CALCULATION OF HIRZEBRUCH GENERA FOR MANIFOLDS ACTED ON BY THE GROUP $\mathbb{Z}/p$ VIA INVARIANTS OF THE ACTION

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Abstract. We obtain general formulae expressing Hirzebruch genera of a manifold with $\mathbb{Z}/p$-action in terms of invariants of this action (the sets of weights of fixed points). As an illustration, we consider numerous particular cases of well-known genera, in particular, the elliptic genus. We also describe the connection with the so-called Conner–Floyd equations for the weights of fixed points.

Introduction

In this paper we obtain general formulae expressing Hirzebruch genera of a manifold acted on by $\mathbb{Z}/p$ with finitely many fixed points or fixed submanifolds with trivial normal bundle via invariants of this action. We also describe the connection with the so-called Conner–Floyd equations for the weights of fixed points.

Actions of $\mathbb{Z}/p$ were studied in [12], [13], [11], [8], where the so-called Conner–Floyd equations were deduced within cobordism theory (see formulae (31), (32)). These equations form necessary and sufficient conditions for a sets of elements of $\mathbb{Z}/p$ to be the set of weights of some $\mathbb{Z}/p$-action (see § 3 for the definition). Two approaches for the calculation of Hirzebruch genera of a stably complex manifold with a $\mathbb{Z}/p$-action were proposed in [5].

The first approach is based on the application of the Atiyah–Bott–Lefschetz fixed point formula [1], and so for its realization it is necessary to have an elliptic complex of bundles that are associated to the tangent bundle of the manifold. The Atiyah–Bott–Lefschetz formula obtained in [1] generalizes the classical Lefschetz formula for the number of fixed points and enables us to calculate the equivariant index of an elliptic complex of bundles over a manifold by means of certain contribution functions of the fixed submanifolds (see the details below). In particular, if an operator acts on a manifold with finitely many fixed points, the corresponding equivariant index can be expressed in terms of the fixed point weights. It was shown in [5] how to express the Todd genus, which is the index of an elliptic complex (namely, the Dolbeault complex) over a manifold with $\mathbb{Z}/p$-action, via the equivariant index of the same complex for the action of the generator of $\mathbb{Z}/p$. This equivariant index enters into the Atiyah–Bott–Lefschetz formula. In this way one deduces the formulae expressing the Todd genus in terms of the weights of

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fixed points for this $\mathbb{Z}/p$-action. The formulae obtained by this method contain the number-theoretical trace of a certain algebraic extension of fields of degree $(p-1)$. In this paper, we use the same approach to obtain the formulae for other genera of manifolds with $\mathbb{Z}/p$-action: the signature (or the L-genus), the Euler number, the $\hat{A}$-genus, the general χ_y-genus, and the elliptic genus. By this method we also obtain some general equations (see. § 5) for an arbitrary Hirzebruch genus having the property to be the index of a certain elliptic complex of bundles associated to the tangent bundle of a manifold with $\mathbb{Z}/p$-action.

In this paper we use the generalized Lefschetz formula in the somewhat different formulation, stated in [3]. This formula and especially the "recipe" it suggests for calculating the equivariant index of a complex via contribution functions of fixed points (see. § 5) are more convenient for applications than the formula from [1] used in [5]. The generalized Lefschetz formula was deduced in [3] from the cohomological form of the Atiyah–Singer index theorem, which was also proved there. We apply both formulae: some of the results (see § 4) we obtain are based on the "old" Lefschetz formula of [1], while others use the "recipe" in § 5 based on the formula from [3].

Another approach to the equations for Hirzebruch genera, also proposed in [5], is based on an application of cobordism theory in the same way as in the derivation of the Conner–Floyd equations in [12], [13]. In [5], the authors give a formula expressing the $\bmod p$ cobordism class of a stably complex manifold with a $\mathbb{Z}/p$-action in terms of invariants of the action. Then they show that the difference between the two formulae for the Todd genus (obtained by the first and second methods) is exactly the sum of the Conner–Floyd equations for the Todd genus. In this paper we show (see Theorem 7.1) that the difference between the two seemingly different formulae deduced by these two methods for an arbitrary genus is a weighted sum (with integer coefficients) of the Conner–Floyd equations for this genus.

