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ON THE COBORDISM CLASSIFICATION OF MANIFOLDS WITH $\mathbb{Z}/p$-ACTION WHOSE FIXED-POINT SET HAS TRIVIAL NORMAL BUNDLE

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Definition 1. An action of the group $\mathbb{Z}/p$ on a stably complex manifold M^{2n} is said to be *simple* if the fixed-point set consists of finitely many connected submanifolds with trivial normal bundle. A simple action is said to be *strictly simple* if the weight sets of action (i.e., the sets of eigenvalues for the differential of the action of the generator $\rho \in \mathbb{Z}/p$ at the fixed points) are identical for all fixed submanifolds of the same dimension. Here "a stably complex manifold" stands for "a manifold with complex structure in the stable tangent bundle."

We obtain a complete classification of the complex cobordism classes $\sigma \in \Omega_U$ containing a manifold with simple action of $\mathbb{Z}/p$. The description is given both in terms of coefficients of the universal formal group of "geometric cobordisms" (Theorem 1) and in terms of characteristic numbers (Theorem 2 and Corollary 2).

The classification problem for strictly simple actions of $\mathbb{Z}/p$ was completely solved by Conner and Floyd in [1]. (In [1], the strictly simple action from Definition 1 was simply called the "action of $\mathbb{Z}/p$ with fixed-point set having a trivial normal bundle.") Note that even in the special case of action with a finite number of isolated fixed points the notions of simple and strictly simple action are distinct (see examples below). The Conner–Floyd results follow from the results of our paper. At the same time, we believe that the approach used in [1] does not allow one to obtain our more general result.

The applications of the formal group theory to problems connected with $\mathbb{Z}/p$-actions were first discussed in the pioneering article [2]. The formal group theory itself comes to topology due to the so-called formal group of geometric cobordisms. The problem solved here was first stated in [3]. There one obtained a formula expressing the $\operatorname{mod} p$ cobordism class of manifold M^{2n} with simple action of $\mathbb{Z}/p$ via certain action invariants (see (8)). Actually, the first results on the problem were obtained even earlier, in [4]. In particular, the statement mentioned in our article as Corollary 3 was proved. In [4], as well as in our paper, the set of cobordism classes of manifolds with simple $\mathbb{Z}/p$-action is described as an Ω_U-module spanned by certain coefficients of the power system defined by the formal group of geometric cobordisms. (Here Ω_U is the complex cobordism ring of points, which is isomorphic to the polynomial ring $\mathbb{Z}[a_1, a_1, \dots]$, $\deg a_i = -2i$ as was shown by Milnor and Novikov.) In this article, we propose a new choice of generators for the above Ω_U-module. Moreover, this choice allows us to solve the classification problem in terms of the characteristic numbers.

Therefore, we consider an operator g of prime period $p > 2$ (i.e., $g^p = \mathrm{id}$) acting on a stably complex manifold M^{2n} in such a way that the fixed-point set is a union of connected submanifolds with trivial normal bundle (e.g., is a finite number of fixed points). This means that we have a simple $\mathbb{Z}/p$-action. Let the fixed submanifolds represent the cobordism classes $\lambda_j \in \Omega_U$ and have weights $(x_k^{(j)}) \in (\mathbb{Z}/p)^*$ (these are the nonunit eigenvalues for the differential of g at the fixed points) in their trivial normal bundles. These data define the cobordism class of M^{2n} in Ω_U up to elements from $p\Omega_U$ (see [3]). This follows from the fact that the cobordism class of manifolds with free $\mathbb{Z}/p$-action (i.e., without fixed points) belongs to $p\Omega_U$, and vice versa, in any cobordism class from $p\Omega_U$, one could obviously find a manifold with free action of $\mathbb{Z}/p$ (e.g., rotating p components).

With each fixed submanifold of the cobordism class $\lambda_j \in \Omega_U$ with weights $(x_1^{(j)}, \dots, x_{m_j}^{(j)})$, $2m_j + \dim \lambda_j = 2n$, one could associate the so-called "Conner–Floyd invariant" $\alpha_{2m_j-1}(x_1^{(j)}, \dots, x_{m_j}^{(j)}) \in U_{2m_j-1}(B\mathbb{Z}/p)$ (see [2]).

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To define it, we mention that the bordism group $U_*(B\mathbb{Z}/p)$ is isomorphic to the group of principal $\mathbb{Z}/p$-equivariant bordisms (see [1]). This isomorphism takes the equivariant bordism class of the manifold N with free action of $\mathbb{Z}/p$ to the bordism class in $U_*(B\mathbb{Z}/p)$ given by the classifying map $N/(\mathbb{Z}/p) \to B\mathbb{Z}/p$. Then $\alpha_{2m_j-1}(x_1^{(j)}, \dots, x_{m_j}^{(j)})$ is the equivariant bordism class of the unit sphere in the fiber of the (trivial) normal bundle to λ_j. If we consider this unit sphere as

$$\Big\{(z_1, \dots, z_{m_j}) \in \mathbb{C}^{m_j} \ \Big| \ |z_1|^2 + \dots + |z_{m_j}|^2 = 1\Big\},$$

then the (free) action of $\mathbb{Z}/p$ on it is given by

$$g : (z_1, \dots, z_{m_j}) \to (\exp(2\pi i x_1^{(j)}/p)z_1, \dots, \exp(2\pi i x_{m_j}^{(j)}/p)z_{m_j}).$$

