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TORUS ACTIONS, EQUIVARIANT MOMENT-ANGLE COMPLEXES, AND COORDINATE SUBSPACE ARRANGEMENTS

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We show that the cohomology algebra of the complement of a coordinate subspace arrangement in the m-dimensional complex space is isomorphic to the cohomology algebra of the Stanley–Reisner face ring of a certain simplicial complex on m vertices. (The face ring is regarded as a module over the polynomial ring on m generators.) After that we calculate the latter cohomology algebra by means of the standard Koszul resolution of a polynomial ring. To prove these facts, we construct a homotopy equivalence (equivariant with respect to the torus action) between the complement of a coordinate subspace arrangement and the moment-angle complex defined by a simplicial complex. The moment-angle complex is a certain subset of the unit polydisk in the m-dimensional complex space invariant with respect to the action of the m-dimensional torus. This complex is a smooth manifold provided that the simplicial complex is a simplicial sphere; otherwise, the complex has a more complicated structure. Then we investigate the equivariant topology of the moment-angle complex and apply the Eilenberg–Moore spectral sequence. We also relate our results with well-known facts in the theory of toric varieties and symplectic geometry. Bibliography: 23 titles.

1. Introduction

In this paper, we apply the results of our previous work [6] to describing the topology of the complement of a complex coordinate subspace arrangement. A coordinate subspace arrangement $\mathcal{A}$ is a set of coordinate subspaces L of the complex space $\mathbb{C}^m$, and its complement is the set $U(\mathcal{A}) = \mathbb{C}^m \setminus \bigcup_{L\in\mathcal{A}} L$. The complement $U(\mathcal{A})$ decomposes as $U(\mathcal{A}) = U(\mathcal{A}') \times (\mathbb{C}^*)^k$, where $\mathcal{A}'$ is a coordinate arrangement in $\mathbb{C}^{m-k}$ containing no hyperplanes. There is a one-to-one correspondence between coordinate subspace arrangements in $\mathbb{C}^m$ without hyperplanes and simplicial complexes on m vertices $v_1, \dots, v_m$; each arrangement $\mathcal{A}$ defines a simplicial complex $K(\mathcal{A})$ and vice versa. Namely, let $|\mathcal{A}|$ denote the support $\bigcup_{L\in\mathcal{A}} L$ of the coordinate subspace arrangement $\mathcal{A}$; then a subset $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ is a $(k-1)$-simplex of $K(\mathcal{A})$ if and only if the $(m-k)$-dimensional coordinate subspace $L_I \subset \mathbb{C}^m$ defined by the equations $z_{i_1} = \ldots = z_{i_k} = 0$ does not belong to $|\mathcal{A}|$. An arrangement $\mathcal{A}$ is obviously recovered from its simplicial complex $K(\mathcal{A})$. For this reason, we write $U(K)$ instead of $U(\mathcal{A}(K))$ throughout this paper. (For more information about relations between arrangements and simplicial complexes, see the beginning of Sec. 2.)

Subspace arrangements and their complements play a pivotal role in many constructions of combinatorics, algebraic and symplectic geometry, mechanics, etc. They also arise as configuration spaces of various classical systems. For this reason, the topology of complements of arrangements attracted the attention of many mathematicians during the last two decades. The first important result in this field deals with arrangements of hyperplanes (not necessarily coordinate) in $\mathbb{C}^m$. Arnold [1] and Brieskorn [4] showed that the cohomology algebra of the corresponding complement $U(\mathcal{A})$ is isomorphic to the algebra of differential forms generated by the closed forms $\frac{1}{2\pi i}\frac{dF_A}{F_A}$, where F_A is a linear form defining the hyperplane A of the arrangement. Orlik and Solomon [18] proved that the cohomology algebra of the complement of a hyperplane arrangement depends only on the combinatorics of intersections of hyperplanes and represented $H^*(U(\mathcal{A}))$ by generators and relations. In the general situation, the Goresky–MacPherson theorem [15, Part III] expresses the cohomology groups $H^i(U(\mathcal{A}))$ (without ring structure) as the sum of homology groups of subcomplexes of a certain simplicial complex. This complex, called the *order* (or *flag*) complex, is defined via the combinatorics of intersections of subspaces of $\mathcal{A}$. The proof of the above-mentioned result uses the stratified Morse theory developed in [15]. Another way of describing the cohomology algebra of the complement of a subspace arrangement was recently presented by De Concini and Procesi [12]. They proved that the rational cohomology ring of $U(\mathcal{A})$ is also determined by the combinatorics of intersections. This result was extended by Yuzvinsky in [23]. In the case of coordinate subspace arrangements, the order complex is the barycentric subdivision of a simplicial complex $\widetilde{K}$, while the summands in the Goresky–MacPherson formula are homology groups of links of simplices of $\widetilde{K}$. The complex $\widetilde{K}$ has the same vertex set $v_1, \dots, v_m$ as our simplicial complex K and is "dual" to the latter in the following sense: a set $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ spans a simplex of $\widetilde{K}$ if and only if the complement $\{v_1, \dots, v_m\} \setminus v_I$ does not span a simplex

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of K. The product of cohomology classes of the complement of a coordinate subspace arrangement was described in [13] in combinatorial terms using the complex $\widetilde{K}$ and the above interpretation of the Goresky–MacPherson formula.

In this paper, we prefer to describe a coordinate subspace arrangement in terms of the simplicial complex K instead of $\widetilde{K}$, since our approach clarifies new connections between the topology of complements of subspace arrangements, commutative algebra, and the geometry of *toric varieties*. We show that the complement $U(K)$ is homotopically equivalent to the so-called *moment-angle complex* $\mathcal{Z}_K$ defined by the simplicial complex K. This complex $\mathcal{Z}_K$ is a compact subset of the unit polydisk $(D^2)^m \subset \mathbb{C}^m$ invariant with respect to the standard T^m-action on $(D^2)^m$. At the same time, the complex $\mathcal{Z}_K$ is a homotopy fiber of the cellular embedding $i : \widetilde{B_T K} \hookrightarrow BT^m$, where BT^m is the T^m-classifying space with standard cellular structure and $\widetilde{B_T K}$ is a cell subcomplex whose cohomology is isomorphic to the *Stanley–Reisner face ring* $\mathbf{k}(K)$ of the simplicial complex K. Then we calculate the cohomology algebra of $\mathcal{Z}_K$ (or of $U(K)$) by means of the Eilenberg–Moore spectral sequence. As a result, we obtain an algebraic description of the cohomology algebra of $U(K)$ as the bigraded cohomology algebra $\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$ of the face ring $\mathbf{k}(K)$. By means of the standard Koszul resolution, the latter algebra can be expressed as the cohomology of the differential bigraded algebra $\mathbf{k}(K) \otimes \Lambda[u_1,\dots,u_m]$, where $\Lambda[u_1,\dots,u_m]$ is an exterior algebra and the differential maps the exterior generator u_i to $v_i \in \mathbf{k}(K) = \mathbf{k}[v_1,\dots,v_m]/I$. The rational models of De Concini and Procesi [12] and Yuzvinsky [23] can also be interpreted as an application of the Koszul resolution to the cohomology of the complement subspace arrangement; however the role of the face ring became clear only after our work [6].