A case of particular interest is that of the so-called elliptic genus. In Witten's papers, a certain invariant was assigned to each oriented $2n$-dimensional manifold M^{2n}. This invariant is the equivariant index of the Dirac-like operator for the canonical action of circle S^1 on manifold's loop space. S. Ochanine [14] showed that this index is a Hirzebruch genus corresponding to the elliptic sine; which led to the term "elliptic genus". In [2], [9] and other papers, the rigidity theorem for the elliptic genus of manifolds with S^1-action was proved. This theorem states that if we regard the equivariant elliptic genus $\varphi_{S^1}(M)$ of such a manifold as a character of the group S^1, then $\varphi_{S^1}(M)$ is the trivial character and is equal to the elliptic genus $\varphi(M)$. At the same time, the elliptic genus takes its values in the ring $\mathbb{Z}\left[\frac{1}{2}\right][\delta,\varepsilon]$, and its value on any manifold M^{2n} is a modular form of weight n on the subgroup $\Gamma_0(2)\subset \mathrm{SL}_2(\mathbb{Z})$ (cf. [7]). In this paper we obtain formulae for the elliptic genus of a manifold with a $\mathbb{Z}/p$-action having finitely many fixed points. A summary of results of this part of the paper has already been published in [15]. As an application we deduce certain relations between the Legendre polynomials by applying our formulae to a special action of $\mathbb{Z}/p$ on $\mathbb{C}P^n$ (see § 8).

In the remaining part of this article (see § 9) we generalize our constructions to the case of $\mathbb{Z}/p$-actions with fixed submanifolds whose normal bundles are trivial.

§ 1. Necessary information about Hirzebruch genera

Let M^{2n} be a manifold with a complex structure in its stable tangent bundle, that is, there is k such that $TM \oplus 2k$ is a complex bundle. We write the total Chern class of the tangent bundle TM as

$$c(TM) = 1 + c_1(M) + c_2(M) + \cdots + c_n(M) = (1 + x_1)(1 + x_2)\cdots(1 + x_n).$$

This means that the total Chern class of M is written as the product of the Chern classes of "virtual" line bundles whose sum gives TM. Therefore, $c_i(M)$ is the ith elementary symmetric function in $x_1, \dots, x_n$.

To each series of the form $Q(x) = 1 + \cdots$ with coefficients in a certain ring Λ there corresponds the Hirzebruch genus $\varphi_Q(M^{2n}) = \big(\prod_{i=1}^n Q(x_i)\big)[M^{2n}]$ (see [6]). Along with $Q(x)$, we introduce $f(x) = x/Q(x)$ and $g(u) = f^{-1}(u)$. Each Hirzebruch genus φ gives rise to a formal group law $F_\varphi(u, v) = g_\varphi^{-1}\big(g_\varphi(u) + g_\varphi(v)\big)$ with logarithm $g_\varphi(u)$ (see [5]). The corresponding power system $[u]_n^\varphi = g_\varphi^{-1}\big(ng_\varphi(u)\big)$ (the nth power in the formal group law F_φ).

Hirzebruch genera for real orientable manifolds M^{4n} are defined similarly. Here we replace the Chern classes c_i by the Pontryagin classes p_i, and x_i by x_i^2, that is,

$$p(TM) = 1 + p_1(M) + p_2(M) + \cdots + p_n(M) = (1 + x_1^2)(1 + x_2^2)\cdots(1 + x_n^2).$$

Hence we now have $Q(x^2)$ instead of $Q(x)$.

The formulae obtained in this paper refer mainly to the following Hirzebruch genera.

1. The universal genus $\underline{\varphi}$ corresponds to the identity homomorphism $\mathrm{id}\colon \Omega_U \otimes \mathbb{Q} \to \Omega_U \otimes \mathbb{Q}$. The corresponding formal group law of "geometric cobordisms" $\underline{F}(u, v)$ (see [12]) is universal, that is, for any formal group law $F(x, y)$ over a ring Λ there is unique ring homomorphism $\lambda\colon \Omega_U \to \Lambda$ such that $F(x, y) = \lambda\big[\underline{F}(u, v)\big]$ (see [5]). The logarithm of $\underline{F}(u, v)$ is

$$\underline{g}(u) = \sum_{n=0}^\infty \frac{\underline{\varphi}[\mathbb{C}P^n]}{n+1} u^{n+1} = \sum_{n=0}^\infty \frac{\mathbb{C}P^n}{n+1} u^{n+1}.$$