The complex cobordism ring of $B\mathbb{Z}/p$ is $U^*(B\mathbb{Z}/p) = \Omega_U[[u]]/[u]_p = 0$, where $[u]_p = p\Psi^p(u)$ is the p-th power in the (universal) formal group of geometric cobordisms (see [3]). We have $D(\alpha_i(1, \dots, 1)) = u^{n-i}$, where D is the Poincaré–Atiyah duality operator from $U_*(L_p^{2n-1})$ to $U^*(L_p^{2n-1})$. From this we deduce that the Ω_U-module $\widetilde{U}_*(B\mathbb{Z}/p)$ is generated by the elements $\alpha_{2i-1}(1, \dots, 1)$ with the following relations:

$$0 = \frac{[u]_p}{u} \cap \alpha_{2i-1}(1, \dots, 1). \tag{1}$$

Here $\cap$ is the cobordism $\cap$-product: $u^k \cap \alpha_{2i-1}(1, \dots, 1) = \alpha_{2(i-k)-1}(1, \dots, 1)$.

It was shown in [5] that

$$\alpha_{2k-1}(x_1, \dots, x_k) = \left(\prod_{j=1}^{k} \frac{u}{[u]_{x_j}}\right) \cap \alpha_{2k-1}(1, \dots, 1). \tag{2}$$

Adding the elements $\alpha_{2k-1}(x_1, \dots, x_k)$, $x_j \neq 1 \bmod p$ to the set of generators for the module $\widetilde{U}_*(B\mathbb{Z}/p)$ and adding relations (2) to relations (1), we come to the $\Omega_U \otimes \mathbb{Z}_{(p)}$-free resolution of the module $\widetilde{U}_*(B\mathbb{Z}/p)$ (here $\mathbb{Z}_{(p)}$ is the ring of rational numbers whose denominators are relatively prime with p, i.e., the ring of integer p-adics):

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \widetilde{U}_*(B\mathbb{Z}/p) \longrightarrow 0.$$

Here F_0 is the free $\Omega_U \otimes \mathbb{Z}_{(p)}$-module spanned by $\alpha_{2k-1}(x_1, \dots, x_k)$ and F_1 is the free $\Omega_U \otimes \mathbb{Z}_{(p)}$-module spanned by the relations

$$a(x_1, \dots, x_k) = \alpha_{2k-1}(x_1, \dots, x_k) - (\prod_{j=1}^{k} \frac{u}{[u]_{x_j}}) \cap \alpha_{2k-1}(1, \dots, 1)$$

and

$$a_k = \frac{[u]_p}{u} \cap \alpha_{2k-1}(1, \dots, 1).$$

Therefore, each simple action $\mathbb{Z}/p$ on M^{2n} gives a certain relation between the elements $\alpha_{2k-1}(x_1, \dots, x_k)$ in $\widetilde{U}_*(B\mathbb{Z}/p)$. Since any element from $\widetilde{U}_*(B\mathbb{Z}/p)$ corresponds to the bordism class of a manifold with free $\mathbb{Z}/p$-action, the converse is also true: any relation in $\widetilde{U}_*(B\mathbb{Z}/p)$ of the form $\sum_j \lambda_j \alpha_{2m_j-1}(x_1^{(j)}, \dots x_{m_j}^{(j)}) = 0$, $\lambda_j \in \Omega_U$, $2m_j + \dim \lambda_j = 2n$ is realized on a certain manifold M^{2n} with a simple action of $\mathbb{Z}/p$ whose cobordism class in Ω_U is uniquely determined up to the elements from $p\Omega_U$. This manifold M^{2n} is constructed as follows. The relation in $\widetilde{U}_*(B\mathbb{Z}/p)$ gives us the manifold with free $\mathbb{Z}/p$-action whose boundary is a union of manifolds of the form $\lambda_j \times \mathrm{S}^{2m_j-1}$. Then we glue "covers" of the form $\lambda_j \times \mathrm{D}^{2m_j}$ to the boundary to get the closed manifold M^{2n}, which realizes the above relation. Hence, we define the "realization homomorphism" $\Phi : F_1 \to \Omega_U/p\Omega_U = \Omega_U \otimes \mathbb{Z}/p$. It assigns to the relation between the elements $\alpha_{2k-1}(x_1, \dots, x_k) \in \widetilde{U}_*(B\mathbb{Z}/p)$ a $\bmod p$ cobordism class of the manifold that realizes this relation as described above. The Conner–Floyd results (cf. [1]; see also [4]) give us the following values of Φ on the basic relations from F_1:

$$\Phi(a(x_1, \dots, x_k)) = \Big\langle \prod_{i=1}^{k} \frac{u}{[u]_{x_i}} \Big\rangle_k \bmod p \in \Omega_U/p\Omega_U,$$

$$\Phi(a_k) = -\left\langle \frac{[u]_p}{u} \right\rangle_k \bmod p \in \Omega_U / p\Omega_U,$$