If K is an $(n-1)$-dimensional simplicial sphere (for instance, if K is the boundary complex of an n-dimensional convex simplicial polytope), our moment-angle complex $\mathcal{Z}_K$ is a smooth $(m+n)$-dimensional manifold (hence, $U(K)$ is homotopically equivalent to a smooth manifold). This important particular case of our constructions was studied in detail in [5, 6]. The topological properties of the above-mentioned manifolds $\mathcal{Z}_K$ are of great interest due to their relations with combinatorics of polytopes, symplectic geometry, and geometry of toric varieties; the latter relation was the starting point in our study of coordinate subspace arrangements. The classical definition of toric varieties (see [10, 14]) deals with a combinatorial object known as a *fan*. However, as was recently shown by several authors (see, for example, [2, 3, 9]), in the case where the fan defining a toric variety M is simplicial, M can be defined as the geometric quotient of the complement $U(K)$ with respect to a certain action of the algebraic torus $(\mathbb{C}^*)^{m-n}$ (here K is the simplicial complex defined by the fan). Our moment-angle manifold $\mathcal{Z}_K$ is the preimage of a regular point in the image of the moment map $U(K) \to \mathbb{R}^{m-n}$ for the Hamiltonian action of a compact torus $T^{m-n} \subset (\mathbb{C}^*)^{m-n}$.

In [11], Davis and Januszkiewicz introduced the notion of a toric manifold (now also known as a quasitoric manifold or unitary toric manifold), which can be regarded as a natural topological extension of the notion of a smooth toric variety. A (quasi)toric manifold M^{2n} admits a smooth action of the torus T^n that locally looks like the standard action of T^n on $\mathbb{C}^n$; the orbit space is required to be an n-dimensional ball furnished with the combinatorial structure of a simple convex polytope by the fixed-point sets of appropriate subtori. Topology, geometry, and combinatorics of quasitoric manifolds are very beautiful; after the pioneering paper [11], many new relations were discovered by different authors (see [7, 8, 5, 6, 19, 20] and more references therein). The dual complex to the boundary complex of a simple polytope in the orbit space of a quasitoric manifold is a simplicial sphere. That is why many results from the present paper may be considered as an extension of our previous constructions with simplicial spheres to the case of a general simplicial complex. We also mention that some of our definitions and constructions (such as the Borel construction $B_T P$) first appeared in [11] in a different fashion; in this case, we tried to preserve the initial notation.

2. Homotopic realization of the complement of a coordinate subspace arrangement

Let $\mathbb{C}^m$ be the complex m-dimensional space with coordinates $z_1,\dots,z_m$. For any index subset $I = \{i_1,\dots,i_k\}$, denote by L_I the $(m-k)$-dimensional coordinate subspace defined by the equations $z_{i_1} = \dots = z_{i_k} = 0$. Note that $L_{\{1,\dots,m\}} = \{0\}$ and $L_\varnothing = \mathbb{C}^m$.

Definition 2.1. *A coordinate subspace arrangement $\mathcal{A}$ is a set of coordinate subspaces L_I. The complement of $\mathcal{A}$ is the subset*

$$U(\mathcal{A}) = \mathbb{C}^m \setminus \bigcup_{L_I \in \mathcal{A}} L_I \subset \mathbb{C}^m.$$

In what follows, we distinguish a coordinate subspace arrangement $\mathcal{A}$ regarded as an abstract set of subspaces and its support $|\mathcal{A}|$, i.e., the subset $\bigcup_{L_I \in \mathcal{A}} L_I \subset \mathbb{C}^m$. If $I \subset J$ and $L_I \subset |\mathcal{A}|$, then $L_J \subset |\mathcal{A}|$. If a coordinate subspace arrangement $\mathcal{A}$ contains a hyperplane $z_i = 0$, then its complement $U(\mathcal{A})$ is represented as $U(\mathcal{A}_0) \times \mathbb{C}^*$, where $\mathcal{A}_0$ is a coordinate subspace arrangement in the hyperplane $\{z_i = 0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Thus, for any coordinate subspace arrangement $\mathcal{A}$, the complement $U(\mathcal{A})$ decomposes as

$$U(\mathcal{A}) = U(\mathcal{A}') \times (\mathbb{C}^*)^k,$$

where $\mathcal{A}'$ is a coordinate arrangement in $\mathbb{C}^{m-k}$ containing no hyperplanes. Keeping this in mind, we restrict ourselves to coordinate subspace arrangements without hyperplanes.

A coordinate subspace arrangement $\mathcal{A}$ in $\mathbb{C}^m$ (without hyperplanes) defines a simplicial complex $K(\mathcal{A})$ with m vertices $v_1, \dots, v_m$ in the following way: we say that a subset $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ is a $(k-1)$-simplex of $K(\mathcal{A})$ if and only if $L_I \not\subset |\mathcal{A}|$.

Example 2.2. 1. If $\mathcal{A} = \varnothing$, then $K(\mathcal{A})$ is the $(m-1)$-dimensional simplex Δ^{m-1}.

2. If $\mathcal{A} = \{0\}$, then $K(\mathcal{A}) = \partial\Delta^{m-1}$ is the boundary of an $(m-1)$-simplex.

On the other hand, a simplicial complex K on the vertex set $\{v_1, \dots, v_m\}$ defines an arrangement $\mathcal{A}(K)$ such that $L_I \subset |\mathcal{A}|$ if and only if $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ is not a simplex of K. Note that if $K' \subset K$ is a subcomplex, then $\mathcal{A}(K) \subset \mathcal{A}(K')$. Thus, we have a reversing order one-to-one correspondence between simplicial complexes on m vertices and coordinate subspace arrangements in $\mathbb{C}^m$ without hyperplanes.

Let $U(K) = \mathbb{C}^m \setminus |\mathcal{A}(K)|$ denote the complement of a coordinate subspace arrangement $\mathcal{A}(K)$.

Example 2.3. 1. If $K = \Delta^{m-1}$ is the $(m-1)$-simplex, then $U(K) = \mathbb{C}^m$.

2. If $K = \partial\Delta^{m-1}$, then $U(K) = \mathbb{C}^m \setminus \{0\}$.

3. If K is a disjoint union of m vertices, then $U(K)$ is obtained by removing from $\mathbb{C}^m$ all coordinate subspaces $z_i = z_j = 0$, $i, j = 1, \dots, m$, of codimension 2.