Therefore, for any Hirzebruch genus φ we have

$$g_\varphi(u) = \sum_{n=0}^\infty \frac{\varphi[\mathbb{C}P^n]}{n+1} u^{n+1}. \tag{1}$$

2. The Todd genus $\mathrm{td}(M)$ corresponds to the following series:

$$Q_{\mathrm{td}}(x) = \frac{x}{1 - e^{-x}}, \quad f_{\mathrm{td}}(w) = 1 - e^{-w}, \quad g_{\mathrm{td}}(u) = -\ln(1 - u).$$

These formulae could also be deduced from the fact that $\mathrm{td}(\mathbb{C}P^n) = 1$. Indeed, using the identity (1), we obtain

$$g_{\mathrm{td}}(u) = \sum_{n=0}^\infty \frac{u^{n+1}}{n+1} = -\ln(1 - u).$$

The corresponding power system is

$$[u]_n^{\mathrm{td}} = g_{\mathrm{td}}^{-1}\big(ng_{\mathrm{td}}(u)\big) = 1 - e^{n\ln(1-u)} = 1 - (1 - u)^n.$$

Thus, for the Todd genus

$$f_{\mathrm{td}}(w)=1-e^{-w},\quad g_{\mathrm{td}}(u)=-\ln(1-u),\quad [u]_n^{\mathrm{td}}=1-(1-u)^n. \tag{2_{td}}$$

3. The Euler number $e(M)$ (of the tangent bundle). Since $e(\mathbb{C}P^n)=n+1$, we deduce (see [13]) that

$$g_e(u)=\frac{u}{1-u},\quad f_e(w)=g_e^{-1}(w)=\frac{w}{1+w}.$$

So $Q(x)=1+x$ and, as might be expected,

$$e[M]=\left(\prod_{i=1}^n Q(x_i)\right)[M]=(x_1x_2\dots x_n)[M^{2n}]=c_n[M^{2n}].$$

Next,

$$[u]_n^e=g_e^{-1}\big(ng_e(u)\big)=\frac{nu}{1+(n-1)u}.$$

Thus, for the Euler number,

$$f_e(w)=\frac{w}{1+w},\quad g_e(u)=\frac{u}{1-u},\quad [u]_n^e=\frac{nu}{1+(n-1)u}. \tag{2_e}$$

4. The signature (the L-genus) corresponds to the following series:

$$Q_L(x)=\frac{x}{\tanh(x)}=\frac{x(e^x+e^{-x})}{e^x-e^{-x}},\quad f_L(w)=\tanh(w),$$
$$g_L(u)=\operatorname{arctanh}(u)=\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right).$$

This is in accordance with Hirzebruch's theorem (the L-genus equals the signature, see [6]), from which we deduce that $L(\mathbb{C}P^{2n})=1$, $L(\mathbb{C}P^{2n+1})=0$. The corresponding power system is

$$[u]_n^L=g_L^{-1}\big(ng_L(u)\big)=\frac{e^{\frac{n}{2}\ln\frac{1+u}{1-u}}-e^{-\frac{n}{2}\ln\frac{1+u}{1-u}}}{e^{\frac{n}{2}\ln\frac{1+u}{1-u}}+e^{-\frac{n}{2}\ln\frac{1+u}{1-u}}}=\frac{(1+u)^n-(1-u)^n}{(1+u)^n+(1-u)^n}.$$

Thus, for the L-genus,

$$f_L(w)=\tanh(w),\quad g_L(u)=\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right),\quad [u]_n^L=\frac{(1+u)^n-(1-u)^n}{(1+u)^n+(1-u)^n}. \tag{2_L}$$

5. There is a one-parametric genus which generalizes three previous examples, namely, the χ_y-genus. This is the Hirzebruch genus that corresponds to the following series:

$$Q_{\chi_y}=\frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}},$$
$$f_{\chi_y}(w)=\frac{w}{Q_{\chi_y}(w)}=\frac{1-e^{-w(1+y)}}{1+ye^{-w(1+y)}},\quad g_{\chi_y}(u)=\frac{1}{1+y}\ln\frac{1+yu}{1-u}.$$

Neglecting the normalizing condition $f_{\chi_y}(w) = w + \cdots$ and replacing $w(1+y)$ by w, we obtain

$$\hat{f}_{\chi_y}(w) = \frac{1 - e^{-w}}{1 + ye^{-w}}, \quad \hat{g}_{\chi_y}(u) = \ln \frac{1 + yu}{1 - u}.$$

The expression for the power system associated to the χ_y-genus is the same in both cases:

$$[u]_n^{\chi_y} = \frac{(1 + yu)^n - (1 - u)^n}{(1 + yu)^n + y(1 - u)^n} = \frac{1 - \left(\frac{1-u}{1+yu}\right)^n}{1 + y\left(\frac{1-u}{1+yu}\right)^n}. \tag{2_{χ_y}}$$

For $y = 0, -1, 1$ we get respectively the Todd genus, the Euler number and the L-genus.