where $\langle\ \rangle_k$ stands for the coefficient of u^k. Therefore, $\operatorname{Im} \Phi = \widetilde{\Lambda}(1) \otimes \mathbb{Z}/p$, where $\widetilde{\Lambda}(1) = \Lambda^+(1) \cdot \Omega_U$ is the Ω_U-module spanned by the positive part $\Lambda^+(1)$ of the coefficient ring $\Lambda(1)$ of the power system $[u]_k$ (here $[u]_k$ is the kth power in the formal group of geometric cobordisms). The homomorphism Φ lifts to the homomorphism $\Phi : F_1 \to \widetilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ or to the homomorphism $\Phi : F_1 \to \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$, where $\widetilde{\Lambda}_p(1)$ is the Ω_U-module spanned by $\Lambda^+(1)$ and p.

Thus, the problem of description of the cobordism classes of manifolds with simple $\mathbb{Z}/p$-action is equivalent to the problem of description of the $\Omega_U \otimes \mathbb{Z}/p$-module $\widetilde{\Lambda}(1) \otimes \mathbb{Z}/p$ or $\Omega_U \otimes \mathbb{Z}_{(p)}$-modules $\widetilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ and $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$. These modules are ideals in $\Omega_U \otimes \mathbb{Z}/p$ and $\Omega_U \otimes \mathbb{Z}_{(p)}$, respectively.

Let $[u]_k = ku + \sum_{n \geq 1} \alpha_n^{(k)} u^{n+1}$, and so, $\alpha_n^{(k)} \in \Omega_U^{-2n}$ are the coefficients of the power system.

Theorem 1. *One could take the following coefficients $\alpha_n \in \Omega_U^{-2n}$ as generators of the $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\widetilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$:*

$$\alpha_n = \begin{cases} \alpha_n^{(p_1)} & \text{if } n \text{ is not divisible by } p-1, \\ \alpha_n^{(p)} & \text{if } n \text{ is divisible by } p-1. \end{cases}$$

Here p_1 is any generator of the cyclic group $(\mathbb{Z}/p)^*$.

Remark. It follows from the Dirichlet theorem that one could choose a *prime* generator p_1 of the cyclic group $(\mathbb{Z}/p)^*$.

Proof of Theorem 1. First, let us consider the coefficients $\alpha_n^{(r)}$ for nonprime r. Therefore, let $r = p_1 q$ with prime p_1. Since $[x]_r = [[x]_{p_1}]_q$, we have

$$\begin{aligned} rx + \textstyle\sum_n \alpha_n^{(r)} x^{n+1} &= q[x]_{p_1} + \textstyle\sum_n \alpha_n^{(q)} ([x]_{p_1})^{n+1} \\ &= p_1 q x + q \textstyle\sum_n \alpha_n^{(p_1)} x^{n+1} \\ &\quad + \textstyle\sum_n \alpha_n^{(q)} (p_1 x + \sum_m \alpha_m^{(p_1)} x^{m+1})^{n+1}. \end{aligned}$$

Taking the coefficients of x^{r+1} in both sides of the above identity, we get

$$\alpha_n^{(r)} = P(\alpha_1^{(p_1)}, \dots, \alpha_n^{(p_1)}, \alpha_1^{(q)}, \dots, \alpha_n^{(q)}),$$

where P is a certain polynomial with integer coefficients (without constant term). Therefore, we can write $\alpha_n^{(r)} = \lambda_1 \alpha_1^{(p_1)} + \ldots + \lambda_n \alpha_n^{(p_1)} + \mu_1 \alpha_1^{(q)} + \ldots + \mu_n \alpha_n^{(q)}$, $\lambda_i, \mu_i \in \Omega_U$. Hence the coefficients $\alpha_n^{(r)}$, $r = p_1 q$ could be excluded from the set of generators for the $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\widetilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$. Now, if q is still not prime, we repeat the above procedure, and so on. Finally, we obtain the set of generators that consists of only the coefficients $\alpha_n^{(p_1)}$ with prime p_1. Among all prime p_1, there is one particular $p_1 = p$ (the order of the group acting on the manifold). Now, we are going to show that the above set of generators could be restricted to the set described in the statement of the theorem.

Note that for any (prime) generator p_1 of the cyclic group $(\mathbb{Z}/p)^*$, one could take the coefficient $\alpha_1^{(p_1)}$ as a generator of $\widetilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ in dimension 1 (i.e., in Ω_U^{-2}). Indeed, let p_2 be any prime. Then $[[x]_{p_2}]_{p_1} = [[x]_{p_1}]_{p_2}$. Hence

$$\begin{aligned} &p_1 p_2 x + p_1 \textstyle\sum_n \alpha_n^{(p_2)} x^{n+1} + \sum_n \alpha_n^{(p_1)} (p_2 x + \sum_m \alpha_m^{(p_2)} x^{m+1})^{n+1} \\ &= p_2 p_1 x + p_2 \textstyle\sum_n \alpha_n^{(p_1)} x^{n+1} + \sum_n \alpha_n^{(p_2)} (p_1 x + \sum_m \alpha_m^{(p_1)} x^{m+1})^{n+1}. \end{aligned} \tag{3}$$