Let $\mathbf{k}$ be a field. Below we call this field the ground field. Consider a polynomial ring $\mathbf{k}[v_1, \dots, v_m]$, where the v_i are regarded as variables.

Definition 2.4. *The face ring (the Stanley–Reisner ring) $\mathbf{k}(K)$ of the simplicial complex K is the quotient ring $\mathbf{k}[v_1, \dots, v_m]/I$, where*

$$I = \bigl\{v_{i_1} \cdots v_{i_s} : \{v_{i_1}, \dots, v_{i_s}\} \ \textit{does not span a simplex in} \ K\bigr\}.$$

Thus, the face ring is a quotient ring of the polynomial ring by the ideal generated by square-free monomials of degree $\geqslant 2$. We make $\mathbf{k}(K)$ a graded ring, setting $\deg v_i = 2$, $i = 1, \dots, m$.

Example 2.5. 1. If $K = \Delta^{m-1}$, then $\mathbf{k}(K) = \mathbf{k}[v_1, \dots, v_m]$.

2. If $K = \partial\Delta^{m-1}$ is the boundary complex of an $(m-1)$-simplex, then

$$\mathbf{k}(K) = \mathbf{k}[v_1, \dots, v_m]/(v_1 \cdots v_m).$$

A compact torus T^m acts on $\mathbb{C}^m$ diagonally. Since the arrangement $\mathcal{A}(K)$ consists of coordinate subspaces, this action is also defined on $U(K)$. Denote by $B_T K$ the corresponding Borel construction,

$$B_T K = ET^m \times_{T^m} U(K), \tag{1}$$

where ET^m is the contractible space of the universal T^m-bundle $ET^m \to BT^m$ over the classifying space $BT^m = (\mathbb{C}P^\infty)^m$. Thus, $B_T K$ is the total space of the bundle $B_T K \to BT^m$ with fiber $U(K)$.

The space BT^m has a canonical cellular decomposition (i.e., each $\mathbb{C}P^\infty$ has one cell in each even dimension). For any index set $I = \{i_1, \dots, i_k\}$, one may consider the cellular subcomplex $BT^k_I = BT^k_{i_1,\dots,i_k} \subset BT^m$ homeomorphic to BT^k.

Definition 2.6. *Given a simplicial complex K with vertex set $\{v_1, \dots, v_m\}$, define the cellular subcomplex $\widetilde{B_T K} \subset BT^m$ as the union of the subcomplexes BT^k_I over all I such that v_I is a simplex of K.*

Example 2.7. Let K be a disjoint union of m vertices $v_1, \dots, v_m$. Then $\widetilde{B_T K}$ is a bouquet of m copies of $\mathbb{C}P^\infty$.

The cohomology ring of BT^m is isomorphic to the polynomial ring $\mathbf{k}[v_1, \dots, v_m]$ (all cohomologies are with coefficients in the ground field $\mathbf{k}$).

Lemma 2.8. *The cohomology ring of $\widetilde{B_T K}$ is isomorphic to the face ring $\mathbf{k}(K)$. The embedding $i : \widetilde{B_T K} \hookrightarrow BT^m$ induces the quotient epimorphism $i^* : \mathbf{k}[v_1, \dots, v_m] \to \mathbf{k}(K) = \mathbf{k}[v_1, \dots, v_m]/I$ in the cohomology.*

Proof. The proof is by induction on the number of simplices of K. If K is a disjoint union of vertices $v_1, \dots, v_m$, then $\widetilde{B_T K}$ is a bouquet of m copies of $\mathbb{C}P^\infty$ (see Example 2.7). In degree zero, $H^*(\widetilde{B_T K})$ coincides with $\mathbf{k}$, while in degrees $\geqslant 1$, this ring is isomorphic to $\mathbf{k}[v_1] \oplus \dots \oplus \mathbf{k}[v_m]$. Therefore, $H^*(\widetilde{B_T K}) = k[v_1, \dots, v_m]/I$, where I is the ideal generated by all square-free monomials of degree $\geqslant 2$, and i^* is the projection onto the quotient ring. Thus, the lemma holds for $\dim K = 0$.

Now assume that the simplicial complex K is obtained from a simplicial complex K' by adding one $(k-1)$-dimensional simplex $v_I = \{v_{i_1}, \dots, v_{i_k}\}$. By the induction hypothesis, the lemma holds for K', i.e., $i^* H^*(BT^m) = H^*(\widetilde{B_T K'}) = \mathbf{k}(K') = \mathbf{k}[v_1, \dots, v_m]/I'$. By Definition 2.6, $\widetilde{B_T K}$ is obtained from $\widetilde{B_T K'}$ by adding the subcomplex $BT^k_{i_1,\dots,i_k} \subset BT^m$. Then $H^*(\widetilde{B_T K'} \cup BT^k_{i_1,\dots,i_k}) = \mathbf{k}[v_1, \dots, v_m]/I = \mathbf{k}(K' \cup v_I)$, where I is generated by I' and $v_{i_1} v_{i_2} \cdots v_{i_k}$. $\square$

Let I^m be the standard m-dimensional cube in $\mathbb{R}^m$:

$$I^m = \{(y_1, \dots, y_m) \in \mathbb{R}^m \; : \; 0 \leqslant y_i \leqslant 1, \, i = 1, \dots, m\}.$$

A simplicial complex K with m vertices $v_1, \dots, v_m$ defines a cubical complex $\mathcal{C}_K$ embedded canonically into the boundary complex of I^m in the following way.

Definition 2.9. *For any $(k-1)$-dimensional simplex $v_J = \{v_{j_1}, \dots, v_{j_k}\}$ of K, denote by C_J the k-dimensional face of I^m defined by the following $m-k$ equations:*

$$y_i = 1, \quad i \notin \{j_1, \dots, j_k\}.$$

We define a cubical subcomplex $\mathcal{C}_K \subset I^m$ as the union of the faces C_J over all simplices v_J of K.