6. The $\hat{A}$-genus corresponds to the following series:

$$Q_A(x) = \frac{x/2}{\sinh(x/2)} = \frac{x}{e^{x/2} - e^{-x/2}},$$

$$f_A(w) = 2\sinh\left(\frac{w}{2}\right), \quad g_A(u) = 2\operatorname{arcsinh}\left(\frac{u}{2}\right) = \ln\left(\frac{u}{2} + \sqrt{1 + \frac{u^2}{4}}\right).$$

The corresponding power system is

$$\begin{aligned} [u]_n^A &= g_A^{-1}\big(ng_A(u)\big) = 2\sinh\big(n\operatorname{arcsinh}(u/2)\big) \\ &= \left(\frac{u}{2} + \sqrt{1 + \frac{u^2}{4}}\right)^n - \left(\frac{u}{2} + \sqrt{1 + \frac{u^2}{4}}\right)^{-n}. \end{aligned}$$

Thus, for the $\hat{A}$-genus,

$$f_A(w) = 2\sinh(w/2), \quad g_A(u) = 2\operatorname{arcsinh}(u/2), \quad [u]_n^A = 2\sinh\big(n\operatorname{arcsinh}(u/2)\big). \tag{2_A}$$

§ 2. Calculation of Hirzebruch genera by means of the Atiyah–Singer index theorem

We deal with the following elliptic complexes: the de Rham complex Λ^*,

$$\Gamma(\Lambda^0) \xrightarrow{d} \Gamma(\Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Lambda^n),$$

and the Dolbeault complex $\Lambda^{p,*}$,

$$\Gamma(\Lambda^{p,0}) \xrightarrow{d''} \Gamma(\Lambda^{p,1}) \xrightarrow{d''} \cdots \xrightarrow{d''} \Gamma(\Lambda^{p,n}),$$

where $\Lambda^i = \Lambda^i(T^*M)$ is the bundle of differential i-forms on M nd $\Lambda^{p,i} = \Lambda^p(T^*M) \wedge \Lambda^i(\overline{T}^*M) \simeq \Lambda^p(T^*M) \wedge \Lambda^i(TM)$ is the bundle of differential forms of type (p, i). (If the manifold M is endowed with an Hermitian metric, then $\overline{T}^*M \simeq TM$.)

The following is the Atiyah–Singer index theorem in cohomological form (see [3]).

Theorem 2.1. *Let M be a compact, oriented, differentiable manifold of dimension $2n$ and let $E = \{d_i\colon \Gamma E_i \to \Gamma E_{i+1}\}$ be an elliptic complex $(i = 0, \dots, m-1)$ associated to the tangent bundle (that is, all bundles E_i are associated to TM). Then the index of this complex is determined by the following formula:*

$$\operatorname{ind}(E) = (-1)^n \left(\left(\frac{1}{e(TM)} \sum_{i=0}^{m} (-1)^i \operatorname{ch}(E_i) \right) \operatorname{td}(TM \otimes \mathbb{C}) \right) [M],$$

where $\operatorname{ch}(E_i)$ is the Chern character of bundle E_i.

Remark. Formally factoring the Euler class $e(TM) = x_1 \dots x_n$ out of $\operatorname{td}(TM \otimes \mathbb{C}) = \prod_{j=1}^{n} \left(\frac{x_j}{1-e^{-x_j}} \cdot \frac{-x_j}{1-e^{x_j}} \right)$ in the previous expression, we get the following formula:

$$\operatorname{ind}(E) = \left(\left(\sum_{i=0}^{m} (-1)^i \operatorname{ch}(E_i) \right) \prod_{j=1}^{n} \left(\frac{x_j}{1-e^{-x_j}} \cdot \frac{1}{1-e^{x_j}} \right) \right) [M]. \tag{3}$$

Let us consider the elliptic complex $\left\{ \mathrm{X}_i^y = \bigoplus_{p=0}^{n} y^p \Lambda^{p,i} \right\}$. (We note that this object becomes a true elliptic complex only after replacing y by actual integers. But its index is obviously defined for arbitrary y as a polynomial in y. In what follows we regard such objects as elliptic complexes.) Using the Atiyah–Singer index theorem, we easily prove the following fact, which was initially proved by Hirzebruch (see [6]).