Taking the coefficient of x^2 in both sides of the above identity, we get $p_1 \alpha_1^{(p_2)} + p_2^2 \alpha_1^{(p_1)} = p_2 \alpha_1^{(p_1)} + p_1^2 \alpha_1^{(p_2)}$. Hence, $(p_1 - p_1^2) \alpha_1^{(p_2)} = (p_2 - p_2^2) \alpha_1^{(p_1)}$. Since p_1 is a generator of $(\mathbb{Z}/p)^*$, $p_1 - p_1^2$ is invertible in $\mathbb{Z}_{(p)}$. Therefore, $\alpha_1^{(p_2)} = \lambda \alpha_1^{(p_1)}$ with $\lambda \in \mathbb{Z}_{(p)} \subset \Omega_U \otimes \mathbb{Z}_{(p)}$. Thus, for any prime $p_2 \neq p_1$ the coefficient $\alpha_1^{(p_2)}$ is a multiple of $\alpha_1^{(p_1)}$, and that is why it could be excluded from the set of generators of $\widetilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$.

Now, consider the coefficient system $\alpha_1, \dots, \alpha_k, \dots$ introduced in the statement of the theorem (that is, α_i is the coefficient by x^{i+1} in the series $[x]_{p_1}$ if i is not divisible by $p-1$ and is the coefficient by x^{i+1} in the series $[x]_p$ if i is divisible by $p-1$). Suppose that we have proved that this coefficient system is a set of generators for $\widetilde{\Lambda}(1)\otimes\mathbb{Z}_{(p)}$ in all dimensions up to $n-1$. Hence, for any q and $k\le n-1$, one has

$$\alpha_k^{(q)} = \lambda_1^{(q)}\alpha_1 + \ldots + \lambda_k^{(q)}\alpha_k, \tag{4}$$

where $\lambda_i^{(q)}\in\Omega_U\otimes\mathbb{Z}_{(p)}$. We are going to prove that $\alpha_n^{(q)}$ could also be decomposed in such a way. It follows from the above argument that we can consider only prime q.

First, suppose that n is not divisible by $p-1$. Hence $\alpha_n=\alpha_n^{(p_1)}$, where p_1 is a generator of $(\mathbb{Z}/p)^*$. Let p_2 be any prime. Taking the coefficient of x^{n+1} in both sides of (3), we obtain

$$p_1\alpha_n^{(p_2)} + p_2^{n+1}\alpha_n^{(p_1)} + \mu_1\alpha_1 + \ldots + \mu_{n-1}\alpha_{n-1}$$

$$= p_2\alpha_n^{(p_1)} + p_1^{n+1}\alpha_n^{(p_2)} + \nu_1\alpha_1 + \ldots + \nu_{n-1}\alpha_{n-1}.$$

Here we expressed the coefficients $\alpha_k^{(p_1)}$, $\alpha_k^{(p_2)}$, $k<n$, as linear combinations of generators $\alpha_1,\ldots,\alpha_{n-1}$, i.e., $\mu_i,\nu_i\in\Omega_U\otimes\mathbb{Z}_{(p)}$. Therefore,

$$p_1(1-p_1^n)\alpha_n^{(p_2)} = (p_2-p_2^{n+1})\alpha_n^{(p_1)} + (\nu_1-\mu_1)\alpha_1 + \ldots + (\nu_{n-1}-\mu_{n-1})\alpha_{n-1}. \tag{5}$$

Since p_1 is a generator of $(\mathbb{Z}/p)^*$ and n is not divisible by $p-1$, we deduce that $p_1(1-p_1^n)$ is invertible in $\mathbb{Z}_{(p)}$. Thus, from (5) we obtain that $\alpha_n^{(p_2)}$ is a linear combination of $\alpha_1,\ldots,\alpha_{n-1}$ and $\alpha_n=\alpha_n^{(p_1)}$ with coefficients from $\Omega_U\otimes\mathbb{Z}_{(p)}$.

Now, suppose that n is divisible by $p-1$. Before we proceed further, let us make some preliminary remarks. It is well known (Milnor, Novikov) that a complex cobordism coefficient ring Ω_U is a polynomial ring: $\Omega_U=\mathbb{Z}[a_1,a_2,\ldots,a_n,\ldots]$, $a_n\in\Omega_U^{-2n}$. The ring Ω_U is also the coefficient ring of the (universal) formal group of geometric cobordisms. The coefficient ring of the logarithm of this formal group is the ring $\Omega_U(\mathbb{Z})=\mathbb{Z}[b_1,b_2,\ldots,b_n,\ldots]$, where $b_n=\dfrac{\mathbb{CP}^n}{n+1}$. (This logarithm is $g(u)=u+\sum_n\dfrac{\mathbb{CP}^n}{n+1}u^{n+1}$, cf. [2].) One could find two sets $\{a_i^*\}$ and $\{b_i^*\}$ of multiplicative generators in the rings Ω_U and $\Omega_U(\mathbb{Z})$ such that the inclusion $\iota_0:\Omega_U\to\Omega_U(\mathbb{Z})$ is as follows:

$$\iota_0(a_i^*) = \begin{cases} p\cdot b_i^* & \text{if } i=p^k-1 \text{ for some } k>0,\\ b_i^* & \text{otherwise.}\end{cases}$$