Remark. Our cubical subcomplex $\mathcal{C}_K \subset I^m$ is a geometric realization of an abstract cubical complex in the cone over the barycentric subdivision of K (see [11, p. 434]). Indeed, let Δ^{m-1} be an $(m-1)$-dimensional simplex on the vertex set $\{v_1, \dots, v_m\}$ and let $\widehat{\Delta}^{m-1}$ be the barycentric subdivision of Δ^{m-1}, i.e., $\widehat{\Delta}^{m-1}$ has a vertex corresponding to each simplex v_J of Δ^{m-1}. Construct a map $\iota : \widehat{\Delta}^{m-1} \to I^m$ taking a vertex v_J of $\widehat{\Delta}^{m-1}$ to the vertex of I^m having the coordinates $y_j = 0$ for $j \in J$ and $y_j = 1$ for $j \notin J$. Extend this map linearly to each simplex of $\widehat{\Delta}^{m-1}$. The image of $\widehat{\Delta}^{m-1}$ under the constructed map is the union of faces of I^m meeting at zero. Now we construct a map $C\iota$ from the cone $C\widehat{\Delta}^{m-1}$ over $\widehat{\Delta}^{m-1}$ to I^m taking the vertex of the cone to the vertex $(1, \dots, 1)$ of the cube. Extend the latter map linearly to simplices of $C\widehat{\Delta}^{m-1}$. The image of $C\widehat{\Delta}^{m-1}$ under the map $C\iota$ is the whole cube I^m. Now let K be a simplicial complex on the vertex set $\{v_1, \dots, v_m\}$. Once a numeration of vertices is fixed, we may consider K as a simplicial subcomplex of Δ^{m-1}. Then our cubical complex $\mathcal{C}_K \subset I^m$ from Definition 2.9 coincides with the image $C\iota(C\widehat{K})$ of the cone over the barycentric subdivision of K under the map $C\iota$.

Example 2.10. Figures 1a and 1b represent a cubical complex $\mathcal{C}_K$ in the cases where K is a disjoint union of 3 vertices or the boundary complex of a 2-simplex, respectively.

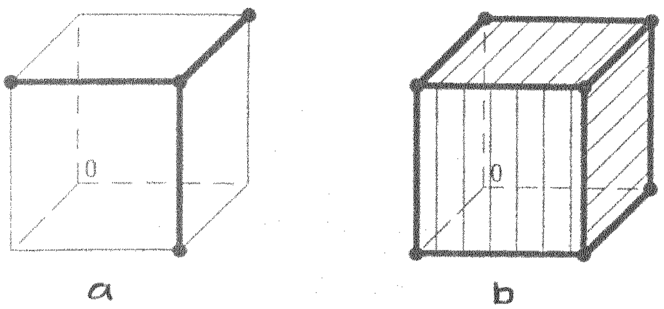

Fig. 1. The cubical complex $\mathcal{C}_K$.

Remark. In the case where K is the dual to the boundary complex of an n-dimensional simple polytope P^n, the cubical complex $\mathcal{C}_K$ coincides with the cubical subdivision of P^n studied in [6].

The orbit space of the diagonal action of T^m on $\mathbb{C}^m$ is the positive cone

$$\mathbb{R}^m_+ = \{(y_1, \dots, y_m) \in \mathbb{R}^m \colon y_i \geqslant 0,\, i = 1, \dots, m\}.$$

The orbit map $\mathbb{C}^m \to \mathbb{R}^m_+$ is determined by $(z_1, \dots, z_m) \to (|z_1|^2, \dots, |z_m|^2)$. If we restrict the above action to the standard polydisk

$$(D^2)^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1,\, i = 1, \dots, m\} \subset \mathbb{C}^m,$$

then the corresponding orbit space is the standard cube $I^m \subset \mathbb{R}^m_+$.

Let $U_{\mathbb{R}}(K) \subset \mathbb{R}^m_+$ denote the orbit space $U(K)/T^m$. Note that if we consider $\mathbb{R}^m_+$ as a subset in $\mathbb{C}^m$, then $U_{\mathbb{R}}(K)$ is the corresponding "real part": $U_{\mathbb{R}}(K) = U(K) \cap \mathbb{R}^m_+$.

Definition 2.11. *The equivariant moment-angle complex $\mathcal{Z}_K \subset \mathbb{C}^m$ corresponding to a simplicial complex K is defined to be the T^m-space determined by the commutative diagram*

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & (D^2)^m \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}_K & \longrightarrow & I^m, \end{array}$$

where the right-hand vertical arrow denotes the orbit map for the diagonal action of T^m and the lower horizontal arrow denotes the embedding of the cubical complex $\mathcal{C}_K$ into I^m.

Lemma 2.12. *We have the inclusions $\mathcal{C}_K \subset U_{\mathbb{R}}(K)$ and $\mathcal{Z}_K \subset U(K)$.*

Proof. Definition 2.11 shows that the second inclusion follows from the first one. To prove the first inclusion, we note that if a point $a = (y_1, \dots, y_m) \in \mathcal{C}_K$ has the coordinates $y_{i_1} = \ldots = y_{i_k} = 0$, then $v_I = \{v_{i_1}, \dots, v_{i_k}\}$ is a simplex of K, hence $L_I \not\subset \mathcal{A}(K)$. □

Lemma 2.13. *The complement $U(K)$ is equivariantly homotopy equivalent to the moment-angle complex $\mathcal{Z}_K$.*

Proof. We construct a retraction $r : U_{\mathbb{R}}(K) \to \mathcal{C}_K$ which is covered by an equivariant retraction $U(K) \to \mathcal{Z}_K$. The latter retraction is the required homotopy equivalence.

The retraction $r : U_{\mathbb{R}}(K) \to \mathcal{C}_K$ is constructed inductively. We start with the boundary complex of an $(m-1)$-simplex and remove simplices of positive dimensions until we obtain K. At each step, we construct a retraction. The composite map gives us the required retraction r. If $K = \partial\Delta^{m-1}$ is the boundary complex of an $(m-1)$-simplex, then $U_{\mathbb{R}}(K) = \mathbb{R}^m_+ \setminus \{0\}$, and the retraction r is shown in Fig. 2.

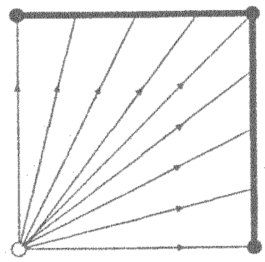

FIG. 2. The retraction $r : U_{\mathbb{R}}(K) \to \mathcal{C}_K$ for $K = \partial\Delta^{m-1}$.

Assume that the simplicial complex K is obtained by removing one $(k-1)$-dimensional simplex $v_J = \{v_{j_1}, \dots, v_{j_k}\}$ from a simplicial complex K'. By the induction hypothesis, the lemma holds for K', i.e., there is a retraction $r' : U_{\mathbb{R}}(K') \to \mathcal{C}_{K'}$ with the required properties. Let us consider the face $C_J \subset I^m$ (see Definition 2.9). Since v_J is not a simplex of K, the point a with the coordinates $y_{j_1} = \ldots = y_{j_k} = 0$, $y_i = 1$, $i \notin \{j_1, \dots, j_k\}$, does not belong to $U(K)$. Hence, we may apply the retraction from Fig. 2 on the face C_J, starting from the point a. Denote this retraction by r_J. Now we take $r = r_J \circ r'$. It is easy to see that r is exactly the required retraction. $\square$

Example 2.14. 1. If $K = \partial\Delta^{m-1}$ is the boundary complex of an $(m-1)$-simplex, then $\mathcal{Z}_K$ is homeomorphic to the $(2m-1)$-dimensional sphere S^{2m-1}.