Theorem 2.2. *The index of the elliptic complex $\{\mathrm{X}_i^y\}$ equals the χ_y-genus (defined above) of the manifold M, that is,*

$$\operatorname{ind}(\mathrm{X}^y) = \chi_y[M] = \left(\prod_{j=1}^{n} \frac{x_j(1+ye^{-x_j})}{1-e^{-x_j}} \right) [M].$$

Proof. Using formula (3), we have:

$$\begin{aligned}
\operatorname{ind}(\mathrm{X}^y) &= \left(\left(\sum_{i=0}^{n} (-1)^i \operatorname{ch}(\mathrm{X}_i^y) \right) \prod_{j=1}^{n} \left(\frac{x_j}{1-e^{-x_j}} \frac{1}{1-e^{x_j}} \right) \right) [M] \\
&= \left(\operatorname{ch}\left(\sum_{i=0}^{n} (-1)^i \Lambda^i TM \right) \operatorname{ch}\left(\sum_{p=0}^{n} y^p \Lambda^p T^* M \right) \prod_{j=1}^{n} \left(\frac{x_j}{1-e^{-x_j}} \frac{1}{1-e^{x_j}} \right) \right) [M] \\
&= \left(\prod_{j=1}^{n} (1-e^{x_j}) \prod_{j=1}^{n} (1+ye^{-x_j}) \prod_{j=1}^{n} \left(\frac{x_j}{1-e^{-x_j}} \frac{1}{1-e^{x_j}} \right) \right) [M] \\
&= \left(\prod_{j=1}^{n} \frac{x_j(1+ye^{-x_j})}{1-e^{-x_j}} \right) [M] = \chi_y[M].
\end{aligned}$$

Here we have used the following fact (see, for example, [7]).

Lemma 2.3. *Let E be a complex n-dimensional bundle over a differentiable manifold X, and let $c(E) = 1 + c_1(E) + \dots + c_n(E) = (1 + x_1) \dots\dots (1 + x_n)$ be the formal factorization of the total Chern class. We define*

$$\Lambda_t E := \sum_{k=0}^{\infty} (\Lambda^k E) t^k, \quad S_t E := \sum_{k=0}^{\infty} (S^k E) t^k.$$

Then

$$\mathrm{ch}(\Lambda_t E) = \prod_{i=1}^{n} (1 + t e^{x_i}), \quad \mathrm{ch}(S_t E) = \prod_{i=1}^{n} \frac{1}{1 - t e^{x_i}}. \tag{4}$$

In particular, if we consider the complexes $\{\Lambda^{0,i}\}$, $\left\{\mathcal{E}_i = \sum_{p=0}^{n} (-1)^p \Lambda^{p,i}\right\}$, $\left\{L_i = \sum_{p=0}^{n} \Lambda^{p,i}\right\}$, we see that their indexes are respectively $\mathrm{td}[M] = \mathrm{ind}(\Lambda^{0,*})$, $e[M] = \mathrm{ind}(\mathcal{E})$, and $L[M] = \mathrm{ind}(L)$.

Now we construct an elliptic complex whose index, under some additional assumptions, equals the $\hat{A}$-genus of the manifold M. Suppose that $c_1(M) \equiv 0 \mod 2$. (This is equivalent to the condition $w_2(M) = 0$, and hence to the existence of a spinor structure on M.) Then there is a line bundle $\mathcal{L}$ over M such that $\mathcal{L} \otimes \mathcal{L} = \Lambda^n T^* M$, that is, for $c(M) = (1+x_1) \dots (1+x_n)$ we have $c(\mathcal{L}) = 1 - \frac{x_1 + \dots + x_n}{2}$ and $\mathrm{ch}(\mathcal{L}) = \exp\left(-\frac{x_1+\dots+x_n}{2}\right)$. We introduce the complex $\{A_i = \Lambda^{0,i} \otimes \mathcal{L}\}$.