Let B^+ be the set of elements of degree >0 in the ring $B=\Omega_U(\mathbb{Z})$. Then $(B^+)^2$ consists of elements in $\Omega_U(\mathbb{Z})$ that can be decomposed into the product of two nontrivial factors. The map $\iota_0:\Omega_U\to\Omega_U(\mathbb{Z})$ sends the coefficients $\alpha_n^{(p)}$ of the series $[x]_p$ to the element of the form $(p-p^{n+1})b_n+((B^+)^2)$. Therefore, the coefficients $\alpha_{p^k-1}^{(p)}$ could be taken as multiplicative generators of $\Omega_U\otimes\mathbb{Z}_{(p)}$ in dimensions p^k-1. In other dimensions $l\ne p^k-1$, we have $\alpha_l^{(p)}\in p\Omega_U$, i.e., $\alpha_l^{(p)}$ is divisible by p in Ω_U.

Now let us return to the proof of Theorem 1. We rewrite identity (3), replacing p_1 by p:

$$\begin{aligned} &pp_2x + p\textstyle\sum_m \alpha_m^{(p_2)}x^{m+1} + \sum_m \alpha_m^{(p)}(p_2x+\alpha_1^{(p_2)}x^2+\alpha_2^{(p_2)}x^3+\ldots)^{m+1}\\ &= p_2px + p_2\textstyle\sum_m \alpha_m^{(p)}x^{m+1} + \sum_m \alpha_m^{(p_2)}(px+\alpha_1^{(p)}x^2+\alpha_2^{(p)}x^3+\ldots)^{m+1}. \end{aligned} \tag{6}$$

Taking the coefficient of x^{n+1} in both sides, we get

$$\begin{aligned} &p\alpha_n^{(p_2)} + p_2^{n+1}\alpha_n^{(p)} + \left\langle \textstyle\sum_{m<n}\alpha_m^{(p)}(p_2x+\alpha_1^{(p_2)}x^2+\alpha_2^{(p_2)}x^3+\ldots)^{m+1}\right\rangle_{n+1}\\ &= p_2\alpha_n^{(p)} + p^{n+1}\alpha_n^{(p_2)} + \left\langle \textstyle\sum_{m<n}\alpha_m^{(p_2)}(px+\alpha_1^{(p)}x^2+\alpha_2^{(p)}x^3+\ldots)^{m+1}\right\rangle_{n+1}, \end{aligned}$$

where $\langle\cdot\rangle_{n+1}$ stands for the coefficient of x^{n+1}. Let us again write the coefficients $\alpha_m^{(p_2)}$ for $m<n$ as linear combinations of generators $\alpha_1,\ldots,\alpha_m$. Since $\alpha_m^{(p)}\in p\Omega_U$ for $m\ne p^k-1$, we could rewrite the last identity

as

$$
\begin{aligned}
p(1-p^n)\alpha_n^{(p_2)} &= p_2(1-p_2^n)\alpha_n^{(p)} + p(\mu_1\alpha_1 + \ldots \mu_n\alpha_n) \\
&\quad - \left\langle \sum_{k:p^k-1<n} \alpha_{p^k-1}^{(p)} \left(p_2 x + \alpha_1^{(p_2)}x^2 + \alpha_2^{(p_2)}x^3 + \ldots\right)^{p^k} \right\rangle_{n+1} \\
&\quad + \left\langle \sum_{m<n} \alpha_m^{(p_2)} \left(\alpha_{p-1}^{(p)}x^p + \alpha_{p^2-1}^{(p)}x^{p^2} + \ldots + \alpha_{p^k-1}^{(p)}x^{p^k} + \ldots\right)^{m+1} \right\rangle_{n+1}.
\end{aligned} \tag{7}
$$

The last two summands in the above formula can be rewritten as $\alpha_{p-1}^{(p)}\nu_1 + \alpha_{p^2-1}^{(p)}\nu_2 + \ldots + \alpha_{p^k-1}^{(p)}\nu_k$, where $p^k - 1 < n$, $\nu_i \in \Omega_U$. The other terms in the above identity belong to $p\Omega_U \otimes \mathbb{Z}_{(p)}$ (i.e., are divisible by p). The coefficients $\alpha_{p^i-1}^{(p)}$ are the multiplicative generators of $\Omega_U \otimes \mathbb{Z}_{(p)}$ in the dimensions $p^i - 1$. Therefore, we also have $\nu_i \in p\Omega_U \otimes \mathbb{Z}_{(p)}$, i.e., ν_i is divisible by p in $\Omega_U \otimes \mathbb{Z}_{(p)}$. Let $\nu_i = p\kappa_i$ with $\kappa_i \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Then from (7), we get

$$
\begin{aligned}
p(1-p^n)\alpha_n^{(p_2)} &= p_2(1-p_2^n)\alpha_n^{(p)} + p(\mu_1\alpha_1 + \ldots + \mu_n\alpha_n) \\
&\quad + p(\alpha_{p-1}^{(p)}\kappa_1 + \alpha_{p^2-1}^{(p)}\kappa_2 + \ldots + \alpha_{p^k-1}^{(p)}\kappa_k),
\end{aligned}
$$

