg v_{\tau_2} = 4$.

It was pointed out in [2, Ch. 3] that a simplicial mapping $\phi\colon K_1 \to K_2$ of simplicial complexes induces a mapping of the face rings $\phi^*\colon \mathbb{Z}[K_2] \to \mathbb{Z}[K_1]$, i.e., the face ring has a contravariant character. A similar property also holds for arbitrary simplicial cell complexes.

**Proposition 5.3.** *Let $\phi\colon \mathcal{S}_1 \to \mathcal{S}_2$ be a simplicial mapping. Define a mapping of the graded polynomial rings*

$$\phi^*\colon \mathbb{Z}\bigl[v_\tau\colon \tau \in \mathcal{S}_2 \setminus \widehat{0}\bigr] \to \mathbb{Z}\bigl[v_\sigma\colon \sigma \in \mathcal{S}_1 \setminus \widehat{0}\bigr], \qquad v_\tau \mapsto \sum v_\sigma,$$

*where the sum is over all simplices $\sigma \in \mathcal{S}_1$ such that $\phi(\sigma) = \tau$ and $\dim \sigma = \dim \tau$. Then, the mapping $\phi^*$ induces a mapping of the face rings $\mathbb{Z}[\mathcal{S}_2] \to \mathbb{Z}[\mathcal{S}_1]$ (which will be denoted by the same symbol).*

**Proof.** One can directly verify that $\phi^*(\mathcal{I}_{\mathcal{S}_2}) \subset \mathcal{I}_{\mathcal{S}_1}$. $\square$

Let $\mathcal{S}$ be a simplicial cell complex of dimension $n$. By analogy with the above procedure, we define a face ring $\mathbf{k}[\mathcal{S}]$ over an arbitrary commutative ring $\mathbf{k}$ with unity (along with $\mathbf{k} = \mathbb{Z}$, we are interested in the case $\mathbf{k} = \mathbb{Q}$). A sequence $t_1, \dots, t_n$ of algebraically independent homogeneous elements of the ring $\mathbf{k}[\mathcal{S}]$ is called a *homogeneous system of parameters* if $\mathbf{k}[\mathcal{S}]$ is a finitely generated $\mathbf{k}[t_1, \dots, t_n]$-module. Thus, the sequence of homogeneous elements $t_1, \dots, t_k$ of length $k \le n$ is a part of the homogeneous system of parameters if and only if $\dim \mathbf{k}[\mathcal{S}]/(t_1, \dots, t_k) = n - k$, where dim denotes the *Krull dimension*. A system of parameters that consists of degree-2 elements is called *linear*. A sequence of elements $t_1, \dots, t_k \in \mathbf{k}[\mathcal{S}]$ is called *regular* if $\mathbf{k}[\mathcal{S}]$ is a free $\mathbf{k}[t_1, \dots, t_k]$-module. A homogeneous regular sequence is a part of the homogeneous system of parameters; however, the converse is not true. A ring $\mathbf{k}[\mathcal{S}]$ is called a *Cohen–Macaulay ring* (and a complex $\mathcal{S}$ is called a *Cohen–Macaulay complex*) if it admits a regular sequence of length $n = \dim \mathbf{k}[\mathcal{S}] = \dim \mathcal{S} + 1$. An example of the Cohen–Macaulay ring is given by a simplicial cellular decomposition of a sphere [14].

For every simplex $\sigma \in \mathcal{S}$, define the *restriction homomorphism*

$$s_\sigma \colon \mathbf{k}[\mathcal{S}] \to \mathbf{k}[\mathcal{S}]/(v_\tau \colon \tau \not\le \sigma).$$

Let $\dim \sigma = k - 1$ and $\{i_1, \dots, i_k\}$ be the vertex set of the simplex $\sigma$. Then, it is obvious that the image of the homomorphism $s_\sigma$ is a polynomial ring $\mathbf{k}[v_{i_1}, \dots, v_{i_k}]$ of $k$ degree-2 generators. The following proposition gives a characterization of linear systems of parameters in the face rings of simplicial cell complexes and generalizes the corresponding proposition [15, Lemma III.2.4] for simplicial complexes (see also [6, Theorem 7.2]).