2. If K is the dual to the boundary complex of an n-dimensional simple polytope P^n, then $\mathcal{Z}_K$ is homeomorphic to a smooth $(m+n)$-dimensional manifold. This manifold, denoted $\mathcal{Z}_P$, is the main object of study in [6].

Corollary 2.15. *The Borel construction $ET^m \times_{T^m} \mathcal{Z}_K$ is homotopically equivalent to $B_T K$.*

Proof. The retraction $r : U(K) \to \mathcal{Z}_K$ constructed in the proof of Lemma 2.13 is equivariant with respect to the T^m-actions on $U(K)$ and $\mathcal{Z}_K$. The equality $B_T K = ET^m \times_{T^m} U(K)$ proves our corollary. $\square$

In what follows, we do not distinguish the Borel constructions $ET^m \times_{T^m} \mathcal{Z}_K$ and $B_T K = ET^m \times_{T^m} U(K)$.

Theorem 2.16. *The cellular embedding $i : \widetilde{B_T K} \hookrightarrow BT^m$ (see Definition 2.6) and the fibration $p : B_T K \to BT^m$ (see (1)) are homotopically equivalent. In particular, $\widetilde{B_T K}$ and $B_T K$ are of the same homotopy type.*

Proof. Let $\pi : \mathcal{Z}_K \to \mathcal{C}_K$ denote the orbit map for the torus action on the moment-angle complex $\mathcal{Z}_K$ (see Definition 2.11). For any subset $I = \{i_1, \dots, i_k\} \subset \{1, \dots, m\}$, denote by B_I the following subset of the polydisk $(D^2)^m$: $B_I = B_1 \times \dots \times B_m \subset D^2 \times \dots \times D^2 = (D^2)^m$, where B_i is the disk D^2 if $i \in I$ and B_i is the boundary S^1 of D^2 if $i \notin I$. Thus, $B_I \cong (D^2)^k \times T^{m-k}$, where $k = |I|$. It is easy to see that if C_I is a face of the cubical complex $\mathcal{C}_K$ (see Definition 2.9), then $\pi^{-1}(C_I) = B_I$. Since B_I is canonically identified with a subset of B_J for $I \subset J$, we see that the subsets B_I such that v_I is a simplex of K fit together to compose $\mathcal{Z}_K$. (This idea can be used to prove that $\mathcal{Z}_K$ is a smooth manifold provided that K is the dual to the boundary complex of a simple polytope; see [6, Theorem 2.4].)

For any simplex $v_I \subset K$, the subset $B_I \subset \mathcal{Z}_K$ is invariant with respect to the T^m-action on $\mathcal{Z}_K$. Hence, the Borel construction $B_T K = ET^m \times_{T^m} \mathcal{Z}_K$ is glued from the Borel constructions $ET^m \times_{T^m} B_I$ (cf. the local construction of $B_T P$ of [11, p. 435]). The latter space can be factorized as $ET^m \times_{T^m} B_I = \left(ET^k \times_{T^k} (D^2)^k\right) \times ET^{m-n}$, which is homotopically equivalent to BT^k_I. Hence, the restriction of the projection $p : B_T K \to BT^m$ to $ET^m \times_{T^m} B_I$ is homotopically equivalent to the embedding $BT^k_I \hookrightarrow BT^m$. These homotopy equivalences for all of the simplices $v_I \subset K$ are glued together to give us the required homotopy equivalence between $p : B_T K \to BT^m$ and $i : \widetilde{B_T K} \hookrightarrow BT^m$. $\square$

Corollary 2.17. *The complement $U(K)$ of a coordinate subspace arrangement is a homotopy fiber of the cellular embedding $i : \widetilde{B_T K} \hookrightarrow BT^m$.*

Corollary 2.18. *The T^m-equivariant cohomology ring $H^*_{T^m}\big(U(K)\big)$ is isomorphic to the face ring $\mathbf{k}(K)$.*

Proof. We have the equalities $H^*_{T^m}\big(U(K)\big) = H^*\big(ET^m \times_{T^m} U(K)\big) = H^*(B_T K)$, hence the corollary follows from Lemma 2.8 and Theorem 2.16. $\square$

3. Cohomology ring of $U(K)$

Consider a $\mathbf{k}[v_1, \dots, v_m]$-free resolution of the face ring $\mathbf{k}(K)$ as a graded module over the polynomial ring $\mathbf{k}[v_1, \dots, v_m]$:

$$0 \to R^{-h} \xrightarrow{d^{-h}} R^{-h+1} \xrightarrow{d^{-h+1}} \dots \to R^{-1} \xrightarrow{d^{-1}} R^0 \xrightarrow{d^0} \mathbf{k}(K) \to 0 \tag{2}$$

(note that the Hilbert syzygy theorem implies that $h \leqslant m$ in (2)). Applying the functor $\otimes_{\mathbf{k}[v_1,\dots,v_m]}\mathbf{k}$ to (2), we obtain the following cochain complex:

$$0 \longrightarrow R^{-h} \otimes_{\mathbf{k}[v_1,\dots,v_m]} \mathbf{k} \longrightarrow \dots \longrightarrow R^0 \otimes_{\mathbf{k}[v_1,\dots,v_m]} \mathbf{k} \longrightarrow 0.$$

Its cohomology modules are denoted by $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big)$. Since all of the R^{-i} in (2) are graded $\mathbf{k}[v_1, \dots, v_m]$-modules,

$$\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big) = \bigoplus_j \mathrm{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big)$$

is a graded $\mathbf{k}$-module, and

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)=\bigoplus_{i,j}\operatorname{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$$

is a *bigraded* $\mathbf{k}$-module. Note that nonzero elements of the latter module have nonpositive first grading and nonnegative even second grading (since $\deg v_i=2$). The bigraded $\mathbf{k}$-module (3) can also be regarded as a one-graded module with respect to the total degree $-i+j$. The Betti numbers

$$\beta^{-i}\big(\mathbf{k}(K)\big)=\dim_{\mathbf{k}}\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$$

and

$$\beta^{-i,2j}\big(\mathbf{k}(K)\big)=\dim_{\mathbf{k}}\operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$$

are of great interest in geometric combinatorics; these numbers were studied by various authors (see, for example, [22]). We mention only one theorem due to Hochster, which reduces the calculation of $\beta^{-i,2j}\big(\mathbf{k}(K)\big)$ to calculating the homology of subcomplexes of K.