where $p^k - 1 < n$, $\mu_i, \kappa_i \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Since n is divisible by $p-1$, we obtain that $1 - p_2^n$ is divisible by p (for $p_2 \neq p$). Hence the entire identity above is divisible by p. Dividing it by p and noting that $1-p^n$ is invertible in $\Omega_U \otimes \mathbb{Z}_{(p)}$, we obtain that $\alpha_n^{(p_2)} = \frac{p_2(1-p_2^n)}{p(1-p^n)}\alpha_n^{(p)} + \lambda_1\alpha_1 + \ldots + \lambda_{n-1}\alpha_{n-1}$. Thus, setting $\lambda_n = \frac{p_2(1-p_2^n)}{p(1-p^n)}\Omega_U \otimes \mathbb{Z}_{(p)}$, we obtain the decomposition of type (4) for $\alpha_n^{(p_2)}$ (since now $\alpha_n = \alpha_n^{(p)}$), which completes the proof of Theorem 1.

Corollary 1. *Let p_1 be a generator of the cyclic group $(\mathbb{Z}/p)^*$. The following set could be taken as a set of generators of the $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$:*
$\alpha_0 = p$;
$\alpha_n = \alpha_n^{(p_1)}$ *if n is not divisible by $p-1$;*
$\alpha_{p^k-1} = \alpha_{p^k-1}^{(p)}$, $k = 1, 2, \ldots$
(*and no generators in other dimensions*).

The following set could be taken as a set of generators of the $\Omega_U \otimes \mathbb{Z}/p$-module $\widetilde{\Lambda}(1) \otimes \mathbb{Z}/p$:
$\alpha_n = \alpha_n^{(p_1)}$ *if n is not divisible by $p-1$;*
$\alpha_{p^k-1} = \alpha_{p^k-1}^{(p)}$, $k = 1, 2, \ldots$
(*and no generators in other dimensions*).

Proof. Let us consider the set of generators of $\widetilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ constructed in Theorem 1. If n is divisible by $p-1$ and $n \neq p^k - 1$, then the elements α_n are divisible by p, i.e., belong to $p\Omega_U$. All other α_n do not belong to $p\Omega_U$. Therefore, one could take the sets described in the corollary as generators of the corresponding modules. The corollary is proved.

Below we are going to use the description of the $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ in order to prove a result similar to the well-known Stong–Hattori theorem [1]. Namely, we are going to describe the set of cobordism classes of manifolds with a simple $\mathbb{Z}/p$-action in terms of characteristic numbers.

As was mentioned in [3], the homomorphism $\Phi : F_1 \to \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ can be extended up to a homomorphism $\gamma_p : F_0 \to \Omega_U(\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$. For this γ_p one has

$$
\gamma_p(x_1, x_2, \ldots, x_k) = \left\langle \left(\prod_{j=1}^{k} \frac{u}{[u]_{x_j}} \right) \frac{pu}{[u]_p} \right\rangle_k,
$$

where $\gamma_p(x_1, x_2, \ldots, x_k) := \gamma_p(\alpha_{2k-1}(x_1, x_2, \ldots, x_k))$, $\alpha_{2k-1}(x_1, x_2, \ldots, x_k) \in F_0$. In particular, $\gamma_p(1, \ldots, 1) = \left\langle \frac{pu}{[u]_p} \right\rangle_k$.

Thus, for any (simple) action of $\mathbb{Z}/p$ on M^{2n}, the mod p cobordism class of M^{2n} is expressed in terms of cobordism classes $\lambda_j \in \Omega_U$ of fixed submanifolds and weights $(x_k^{(j)}) \in (\mathbb{Z}/p)^*$ in the corresponding (trivial) normal bundles as follows:

$$[M^{2n}] \equiv \sum_j \lambda_j \gamma_p(x_1^{(j)}, \dots, x_{m_j}^{(j)}) \bmod p\Omega_U \tag{8}$$

Now, the following question arises: which elements of the form $\sum_j \lambda_j \gamma_p(x_1^{(j)}, \dots, x_{m_j}^{(j)}) \in \Omega_U(\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$ represent cobordism classes of manifolds with simple $\mathbb{Z}/p$-action? This question was first posed in [3] and is analogous to the Milnor–Hirzebruch problem of describing the set of elements in $\Omega_U(\mathbb{Z})$ representing the cobordism classes of (stably complex) manifolds. While the Milnor–Hirzebruch problem is solved in the Stong–Hattori theorem, the answer to the above question is given in our Theorem 2. We will need the following definition.

Definition 2. Let $\omega = \sum_i k_i \cdot (i)$, $i, k_i \in \mathbb{Z}$, $i > 0$, $k_i \geq 0$, be a partitioning of $n = \|\omega\| = \sum_i k_i \cdot i$ (i.e., n is decomposed into a sum of positive integers, and the number i enters this sum k_i times). We say that the partitioning ω is *divisible by* $p-1$ if all i such that $k_i \neq 0$ are divisible by $p-1$ (i.e., all the summands are divisible by $p-1$; obviously, such partitionings exist only for those n divisible by $p-1$). We say that the partitioning ω is *non-p-adic* if, for any $j > 0$, one has $k_{p^j-1} = 0$ (i.e., there are no summands of the form $p^j - 1$).