Theorem 2.4. *The index of the above elliptic complex A equals the $\hat{A}$-genus of M:*

$$\mathrm{ind}(A) = \hat{A}[M] = \left(\prod_{j=1}^{n} \frac{x_j}{2 \sinh(x_j/2)} \right) [M].$$

Proof. We again use formulae (3) and (4):

$$\begin{aligned}
\mathrm{ind}(A) &= \left(\mathrm{ch}(\mathcal{L})\, \mathrm{ch}\left(\sum_{i=0}^{n} (-1)^i \Lambda^{0,i} \right) \prod_{j=1}^{n} \left(\frac{x_j}{1 - e^{-x_j}} \frac{1}{1 - e^{x_j}} \right) \right) [M] \\
&= \left(\exp\left(-\frac{x_1 + \dots + x_n}{2} \right) \mathrm{ch}\left(\sum_{i=0}^{n} (-1)^i \Lambda^i TM \right) \prod_{j=1}^{n} \left(\frac{x_j}{1 - e^{-x_j}} \frac{1}{1 - e^{x_j}} \right) \right) [M] \\
&= \left(\prod_{j=1}^{n} e^{-x_j/2} \prod_{j=1}^{n} (1 - e^{x_j}) \prod_{j=1}^{n} \left(\frac{x_j}{1 - e^{-x_j}} \frac{1}{1 - e^{x_j}} \right) \right) [M] \\
&= \left(\prod_{j=1}^{n} \frac{x_j e^{-x_j/2}}{1 - e^{-x_j}} \right) [M] = \left(\prod_{j=1}^{n} \frac{x_j}{2 \sinh(x_j/2)} \right) [M] = \hat{A}(M).
\end{aligned}$$

Thus we have constructed elliptic complexes associated to the tangent bundle of M that enable us to calculate all the Hirzebruch genera in § 1.

§ 3. The problem of calculating Hirzebruch genera for manifolds with $\mathbb{Z}/p$-action in terms of invariants of the action

Let g be a transversal endomorphism (that is, g has only finitely many fixed points) acting on a manifold M^{2n} such that $g^p = 1$ for some prime p. (Hence an action of $\mathbb{Z}/p$ is given.) Let $\mathcal{P}_1, \dots, \mathcal{P}_q$ be the fixed points, and the Jacobi matrix $\mathcal{J}_{\mathcal{P}_j}(g)$ of the map g at the point $\mathcal{P}_j$ ($j = 1, \dots, q$) has eigenvalues

$$\lambda_k^{(j)} = \exp\left(\frac{2\pi i x_k^{(j)}}{p}\right), \quad x_k^{(j)} \neq 0 \mod p, \quad k = 1, \dots, n.$$

Suppose also that there is given an elliptic complex E on the manifold M. Let us give the following definition, which is taken from [1].

Definition 3.1. *A lifting of g to the components of the elliptic complex E* is a set of linear differential operators $\varphi_i\colon \Gamma(g^*E_i) \to \Gamma(E_i)$. Here g^*E_i is the pullback of the bundle E_i under the map g, and $\Gamma(E_i)$ denotes the linear space of sections of the bundle E_i.

Using φ_i, one can define the "geometric" endomorphisms $T_i(g, \varphi)\colon \Gamma(E_i) \to \Gamma(E_i)$ to be the composite of φ and Γ_g, $T_i(g, \varphi) = \varphi_i \circ \Gamma_g$, where $\Gamma_g\colon \Gamma(E_i) \to \Gamma(g^*E_i)$ is the natural map from the sections of the bundle E_i into the sections of the pullback g^*E_i. Under these assumptions we have the following general Atiyah–Bott–Lefschetz theorem (see [1]).

Theorem 3.2. *Suppose that $g\colon M \to M$ is a transversal endomorphism of a compact oriented manifold M. Let E be an elliptic complex on M and let $\varphi_i\colon \Gamma(g^*E_i) \to \Gamma(E_i)$ be a lifting of g to the components of the complex E such that the corresponding "geometric" endomorphisms $T_i(g, \varphi)\colon \Gamma(E_i) \to \Gamma(E_i)$ define an endomorphism $T(g, \varphi)$ of the complex E (that is, the T_i commute with the differentials d_i). Then the "equivariant index"* $\operatorname{ind}(g, E) := \sum_{i=1}^n (-1)^i \operatorname{tr} T_i^*$ *(where $T_i^*\colon H^i(E) \to H^i(E)$) is given by the formula*

$$\operatorname{ind}(g, E) = \sum_{j=1}^q \sigma(\mathcal{P}_j), \tag{5}$$

where $\sigma(\mathcal{P}_j) \in \mathbb{C}$ depends only on local properties of T and φ_i at the point $\mathcal{P}_j$.