**Theorem 5.4.** *A sequence $\boldsymbol{t} = (t_1, \dots, t_n)$ of degree-2 elements of the ring $\mathbf{k}[\mathcal{S}]$ is a linear system of parameters if and only if, for every simplex $\sigma \in \mathcal{S}$, the sequence $s_\sigma(\boldsymbol{t})$ generates the ring of polynomials $\mathbf{k}[v_i \colon i \in \sigma]$.*

**Proof.** First, suppose that $\boldsymbol{t}$ is a linear system of parameters. The mapping $s_\sigma$ induces an epimorphism of quotient rings:

$$\mathbf{k}[\mathcal{S}]/(\boldsymbol{t}) \to \mathbf{k}[v_i \colon i \in \sigma]/s_\sigma(\boldsymbol{t}).$$

Since $\boldsymbol{t}$ is a system of parameters, it follows that $\dim \mathbf{k}[\mathcal{S}]/(\boldsymbol{t}) = 0$, i.e., $\dim_{\mathbf{k}} \mathbf{k}[\mathcal{S}]/(\boldsymbol{t}) < \infty$. Therefore, $\mathbf{k}[v_i \colon i \in \sigma]/s_\sigma(\boldsymbol{t}) < \infty$. However, this happens only when $s_\sigma(\boldsymbol{t})$ multiplicatively generates a polynomial ring over $\mathbf{k}$.

Now, suppose that, for every $\sigma \in \mathcal{S}$, the set $s_\sigma(\boldsymbol{t})$ generates the ring $\mathbf{k}[v_i \colon i \in \sigma]$. Then, we have

$$\dim_{\mathbf{k}} \bigoplus_{\sigma \in \mathcal{S}} \mathbf{k}[v_i \colon i \in \sigma]/s_\sigma(\boldsymbol{t}) < \infty.$$

Moreover, the sum $s \colon \mathbf{k}[\mathcal{S}] \to \bigoplus_{\sigma \in \mathcal{S}} \mathbf{k}[v_i \colon i \in \sigma]$ of restriction homomorphisms is a monomorphism [2, Theorem 4.8]. Hence, $\dim_{\mathbf{k}} \mathbf{k}[\mathcal{S}]/(\boldsymbol{t}) < \infty$ as well (see [5, Lemma 4.7.1]). Thus, $\boldsymbol{t}$ is a linear system of parameters. $\square$

It is clear that the theorem remains valid if we consider only restrictions $s_\sigma(\boldsymbol{t})$ onto maximal simplices $\sigma \in \mathcal{S}$. In particular, if $\mathcal{S}$ is a pure complex (all maximal simplices have dimension $n - 1$), then a sequence $t_1, \dots, t_n$ is a linear system of parameters if and only if its restriction onto each $(n-1)$-simplex yields a basis in the space of linear forms (over $\mathbf{k}$).

Let $K_{\mathcal{S}}$ be the simplicial complex constructed by the simplicial cell complex $\mathcal{S}$ in the proof of Theorem 4.1. It is obvious that the ring $\mathbf{k}[K_{\mathcal{S}}]$ coincides with a subring of the ring $\mathbf{k}[\mathcal{S}]$ generated by degree-2 elements.

**Lemma 5.5.** *The ring $\mathbb{Q}[\mathcal{S}]$ admits a linear system of parameters.*

**Proof.** If $\mathcal{S}$ is a simplicial complex, then the ring $\mathbb{Q}[\mathcal{S}]$ is generated by linear elements, and the assertion of the lemma follows from the Noether normalization lemma (see, e.g., [5, Theorem 1.5.17]). In the general case, Theorem 5.4 implies that the linear system of parameters in the ring $\mathbb{Q}[K_{\mathcal{S}}]$ is also a linear system of parameters for $\mathbb{Q}[\mathcal{S}]$. □

**Example 5.6.** In the face ring from Example 5.2, the set $v_{\sigma_1}, v_{\sigma_2}$ forms a linear system of parameters, while the elements $v_{\tau_1}$ and $v_{\tau_2}$ are the roots of the algebraic equation $x^2-(v_{\sigma_1}v_{\sigma_2})x=0$.

The question of the existence of a liner system of parameters in the ring $\mathbb{Z}[\mathcal{S}]$ is much more delicate even when $\mathcal{S}$ is a simplicial complex (in this case, the Noether normalization lemma states only the existence of a nonlinear homogeneous system of parameters). This question is closely related to the calculation of the rank of a freely acting torus on certain manifolds (see Section 6 below). Here, we only note that an example of a simplicial complex $K$ for which the ring $\mathbb{Z}[K]$ does not admit a linear system of parameters is given by the boundary of a *cyclic polyhedron* $C^n(m)$ with $m \geq 2^n \geq 16$ vertices. This example was constructed in [6, Example 1.22] (it is also presented in [2, Example 6.33]).