Theorem 2. *The element $\sigma \in \Omega_U(\mathbb{Z})^{-2n} \otimes \mathbb{Z}_{(p)}$ belongs to the $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ and, therefore, represents the cobordism class of the manifold with simple $\mathbb{Z}/p$-action if and only if all its K-theory characteristic numbers $s_\omega(\sigma)$, $\omega = \sum_i k_i \cdot (i)$, $\|\omega\| = \sum_i k_i \cdot i \leq n$ belong to $\mathbb{Z}_{(p)}$, and for all partitionings ω divisible by $p-1$ the cohomological characteristic numbers $s_\omega(\sigma)$, $\|\omega\| = n$ are zero* mod p.

Proof. (a) *Necessity.* Let $\sigma \in \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$. Note that the set of generators for the $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ described in Corollary 1 has the following property: each of its elements $\alpha_i \in \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ is at the same time a multiplicative generator of $\Omega_U \otimes \mathbb{Z}_{(p)}$ in dimension $-2i$. However, the above set of generators of $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ has no elements in dimensions $-2i$ such that i is divisible by $p-1$ and $i \neq p^k - 1$. Therefore, we add the missing generators α_i in these dimensions to get the whole set of multiplicative generators for $\Omega_U \otimes \mathbb{Z}_{(p)}$. Now, we have $\Omega_U \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[\alpha_1, \alpha_2, \dots]$.

Since $\sigma \in \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)} \subset \Omega_U \otimes \mathbb{Z}_{(p)}$, we see that all K-characteristic numbers $s_\omega(\sigma)$ are in $\mathbb{Z}_{(p)}$.

If n is not divisible by $p-1$, then there are no partitionings ω divisible by $p-1$.

Now, let $n = m(p-1)$. One could write σ as a homogeneous polynomial of degree $-2m(p-1)$ in α_i:

$$\sigma = \sum_{\|\omega\| = m(p-1)} r_\omega \alpha_\omega = r_{m(p-1)} \alpha_{m(p-1)} + \dots, \tag{9}$$

where $\omega = \sum_i k_i \cdot (i)$, $\alpha_\omega = \alpha_1^{k_1} \cdot \alpha_2^{k_2} \cdot \dots$. Now, it follows from the description of $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ given in Corollary 1 that $\sigma \in \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ if and only if for all non-p-adic and divisible by $p-1$ partitionings ω the coefficients r_ω in decomposition (9) are zero modulo p.

Consider the Chern–Dold character ch_U in cobordisms [3]: $\mathrm{ch}_U(u) = t + \sum_{i \geq 1} \beta_i t^{i+1}$. Here $u = c_1^U(\zeta) \in U^2(\mathbb{CP}^\infty)$ is the first cobordism Chern class of the universal line bundle, $t = c_1^H(\zeta) \in H^2(\mathbb{CP}^\infty)$ is that in cohomologies, and the coefficients β_i are from $\Omega_U(\mathbb{Z})$. Then, for any $\sigma \in \Omega_U^{-2n}$, $\sigma = \sum_{\|\omega\|=n} s_\omega(\sigma)\beta_\omega$ holds and $B = \Omega_U(\mathbb{Z}) = \mathbb{Z}[\beta_1, \beta_2, \dots]$ (i.e., the coefficient ring of the Chern–Dold character coincides with $\Omega_U(\mathbb{Z})$). Moreover, $\alpha_i = e_i \cdot \beta_i + ((B^+)^2)$ if $i \neq p^k - 1$ and $\alpha_i = pe_i \cdot \beta_i + ((B^+)^2)$ if $i = p^k - 1$ with invertible $e_i \in \mathbb{Z}_{(p)}$. Now, let us write σ as a homogeneous polynomial in β_i. Since all β_i have integer homological characteristic numbers, to prove the necessity of the theorem it suffices to show that the coefficient of β_ω in the decomposition of σ is zero modulo p for all partitionings $\omega = \sum_i k_i \cdot (i)$ divisible by $p-1$. This coefficient is the corresponding homological characteristic number $s_\omega(\sigma)$, which could be decomposed as follows:

$$s_\omega(\sigma) = \sum_{\omega' : \omega' \supset \omega} r_{\omega'} s_\omega(\alpha_{\omega'}), \tag{10}$$

where $\omega' \supset \omega$ means that ω refines ω'. This coefficient is divisible by p. Indeed, if the partitioning $\omega' = \sum_i k'_i \cdot (i)$ is divisible by $p-1$ and is non-p-adic, then $r_{\omega'}$ is zero modulo p, since $\sigma \in \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ (see above). If summands of the form $p^k - 1$ enter the partitioning ω', then $\alpha_{\omega'} \in p\Omega_U(\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$, i.e., $s_\omega(\alpha_{\omega'})$ is divisible by p. The necessity of the theorem is proved.