Theorem 3.1 (Hochster [16, 22])**.** *The Hilbert series*

$$\sum_j \beta^{-i,2j}(\mathbf{k}(K))t^{2j}$$

of $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k})$ *can be calculated as follows:*

$$\sum_j \beta^{-i,2j}\big(\mathbf{k}(K)\big)t^{2j}=\sum_{I\subset\{v_1,\dots,v_m\}}\big(\dim_{\mathbf{k}}\widetilde{H}_{|I|-i-1}(K_I)\big)t^{2|I|},$$

where K_I is the subcomplex of K consisting of all simplices with vertices in I.

Note that the calculation of $\beta^{-i,2j}\big(\mathbf{k}(K)\big)$ using this theorem is very complicated even for small K.

It turns out that $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big)$ is a bigraded *algebra* in a natural way, and the associated one-graded algebra is exactly $H^*\big(U(K)\big)$.

Theorem 3.2. *The following isomorphism of graded algebras holds:*

$$H^*\big(U(K)\big)\cong\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big).$$

Proof. Let us consider the commutative diagram

$$\begin{array}{ccc}\widetilde{U(K)} & \longrightarrow & ET^m\\ \big\downarrow & & \big\downarrow\\ \widetilde{B_TK} & \xrightarrow{\ i\ } & BT^m,\end{array}\qquad(4)$$

where the left-hand vertical arrow is the induced fiber bundle. Corollary 2.17 shows that $\widetilde{U(K)}$ is homotopically equivalent to $U(K)$.

From (4) we deduce that the cellular cochain algebras $C^*(\widetilde{B_TK})$ and $C^*(ET^m)$ are modules over $C^*(BT^m)$. It is clear from the proof of Lemma 2.8 that $C^*(\widetilde{B_TK})=\mathbf{k}(K)$ and that $i^*:C^*(BT^m)=\mathbf{k}[v_1,\dots,v_m]\to\mathbf{k}(K)=C^*(\widetilde{B_TK})$ is the quotient epimorphism. Since ET^m is contractible, we have a chain equivalence $C^*(ET^m)\to\mathbf{k}$. Therefore, there is an isomorphism

$$\operatorname{Tor}_{C^*(BT^m)}\big(C^*(\widetilde{B_TK}),C^*(ET^m)\big)\cong\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big).\qquad(5)$$

The Eilenberg–Moore spectral sequence (see [21, Theorem 1.2]) of commutative square (4) has the E_2-term,

$$E_2=\operatorname{Tor}_{H^*(BT^m)}\big(H^*(\widetilde{B_TK}),H^*(ET^m)\big),$$

and converges to $\mathrm{Tor}_{C^*(BT^m)}(C^*(\widetilde{B_TK}), C^*(ET^m))$. Since

$$\mathrm{Tor}_{H^*(BT^m)}(H^*(\widetilde{B_TK}), H^*(ET^m)) = \mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big),$$

it follows from (5) that the spectral sequence collapses at the E_2 term, i.e., $E_2 = E_\infty$. Proposition 3.2 of [21] shows that the module $\mathrm{Tor}_{C^*(BT^m)}\big(C^*(\widetilde{B_TK}), C^*(ET^m)\big)$ is an algebra isomorphic to $H^*\big(\widetilde{U(K)}\big)$, which concludes the proof. □

Our next theorem gives us an explicit description of the algebra $H^*\big(U(K)\big)$ as the cohomology algebra of a simple differential bigraded algebra. We consider the tensor product $\mathbf{k}(K) \otimes \Lambda[u_1,\dots,u_m]$ of the face ring $\mathbf{k}(K) = \mathbf{k}[v_1,\dots,v_m]/I$ and an exterior algebra $\Lambda[u_1,\dots,u_m]$ on m generators. We make this tensor product a differential bigraded algebra by setting

$$\operatorname{bideg} v_i = (0,2), \quad \operatorname{bideg} u_i = (-1,2), \quad d(1 \otimes u_i) = v_i \otimes 1, \quad d(v_i \otimes 1) = 0, \tag{6}$$

and requiring that d is a derivation of algebras.

Theorem 3.3. *The following isomorphism of graded algebras holds:*

$$H^*\big(U(K)\big) \cong H\big[\mathbf{k}(K) \otimes \Lambda[u_1,\dots,u_m], d\big],$$

where the right-hand side is the one-graded algebra associated with the bigraded cohomology algebra.

Proof. One can make $\mathbf{k}$ a $\mathbf{k}[v_1,\dots,v_m]$-module by applying the homomorphism mapping 1 to 1 and v_i to 0. Let us consider the Koszul resolution (see, for example, [17, Chapter VII, Section 2]) of $\mathbf{k}$ regarded as a $\mathbf{k}[v_1,\dots,v_m]$-module:

$$\big[\mathbf{k}[v_1,\dots,v_m] \otimes \Lambda[u_1,\dots,u_m], d\big],$$

where the differential d is defined similarly to (6).

Since the bigraded torsion product $\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\ ,\ )$ is a symmetric function of its arguments, one has the equalities

$$\mathrm{Tor}_\Gamma\big(\mathbf{k}(K), \mathbf{k}\big) = H\big[\mathbf{k}(K) \otimes_\Gamma \Gamma \otimes \Lambda[u_1,\dots,u_n], d\big] = \big[\Gamma \otimes \Lambda[u_1,\dots,u_m], d\big],$$

where we denoted $\Gamma = \mathbf{k}[v_1,\dots,v_m]$. Since $H^*\big(U(K)\big) \cong \mathrm{Tor}_\Gamma\big(\mathbf{k}(K), \mathbf{k}\big)$ by Theorem 3.2, we obtain the required isomorphism. □

Note that the above-formulated theorem not only calculates the cohomology algebra of $U(K)$ but also makes this algebra *bigraded*.

Corollary 3.4. *The Leray–Serre spectral sequence of the bundle* $\widetilde{U(K)} \to \widetilde{B_TK}$ *with fiber* T^m *(see* (4)*) collapses at the* E_3 *term.*

Proof. The spectral sequence under consideration converges to $H^*\big(\widetilde{U(K)}\big) = H^*\big(U(K)\big)$, and the following equalities hold:

$$E_2 = H^*(\widetilde{B_TK}) \otimes H^*(T^m) = \mathbf{k}(K) \otimes \Lambda[u_1,\dots,u_m].$$

It is easy to see that the differential in the E_2 term acts similarly to (6). Hence, $E_3 = H[E_2, d] = H\big[\mathbf{k}(K) \otimes \Lambda[u_1,\dots,u_m]\big] = H^*\big(U(K)\big)$ by Theorem 3.3. □