In particular, if the operator φ_i induces an endomorphism $\varphi_i(\mathcal{P}_j)\colon E_{i,\mathcal{P}_j} \to E_{i,\mathcal{P}_j}$ at each fixed point $\mathcal{P}_j$, then

$$\sigma(\mathcal{P}_j) = \sum_{i=0}^n (-1)^i \frac{\operatorname{tr} \varphi_i(\mathcal{P}_j)}{\left|\det\left(1 - \mathcal{J}_{\mathcal{P}_j}(g)\right)\right|}. \tag{6}$$

Example 3.1. If $E_i = \Lambda^i(M)$ is the de Rham complex, then $\varphi_i(\mathcal{P}_j) = \Lambda^i \mathcal{J}_{\mathcal{P}_j}(g)$.

Example 3.2. If $E_i = \Lambda^{p,i}(M)$ is the Dolbeault complex, then

$$\varphi_i(\mathcal{P}_j) = \Lambda^p \mathcal{J}'_{\mathcal{P}_j}(g) \wedge \Lambda^i \bar{\mathcal{J}}'_{\mathcal{P}_j}(g),$$

where $\mathcal{J}'_{\mathcal{P}_j}(g)$ is the holomorphic part of the Jacobi matrix $\mathcal{J}_{\mathcal{P}_j}(g)$.

Let us find $\sigma(\mathcal{P}_j)$ for the Dolbeault complex $\Lambda^{p,*}$. Since $\left|\det\bigl(1-\mathcal{J}_{\mathcal{P}_j}(g)\bigr)\right| = \det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)\det\bigl(1-\bar{\mathcal{J}}'_{\mathcal{P}_j}(g)\bigr)$, we deduce from (6) that

$$\sigma(\mathcal{P}_j) = \frac{\operatorname{tr}\left(\Lambda^p \mathcal{J}'_{\mathcal{P}_j}(g) \wedge \sum_i (-1)^i \Lambda^i \bar{\mathcal{J}}'_{\mathcal{P}_j}(g)\right)}{\det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)\det\bigl(1-\bar{\mathcal{J}}'_{\mathcal{P}_j}(g)\bigr)}$$
$$= \frac{\operatorname{tr}\bigl(\Lambda^p \mathcal{J}'_{\mathcal{P}_j}(g)\bigr)\det\bigl(1-\bar{\mathcal{J}}'_{\mathcal{P}_j}(g)\bigr)}{\det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)\det\bigl(1-\bar{\mathcal{J}}'_{\mathcal{P}_j}(g)\bigr)} = \frac{\operatorname{tr}\Lambda^p \mathcal{J}'_{\mathcal{P}_j}(g)}{\det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)}.$$

Here we have used the following well-known identity from linear algebra:

$$\det(1-A) = \sum_i (-1)^i \operatorname{tr}\Lambda^i A \tag{7}$$

for any linear operator A. Thus,

$$\sigma_p(\mathcal{P}_j) = \frac{\operatorname{tr}\Lambda^p \mathcal{J}'_{\mathcal{P}_j}(g)}{\det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)}. \tag{8}$$

Theorem 3.2 and formula (8) enable us to find the fixed point contribution functions $\sigma(\mathcal{P}_j)$ for the complexes $\Lambda^{0,*}$, $\mathcal{E}$, L, X^y. These complexes calculate respectively the Todd genus, the Euler number, the L-genus and the χ_y-genus of the manifold M.