(b) *Sufficiency.*

Since all K-characteristic numbers of σ are in $\mathbb{Z}_{(p)}$, one could deduce from the Stong–Hattori theorem [6] that $\sigma \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Suppose, moreover, that the characteristic number $s_\omega(\sigma)$ is zero modulo p for any divisible by $p-1$ partitioning $\omega = \sum_i k_i \cdot (i)$, $\|\omega\| = n$.

Consider again the generator set $\alpha_1, \alpha_2, \dots$ for $\Omega_U \otimes \mathbb{Z}_{(p)}$ constructed above. In order to prove that $\sigma \in \widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$, one needs to show that for every divisible by $p-1$ and non-p-adic partitioning $\omega = \sum_i k_i \cdot (i)$, the coefficient r_ω in decomposition (9) is zero modulo p. Let ω be such a partitioning. We can rewrite identity (10) as follows:

$$s_\omega(\sigma) = r_\omega s_\omega(\alpha_\omega) + \sum_{\omega' \supset \omega, \omega' \neq \omega} r_{\omega'} s_\omega(\alpha_{\omega'}). \tag{11}$$

One can assume by induction that if a partitioning ω' such that $\omega' \supset \omega$, $\omega' \neq \omega$, $\|\omega'\| = m(p-1)$ is non-p-adic, then the coefficient $r_{\omega'}$ is divisible by p. If the partitioning $\omega' = \sum_i k'_i \cdot (i)$ is not non-p-adic (i.e., there are some summands of the form $p^k - 1$), then $s_\omega(\alpha_{\omega'})$ is divisible by p. In any case, the second summand on the right-hand side of (11) is zero modulo p. The left-hand side of (11) is zero modulo p by assumption. Since ω is non-p-adic, we have $\alpha_\omega = e \cdot \beta_\omega + \dots$ with invertible $e \in \mathbb{Z}_{(p)}$. Therefore, $s_\omega(\alpha_\omega)$ is not divisible by p. Thus, it follows from (11) that r_ω is zero modulo p. The theorem is proved.

Corollary 2. *The element $\sigma \in \Omega_U$ represents the cobordism class of manifolds with $\mathbb{Z}/p$-action whose fixed-point set has trivial normal bundle if and only if for all divisible by $p-1$ partitionings ω, the cohomological characteristic numbers $s_\omega(\sigma)$, $\|\omega\| = n$, are zero modulo p.*

Corollary 3. *Each cobordism class of dimension $n \leq 4p - 6$ contains a manifold M^n with simple action of $\mathbb{Z}/p$.*

In dimension $n = 4p - 4$, there exist manifolds (e.g., $\mathbb{CP}^{2p-2}$) whose cobordism class does not contain a manifold with a simple action of $\mathbb{Z}/p$.

In [1], it was shown by means of methods not involving the formal group theory that a cobordism class $\sigma \in \Omega_U$ contains a manifold with *strictly* simple action of $\mathbb{Z}/p$ if and only if *all* the characteristic numbers $\sigma_\omega(\sigma)$ are zero modulo p. More precisely, it was shown in [1] that the set of cobordism classes of manifolds with strictly simple $\mathbb{Z}/p$-action coincides with the Ω_U-module spanned by the set $Y^0 = p, Y^1, Y^2, \dots$, where $Y^i \in \Omega_U^{p^i - 1}$ are the so-called "Milnor manifolds." The manifold Y^i is uniquely determined by the following conditions: $s_{(p^i-1)}(Y^i) = p$ and $s_\omega(Y^i)$ is divisible by p for every ω. For the purposes of description of the set of cobordism classes in terms of characteristic numbers, one could consider $\Omega_U \otimes \mathbb{Z}_{(p)}$-modules instead of Ω_U-modules. Hence one could take the elements $\alpha^{(p)}_{p^i-1}$ from Corollary 1 as the representatives of the cobordism classes of Y^i. Now we see that the $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\Omega_U[p, Y^1, Y^2, \dots] \otimes \mathbb{Z}_{(p)}$ studied by Conner and Floyd is included in our $\Omega_U \otimes \mathbb{Z}_{(p)}$-module $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$. Indeed, the set of generators for the Conner–Floyd module is a subset of the generator set for $\widetilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$.

Finally, we note that if a certain cobordism class $\sigma \in \Omega_U$ contains a representative M with strictly simple action of $\mathbb{Z}/p$, then any simple action of $\mathbb{Z}/p$ on M need not be strictly simple. Indeed, let us consider $M_1 = \mathbb{CP}^{p-1}$ on which the generator $\rho \in \mathbb{Z}/p$ acts as follows: $\rho(z_1 : \ldots : z_p) = (z_1 : \rho z_2 : \ldots : \rho^{p-1} z_p)$ (this simple action with p fixed points is strictly simple as well), and $M_2 = \mathbb{CP}^1$, $\rho(z_1 : z_2) = (z_1 : \rho z_2)$ (this simple action with 2 fixed points is not strictly simple). Then one has two simple actions of $\mathbb{Z}/p$ on $M = M_1 \times M_2$: $\rho(a, b) = (\rho a, b)$ and $\rho(a, b) = (a, \rho b)$, $a \in \mathbb{CP}^{p-1}$, $b \in \mathbb{CP}^1$. The first one is strictly simple, while the second one is not.

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