**Lemma 5.7.** *Let $v_1, \ldots, v_m$ be degree-2 elements in the ring $\mathbb{Z}[\mathcal{S}]$ that correspond to the vertices of the complex $\mathcal{S}$. Then, the following identity holds in the ring $\mathbb{Z}[\mathcal{S}]$:*

$$(1+v_1)\cdot \ldots \cdot (1+v_m) = \sum_{\sigma\in\mathcal{S}} v_\sigma. \tag{5.2}$$

**Proof.** Let $\operatorname{ver}\sigma$ denote the vertex set of the element $\sigma\in\mathcal{S}$. Relations (5.1) in the ring $\mathbb{Z}[\mathcal{S}]$ imply the relations

$$v_{i_1}\cdot \ldots \cdot v_{i_k} = \sum_{\sigma\colon \operatorname{ver}\sigma=\{i_1,\ldots,i_k\}} v_\sigma. \tag{5.3}$$

Summing up these relations over all $\sigma\in\mathcal{S}$, we obtain the required identity. □

The ring $\mathbb{Z}[\mathcal{S}]$ admits a canonical $\mathbb{N}^m$-grading defined as $\operatorname{mdeg} v_\sigma = 2\sum_{i\in\operatorname{ver}\sigma} e_i$, where $e_i\in\mathbb{N}^m$ is the $i$th basis vector. In particular, $\operatorname{mdeg} v_i = 2e_i$. If all the rings are assumed to be multigraded, then formula (5.2) is equivalent to the set of relations (5.3). It follows from Proposition 5.1 that relations (5.3) generate the ideal $\mathcal{I}_{\mathcal{S}}$ in the case when $\mathcal{S}$ is a simplicial complex. However, the following example shows that, for an arbitrary simplicial partially ordered set $\mathcal{S}$, the face ring $\mathbb{Z}[\mathcal{S}]$ may not be isomorphic to the quotient ring of the $\mathbb{N}^m$-graded polynomial ring $\mathbb{Z}[v_\sigma\colon \sigma\in\mathcal{S}]$ by relation (5.2).

**Example 5.8.** Consider a simplicial cell complex $\mathcal{S}$ obtained by identifying two 2-simplices $\tau_1$ and $\tau_2$ at three their vertices. Consider two edges $\varepsilon_1\subset\tau_1$ and $\varepsilon_2\subset\tau_2$ that have common vertices, and let $\pi$ be the opposite vertex of the complex $\mathcal{S}$. Then, the following relations hold in the ring $\mathbb{Z}[\mathcal{S}]$: $v_{\varepsilon_1}v_\pi = v_{\tau_1}$ and $v_{\varepsilon_2}v_\pi = v_{\tau_2}$; however, relation (5.2) only implies $v_{\varepsilon_1}v_\pi + v_{\varepsilon_2}v_\pi = v_{\tau_1}+v_{\tau_2}$.

The polynomial $P_{\mathcal{S}} = \sum_{\sigma\in\mathcal{S}} v_\sigma$ was considered by Alexander in [4] (1930) in relation to the formalization of the concept of simplicial complex in combinatorial topology (however, as we pointed out in the introduction, the concept of face ring appeared much later).

A natural question arises: To what extent a simplicial partially ordered set $\mathcal{S}$ is determined by its face ring $\mathbb{Z}[\mathcal{S}]$? In the case of simplicial complexes, the following simple proposition holds.