Proposition 3.5. *Assume that a monomial*

$$v_{i_1}^{\alpha_1} \dots v_{i_p}^{\alpha_p} u_{j_1} \dots u_{j_q} \in \mathbf{k}(K) \otimes \Lambda[u_1,\dots,u_m],$$

where $i_1 < \dots < i_p$ *and* $j_1 < \dots < i_q$*, represents a nontrivial cohomology class in* $H^*\big(U(K)\big)$*. Then* $\alpha_1 = \dots = \alpha_p = 1$*,* $\{v_{i_1},\dots,v_{i_p}\}$ *spans a simplex of* K*, and* $\{i_1,\dots,i_p\} \cap \{j_1,\dots,j_q\} = \varnothing$*.*

Proof. See [6, Lemma 5.3]. □

As was mentioned above (see Example 2.14), if K is the boundary complex of a convex simplicial polytope (or, equivalently, K is the dual to the boundary complex of a simple polytope) or at least a simplicial sphere, then $U(K)$ has homotopy type of a smooth manifold $\mathcal{Z}_K$. It was shown in [6, Theorem 2.10] that the corresponding homotopy equivalence can be treated as the orbit map $U(K) \to U(K)/\mathbb{R}^{m-n} \cong \mathcal{Z}_K$ with respect to a certain action of $\mathbb{R}^{m-n}$ on $U(K)$.

The coordinate subspace arrangement $\mathcal{A}(K)$ and its complement $U(K)$ play an important role in the theory of toric varieties and symplectic geometry (see, for example, [2, 3, 9]). More precisely, any n-dimensional simplicial toric variety M defined by a (simplicial) fan Σ in $\mathbb{Z}^n$ with m one-dimensional cones can be obtained as the geometric quotient $U(K_\Sigma)/G$. Here G is a subgroup of the complex torus $(\mathbb{C}^*)^m$ isomorphic to $(\mathbb{C}^*)^{m-n}$ and K_Σ is the simplicial complex defined by the fan Σ (i-simplices of K_Σ correspond to $(i+1)$-dimensional cones of Σ). A smooth projective toric variety M is a symplectic manifold of real dimension $2n$. This manifold can be constructed by the process of *symplectic reduction* in the following way. Let $G_\mathbb{R} \cong T^{m-n}$ denote the maximal compact subgroup of G and let $\mu : \mathbb{C}^m \to \mathbb{R}^{m-n}$ be the moment map for the Hamiltonian action of $G_\mathbb{R}$ on $\mathbb{C}^m$. Then, for each regular value $a \in \mathbb{R}^{m-n}$ of μ, there is a diffeomorphism

$$\mu^{-1}(a)/G_\mathbb{R} \longrightarrow U(K_\Sigma)/G = M$$

(see [9] for more information). In this situation, it can be seen easily that $\mu^{-1}(a)$ is exactly our manifold $\mathcal{Z}_K$ for $K = K_\Sigma$.

Example 3.6. Let $G \cong \mathbb{C}^*$ be the diagonal subgroup in $(\mathbb{C}^*)^{n+1}$ and let K_Σ be the boundary complex of an n-simplex. Then $U(K_\Sigma) = \mathbb{C}^{n+1} \setminus \{0\}$, and $M = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$ is the complex projective space $\mathbb{C}P^n$. The moment map $\mu : \mathbb{C}^m \to \mathbb{R}$ takes $(z_1, \ldots, z_m) \in \mathbb{C}^m$ to $\frac{1}{2}(|z_1|^2 + \ldots + |z_m|^2)$, and for $a \neq 0$, one has $\mu^{-1}(a) \cong S^{2n+1} \cong \mathcal{Z}_K$ (see Example 2.14).

If K is a simplicial sphere (hence, the complement $U(K)$ is homotopically equivalent to the smooth manifold $\mathcal{Z}_K$), then there is a Poincaré duality defined in the cohomology ring of $U(K)$.

Proposition 3.7. *Assume that K is a simplicial sphere of dimension $n-1$, hence $U(K)$ is homotopically equivalent to the smooth manifold $\mathcal{Z}_K$. Then the following statements hold.*

(1) *The Poincaré duality in $H^*\big(U(K)\big)$ respects the bigraded structure defined by Theorem 3.3. More precisely, if $\alpha \in H^{-i,2j}\big(U(K)\big)$ is a cohomology class, then its Poincaré dual $D\alpha$ belongs to $H^{-(m-n)+i,2(m-j)}$.*

(2) *Let $\{v_{i_1}, ..., v_{i_n}\}$ be an $(n-1)$-simplex of K and let $j_1 < ... < j_{m-n}$ and*

$$\{i_1, ..., i_n, j_1, ..., j_{m-n}\} = \{1, ..., m\}.$$

Then the value of the element

$$v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}} \in H^{m+n}\big(U(K)\big) \cong H^{m+n}(\mathcal{Z}_K)$$

on the fundamental class of $\mathcal{Z}_K$ equals ± 1.

(3) *Let $\{v_{i_1}, \ldots, v_{i_n}\}$ and $\{v_{i_1}, \ldots, v_{i_{n-1}}, v_{j_1}\}$ be two $(n-1)$-simplices of K having a common $(n-2)$-face $\{v_{i_1}, \ldots, v_{i_{n-1}}\}$ and let $j_1, \ldots, j_{m-n}$ be as in statement (2). Then*

$$v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}} = v_{i_1} \cdots v_{i_{n-1}} v_{j_1} u_{i_n} u_{j_2} \cdots u_{j_{m-n}}$$

in $H^{m+n}\big(U(K)\big)$.

Proof. For a proof of statements (1) and (2), see [6, Lemma 5.1]. To prove statement (3), we note that

$$d(v_{i_1} \cdots v_{i_{n-1}} u_{i_n} u_{j_1} u_{j_2} \cdots u_{j_{m-n}}) = v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}} - v_{i_1} \cdots v_{i_{n-1}} v_{j_1} u_{i_n} u_{j_2} \cdots u_{j_{m-n}}$$

in $\mathbf{k}(K) \otimes \Lambda[u_1, \ldots, u_m]$ (see (6)). □

A simplicial complex K is called a *Cohen–Macaulay complex* if its face ring $\mathbf{k}(K)$ is a Cohen–Macaulay algebra, i.e, if $\mathbf{k}(K)$ is a finite-dimensional free module over a polynomial ring $\mathbf{k}[t_1, \ldots, t_n]$ (here n is the maximal number of algebraically independent elements of $\mathbf{k}(K)$). Equivalently, $\mathbf{k}(K)$ is a Cohen–Macaulay algebra if it admits a *regular sequence* $\{\lambda_1, \ldots, \lambda_n\}$, i.e., a set of n homogeneous elements such that λ_{i+1} is not a zero divisor in $\mathbf{k}(K)/(\lambda_1, \ldots, \lambda_i)$ for $i = 0, \ldots, n-1$. If K is a Cohen–Macaulay complex and $\mathbf{k}$ is of infinite characteristic, then $\mathbf{k}(K)$ admits a regular sequence of degree-two elements (recall that we set $\deg v_i = 2$ in $\mathbf{k}(K)$), i.e., $\lambda_i = \lambda_{i1} v_1 + \lambda_{i2} v_2 + \ldots + \lambda_{im} v_m$, $i = 1, \ldots, n$.