**Proposition 5.9.** *The ring homomorphism $F\colon \mathbb{Z}[K_2]\to\mathbb{Z}[K_1]$ is a homogeneous degree-0 isomorphism of $\mathbb{N}^m$-graded face rings of two simplicial complexes if and only if $F$ is induced by a simplicial isomorphism $K_1\to K_2$.*

**Proof.** Let $F$ be a homogeneous degree-0 isomorphism of $\mathbb{N}^m$-graded rings. Then, it establishes a bijection between the vertex sets of the complexes $K_1$ and $K_2$. Therefore, $F(P_{K_1}) = P_{K_2}$, and the assertion follows from relation (5.2). The converse is obvious. □

Even the simplest example of a simplex (when the face ring is a polynomial ring) shows that one cannot replace "isomorphism of $\mathbb{N}^m$-graded rings" by "isomorphism of graded rings" in the above proposition. In order that the isomorphism $F\colon \mathbb{Z}[K_2] \to \mathbb{Z}[K_1]$ of graded rings be induced by a simplicial isomorphism, one should additionally require that $F(P_{K_1}) = P_{K_2}$. At the same time, we do not know whether there exist nonisomorphic simplicial partially ordered sets with isomorphic face rings.

A simplicial mapping is called *nondegenerate* if its restriction onto each simplex is a simplicial isomorphism. The following proposition is obvious.

**Proposition 5.10.** *If $\phi\colon \mathcal{S}_1 \to \mathcal{S}_2$ is a nondegenerate simplicial mapping, then we have $\phi^*(P_{\mathcal{S}_2}) = P_{\mathcal{S}_1}$.*

Simple examples show that the relation $\phi^*(P_{\mathcal{S}_2}) = P_{\mathcal{S}_1}$ does not hold for arbitrary simplicial mappings.

## 6. TORUS ACTIONS

Let $K$ be a simplicial complex with $m$ vertices. In [2, § 5.2], we described a cubic decomposition $\mathrm{cc}(K)$ of a cone $\mathsf{cone}\, K'$ over the barycentric subdivision of the simplicial complex $K$. This cubic decomposition is obtained by identifying the cone $\mathsf{cone}(\Delta^{m-1})'$ over the barycentric subdivision of an $(m-1)$-dimensional simplex with the standard triangulation of a cube $I^m$ and considering the embedding $\mathsf{cone}\, K' \subset \mathsf{cone}(\Delta^{m-1})'$. (This embedding is induced by the canonical embedding $K \subset \Delta^{m-1}$.) Next, in [2, § 7.2], we constructed a *moment–angle complex* $\mathcal{Z}_K$ as the pullback that closes the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_K \arrow[r] \arrow[d] & (D^2)^m \arrow[d] \\ \mathrm{cc}(K) \arrow[r] & I^m \end{tikzcd}$$

where the right arrow is a projection onto the orbit space of the standard action of an $m$-dimensional torus on a polydisk. Thus, if $\dim K = n-1$, then $\mathcal{Z}_K$ is an $(m+n)$-dimensional space with a $T^m$-action. This space was first introduced by Davis and Januszkiewicz in [6] in relation to the construction of topological analogues of algebraic toric varieties. In [2], we devoted Chapters 7 and 8 to the study of various topological properties of $\mathcal{Z}_K$ and their relations to the combinatorics of the complex $K$. For instance, it is easy to prove that if $K$ is a triangulation of an $(n-1)$-dimensional sphere, then $\mathcal{Z}_K$ is an $(m+n)$-dimensional manifold (see, for example, [2, Lemma 7.13]). Further in the present paper, using Theorem 4.1, we generalize some of the constructions related to the space $\mathcal{Z}_K$ to the case of arbitrary simplicial cell complexes.

Let $\mathcal{S}$ be an arbitrary simplicial cell complex with $m$ vertices. Consider the composition $\mathsf{cone}\,\mathcal{S}' \to \mathsf{cone}\, K' \to I^m$ of the mapping induced by the branched combinatorial covering $p\colon \mathcal{S} \to K$ from Theorem 4.1 and the above-described embedding into a cube. Define a space $\mathcal{Z}_\mathcal{S}$ with a $T^m$-action that is induced by this composition:

$$\begin{tikzcd} \mathcal{Z}_\mathcal{S} \arrow[r] \arrow[d, "\rho"'] & \mathcal{Z}_K \arrow[r] \arrow[d] & (D^2)^m \arrow[d] \\ \mathsf{cone}\,\mathcal{S}' \arrow[r] & \mathsf{cone}\, K' \arrow[r] & I^m \end{tikzcd}$$

**Example 6.1.** Let $\mathcal{S}$ be a simplicial cell complex obtained by the identification of two $(n-1)$-dimensional simplices along their boundaries (see Example 2.1). Then, $\mathcal{Z}_\mathcal{S}$ is obtained by the identification of two polydisks $(D^2)^n$ along their boundaries. Hence, $\mathcal{Z}_\mathcal{S} \cong S^{2n}$. In another aspect, this manifold with a $T^n$-action was considered in [11] as the first example of a torus manifold that is not a quasitoric variety in the sense of [6].

The following proposition follows obviously from the construction of the space $\mathcal{Z}_{\mathcal{S}}$.

**Proposition 6.2.** *The isotropy subgroups* (*stabilizers*) *of the $T^m$-action on $\mathcal{Z}_{\mathcal{S}}$ are coordinate subtori of the form $T^{\operatorname{ver}\sigma} \subset T^m$, where $\sigma$ is a certain simplex in $\mathcal{S}$.*

Let $X$ be a space and $K_1$ and $K_2$ be two triangulations of $X$, i.e., two simplicial complexes such that $|K_1| \cong |K_2| \cong X$. Two such triangulations are called *combinatorially equivalent* (or *piecewise linearly isomorphic*) if there exists a simplicial complex $K$ that is a subdivision of both complexes $K_1$ and $K_2$. A triangulation of an $(n-1)$-dimensional sphere $S^{n-1}$ is called *piecewise linear* (or, shortly, a *$PL$-sphere*) if it is combinatorially equivalent to the boundary of the simplex $\partial\Delta^n$. The concepts of combinatorial equivalence and $PL$-sphere are directly carried over to the simplicial cell complexes $\mathcal{S}$ by passing to the barycentric subdivision $\mathcal{S}'$.

**Theorem 6.3.** *Let $\mathcal{S}$ be a piecewise linear simplicial cellular decomposition of a sphere $S^{n-1}$ with $m$ vertices. Then, $\mathcal{Z}_{\mathcal{S}}$ is an $(m+n)$-dimensional manifold.*

**Proof.** Consider the complex dual to the simplicial complex $\mathcal{S}'$. This complex has one hyperface (a face of codimension 1) $F_v$ per each vertex $v$ of the complex $\mathcal{S}'$, where

$$F_v := \operatorname{star}_{\mathcal{S}'} v = \{\sigma \in \mathcal{S}' :\ v \cup \sigma \in \mathcal{S}'\}.$$

Faces of codimension $k$ are defined as nonempty intersections of the sets of $k$ hyperfaces. Since $\mathcal{S}'$ is a $PL$-sphere, each $i$-dimensional face is piecewise linearly homeomorphic to an $i$-dimensional ball. For each vertex $v \in \mathcal{S}'$, denote by $U_v$ an open subset in $\mathsf{cone}(\mathcal{S}')$ obtained by removing from $\mathcal{S}'$ all the faces that do not contain $v$. Then, $\{U_v\}$ is an open covering of the space $\mathsf{cone}(\mathcal{S}')$; moreover, each $U_v$ is homeomorphic to an open subset in $\mathbb{R}^n_+$ while preserving the codimension of faces. Thus, $\mathsf{cone}(\mathcal{S}')$ acquires the structure of a *manifold with corners* (see [2, Definition 6.13]). At the same time, every point of $\mathcal{Z}_{\mathcal{S}} = \rho^{-1}(\mathsf{cone}(\mathcal{S}'))$ lies in one of the subsets $\{\rho^{-1}(U_v)\}$, which is an open subset in $\mathbb{R}^{2n} \times T^{m-n}$. Since the latter is a manifold, so is $\mathcal{Z}_{\mathcal{S}}$. $\square$

A $T^k$-action on the space $X$ is called *almost free* if all the isotropy subgroups are finite. The *toral rank* of the space $X$ is the greatest number $k$ for which there exists an almost free $T^k$-action on $X$. Denote the toral rank by $\operatorname{trk}(X)$.

The following theorem was proved in [6, § 7.1] in the case of a simplicial complex $\mathcal{S}$; however, the arguments also apply to the general case.

**Theorem 6.4.** *Let $\mathcal{S}$ be a simplicial cell complex of dimension $n-1$ with $m$ vertices. Then,* $\operatorname{trk}\mathcal{Z}_{\mathcal{S}} \geq m-n$.