Theorem 3.8. *Assume that K is a Cohen–Macaulay complex and let $J = (\lambda_1, \ldots, \lambda_n)$ be an ideal in $\mathbf{k}(K)$ generated by a regular sequence. Then the following isomorphism of bigraded algebras holds:*

$$H^*\big(U(K)\big) \cong H\big[\,\mathbf{k}(K)/J \otimes \Lambda[u_1, \ldots, u_{m-n}], d\,\big],$$

where the gradings and differential on the right-hand side are defined as follows:

$$\operatorname{bideg} v_i = (0,2), \quad \operatorname{bideg} u_i = (-1,2), \quad (1 \otimes u_i) = \lambda_i \otimes 1, \quad \textit{and} \quad d(v_i \otimes 1) = 0.$$

Hence, in the case where K is a Cohen–Macaulay complex, the cohomology of $U(K)$ can be calculated via the finite-dimensional differential algebra $\mathbf{k}(K)/J \otimes \Lambda[u_1, \ldots, u_{m-n}]$ instead of the infinite-dimensional algebra $\mathbf{k}(K) \otimes \Lambda[u_1, \ldots, u_m]$ (see Theorem 3.3).

Example 3.9. Let K be the boundary complex of an $(m-1)$-dimensional simplex. Then

$$\mathbf{k}(K) = \mathbf{k}[v_1, \ldots, v_m]/(v_1 \cdots v_m).$$

It is easy to check that only nontrivial cohomology classes in $H\big[\mathbf{k}(K) \otimes \Lambda[u_1, \ldots, u_m], d\big]$ (see Theorem 3.3) are represented by the cocycles 1, or $v_1 v_2 \cdots v_{m-1} u_m$, or their multiples. We have the equality $\deg(v_1 v_2 \cdots v_{m-1} u_m) = 2m-1$. Proposition 3.7 shows that $v_1 v_2 \cdots v_{m-1} u_m$ is the fundamental cohomological class of $\mathcal{Z}_K \cong S^{2m-1}$ (see Example 2.14.1).

Example 3.10. Let K be a disjoint union of m vertices. Then $U(K)$ is obtained by removing from $\mathbb{C}^m$ all codimension-two coordinate subspaces $z_i = z_j = 0$, $i, j = 1, \ldots, m$ (see Example 2.3). In this case, $\mathbf{k}(K) = \mathbf{k}[v_1, \ldots, v_m]/I$, where I is the ideal generated by all monomials $v_i v_j$, $i \neq j$. It is easily deduced from Theorem 3.3 and Proposition 3.5 that any cohomology class of $H^*\big(U(K)\big)$ is represented by a linear combination of monomial cocycles $v_{i_1} u_{i_2} u_{i_3} \cdots u_{i_k} \subset \mathbf{k}(K) \otimes \Lambda[u_1, \ldots, u_m]$ such that $k \geqslant 2$ and $i_p \neq i_q$ for $p \neq q$. For each k, we have $m\binom{m-1}{k-1}$ such monomials, and there exist $\binom{m}{k}$ relations between them (each relation is obtained by calculating the differential of $u_{i_1} \cdots u_{i_k}$). Since $\deg(v_{i_1} u_{i_2} u_{i_3} \cdots u_{i_k}) = k+1$, we have the equalities

$$\dim H^0\big(U(K)\big) = 1, \quad H^1\big(U(K)\big) = H^2\big(U(K)\big) = 0,$$

and

$$\dim H^{k+1}\big(U(K)\big) = m\binom{m-1}{k-1} - \binom{m}{k}, \quad 2 \leqslant k \leqslant m,$$

and the multiplication in the cohomology is trivial.

In particular, for $m = 3$, we have 6 three-dimensional cohomology classes $v_i u_j$, $i \neq j$, with 3 relations $v_i u_j = v_j u_i$, and 3 four-dimensional cohomology classes $v_1 u_2 u_3$, $v_2 u_1 u_3$, and $v_3 u_1 u_2$ with one relation

$$v_1 u_2 u_3 - v_2 u_1 u_3 + v_3 u_1 u_2 = 0.$$

Hence, $\dim H^3\big(U(K)\big) = 3$, $\dim H^4\big(U(K)\big) = 2$, and the multiplication is trivial.

Example 3.11. Let K be a boundary complex of an m-gon ($m \geqslant 4$). Then, as was mentioned above, the moment-angle complex $\mathcal{Z}_K$ is a smooth manifold of dimension $m+2$, and $U(K)$ is homotopically equivalent to $\mathcal{Z}_K$. We have the equality $\mathbf{k}(K) = \mathbf{k}[v_1, \ldots, v_m]/I$, where I is generated by monomials $v_i v_j$ such that $i \neq j \pm 1$. (Here we agree that $v_{m+i} = v_i$ and $v_{i-m} = v_i$.) The cohomology rings of these manifolds were calculated in [6]. We have the following relations:

$$\dim H^k\big(U(K)\big) = \begin{cases} 1 & \text{if} \quad k = 0 \quad \text{or} \quad m+2; \\ 0 & \text{if} \quad k = 1, 2, m, \quad \text{or} \quad m+1; \\ (m-2)\binom{m-2}{k-2} - \binom{m-2}{k-1} - \binom{m-2}{k-3} & \text{if} \quad 3 \leqslant k \leqslant m-1. \end{cases}$$

For example, if $m = 5$, then there exist five generators of $H^3\big(U(K)\big)$ represented by the cocycles $v_i u_{i+2} \in \mathbf{k}(K) \otimes \Lambda[u_1, \ldots, u_5]$, $i = 1, \ldots, 5$, and five generators of $H^4\big(U(K)\big)$ represented by the cocycles $v_j u_{j+2} u_{j+3}$, $j = 1, \ldots, 5$. It follows from Proposition 3.7 that the product of the cocycles $v_i u_{i+2}$ and $v_j u_{j+2} u_{j+3}$ represents a nontrivial cohomology class in $H^7\big(U(K)\big)$ (the fundamental cohomology class up to sign) if and only if $\{i, i+$

$2, j, j+2, j+3\} = \{1,2,3,4,5\}$. Hence, for each cohomology class $[v_i u_{i+2}]$, there is a unique (Poincaré dual) cohomology class $[v_j u_{j+2} u_{j+3}]$ such that the product $[v_i u_{i+2}] \cdot [v_j u_{j+2} u_{j+3}]$ is nontrivial.

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