**Proof.** Let us choose a linear system of parameters $t_1, \dots, t_n$ in the ring $\mathbb{Q}[\mathcal{S}]$ according to Lemma 5.5. Write $t_i = \lambda_{i1}v_1 + \ldots + \lambda_{im}v_m$, $i = 1, \dots, n$; then, the matrix $(\lambda_{ij})$ defines a linear mapping $\lambda\colon \mathbb{Q}^m \to \mathbb{Q}^n$. Replacing, if necessary, $\lambda$ by $k\lambda$ with a sufficiently large $k$, we can assume that the mapping $\lambda$ is induced by the mapping $\mathbb{Z}^m \to \mathbb{Z}^n$, which we will also denote by $\lambda$. It follows from Theorem 5.4 that, for any simplex $\sigma \in \mathcal{S}$, the restriction $\lambda|_{\mathbb{Z}^{\operatorname{ver}\sigma}}\colon \mathbb{Z}^{\operatorname{ver}\sigma} \to \mathbb{Z}^n$ of the mapping $\lambda$ onto the coordinate subspace $\mathbb{Z}^{\operatorname{ver}\sigma} \subset \mathbb{Z}^m$ is injective. Denote by $N$ a subgroup in $T^m$ that corresponds to the kernel of the mapping $\lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$. Then, $N$ is a product of an $(m-n)$-dimensional torus and a finite group, and $N$ intersects coordinate subgroups of the form $T^{\operatorname{ver}\sigma} \subset T^m$ along finite subgroups. Proposition 6.2 implies that $N$ acts on $\mathcal{Z}_{\mathcal{S}}$ almost freely; this completes the proof. $\square$

The following *toral rank conjecture* is well-known (Halperin [7]): the inequality

$$\dim H^*(X; \mathbb{Q}) \geq 2^{\operatorname{trk}(X)}$$

holds for any finite-dimensional space $X$. This conjecture has been proved in many particular cases. Our approach provides a rich class of torus actions. Theorem 6.4 implies that if the toral rank conjecture holds, then

$$\dim H^*(\mathcal{Z}_{\mathcal{S}};\mathbb{Q}) \geq 2^{m-n}.$$

In the case of a simplicial complex $K$, we proved in [1] that

$$H^*(\mathcal{Z}_K;\mathbb{Q}) \cong \bigoplus_{\omega\subseteq[m]} \widetilde{H}^*(K_\omega;\mathbb{Q}),$$

where $K_\omega$ is a full subcomplex in $K$ spanned over the vertex subset $\omega \subseteq [m]$. (A stronger result was obtained [2, Corollary 8.8].)

**Corollary 6.5.** *Under the toral rank conjecture, the following inequality holds for any simplicial complex $K$:*

$$\dim \bigoplus_{\omega\subseteq[m]} \widetilde{H}^*(K_\omega;\mathbb{Q}) \geq 2^{m-n}.$$

This assertion, concerning the combinatorial structure of simplicial complexes, has been confirmed in a number of cases.

For free torus actions, the arguments used in the proof of Theorem 6.4 lead to the following proposition.

**Theorem 6.6.** *In the torus $T^m$, a toral subgroup of dimension $m-n$ that acts freely on $\mathcal{Z}_{\mathcal{S}}$ exists if and only if the ring $\mathbb{Z}[\mathcal{S}]$ admits a linear system of parameters.*

The maximal number $k$ for which there exists a toral subgroup of dimension $k$ in $T^m$ that acts freely on $\mathcal{Z}_{\mathcal{S}}$ is denoted by $s(\mathcal{S})$ and is an important combinatorial invariant of a simplicial cell complex $\mathcal{S}$ (for the case of simplicial complexes, see the discussion in [2, § 7.1]). Note that, for simplicial complexes, the diagonal circle subgroup in $T^m$ acts on $\mathcal{Z}_K$ freely (indeed, we have $m > n$, and therefore the diagonal subgroup intersects each isotropy subgroup $T^{\mathrm{ver}\,\sigma}$ only at unity). Thus, $s(K) \geq 1$, and the Euler characteristic of the space $\mathcal{Z}_K$ is equal to zero. Example 6.1 shows that this is not the case for simplicial cell complexes $\mathcal{S}$.

## ACKNOWLEDGMENTS

We are grateful to G. Lupton, who turned our attention to the toral rank conjecture, and to E.V. Shchepin for a helpful suggestion that was used in the proof of Theorem 4.1.

This work was supported by the grant of the President of the Russian Federation (project no. NSh-2185.2003.1) and by the Russian Foundation for Basic Research (project nos. 02-01-00659 and 04-01-00702).

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*Translated by I. Nikitin*