§ 4. Calculations for the Todd genus, Euler number and the L-genus

4.1. Calculations for the Euler number. Let us consider the complex $\mathcal{E}_i = \sum_{p=0}^n (-1)^p \Lambda^{p,i}$, for which $\operatorname{ind}(\mathcal{E}) = e(M)$. In this case

$$\varphi_i(\mathcal{P}_j) = \sum_{p=0}^n (-1)^p \Lambda^p \mathcal{J}'_{\mathcal{P}_j}(g) \wedge \Lambda^i \bar{\mathcal{J}}'_{\mathcal{P}_j}(g),$$

and the fixed point contribution functions are as follows (see (8)):

$$\sigma_e(\mathcal{P}_j) = \sum_{p=0}^n (-1)^p \sigma_p(\mathcal{P}_j) = \frac{\sum_{p=0}^n (-1)^p \operatorname{tr}\Lambda^p \mathcal{J}'_{\mathcal{P}_j}(g)}{\det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)} = 1. \tag{9}$$

Here we have used formula (7). From this and theorem 3.2 we deduce that

$$\operatorname{ind}(g,\mathcal{E}) = \sum_{j=1}^q \sigma_e(\mathcal{P}_j) = q.$$

Since $\frac{1}{p}\sum_{l\in\mathbb{Z}/p} \operatorname{ind}(g^l,\mathcal{E}) = s$ is the alternating sum of dimensions of the invariant subspaces for the action of g on the cohomology of the complex $\mathcal{E}$, and since $\operatorname{ind}(1,\mathcal{E}) = \operatorname{ind}(\mathcal{E}) = e(M)$, we have

$$\operatorname{ind}(1,\mathcal{E}) = e(M) = -\sum_{l=1}^{p-1} \operatorname{ind}(g^l,\mathcal{E}) + ps = ps - q(p-1) = q + p(s-q).$$

Therefore, we obtain the following formula for the Euler number:

$$e(M) \equiv q \pmod{p}. \tag{10}$$

4.2. Calculations for the Todd genus. The calculation of the Todd genus of a stably complex manifold in terms of the $\mathbb{Z}/p$-action was carried out by Buchstaber and Novikov in [5]. To make our exposition complete, we give their results here.

Definition 4.1. *The Atiyah–Bott function* $AB_{\mathrm{td}}(x_1,\dots,x_n)$ of a given fixed point is the following function of the set of weights $x_1,\dots,x_n,\ \ x_i\in\mathbb{Z}/p$:

$$AB_{\mathrm{td}}(x_1,\dots,x_n)=-\operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1}{1-e^{2\pi i x_k/p}}\right), \tag{11}$$

where $\operatorname{Tr}\colon\mathbb{Q}(\zeta)\to\mathbb{Q}$ is the number-theoretical trace, $\zeta:=e^{2\pi i/p}$.

It was shown in [5] that

$$\sum_{j=1}^{q}AB_{\mathrm{td}}(x_1^{(j)},\dots,x_n^{(j)})\equiv\mathrm{td}(M)\mod p, \tag{12}$$

The number-theoretical trace in the definition of Atiyah–Bott functions for the Todd genus was also calculated in [5]:

$$\begin{aligned}AB_{\mathrm{td}}(x_1,\dots,x_n)&\equiv-\left\langle\frac{p[u]^{\mathrm{td}}_{p-1}}{[u]^{\mathrm{td}}_{p}}\prod_{k=1}^{n}\frac{u}{[u]^{\mathrm{td}}_{x_k}}\right\rangle_n\mod p,\\ AB_{\mathrm{td}}(x_1,\dots,x_n)&\equiv\sum_{m=0}^{n}\left\langle\frac{pu}{[u]^{\mathrm{td}}_{p}}\prod_{k=1}^{n}\frac{u}{[u]^{\mathrm{td}}_{x_k}}\right\rangle_m\mod p.\end{aligned} \tag{13}$$

Here $[u]^{\mathrm{td}}_q$ is the qth power in the formal group law corresponding to the Todd genus (see formula (2_T)), and $\langle h(u)\rangle_k$ is the coefficient of u^k in the power series $h(u)$.

4.3. Calculations for the L-genus. Now let us consider the elliptic complex $L=\sum_{p=0}^{n}\Lambda^{p,*}$. Its index is the L-genus: $\mathrm{ind}(L)=L(M)$. In this case, the lifting of g to the components of the complex L has the following form:

$$\varphi_i(\mathcal{P}_j)=\sum_{p=0}^{n}\Lambda^p\mathcal{J}'_{\mathcal{P}_j}(g)\wedge\Lambda^i\bar{\mathcal{J}}'_{\mathcal{P}_j}(g),$$

and the fixed point contribution functions are (see formula (8))

$$\begin{aligned}\sigma_L(\mathcal{P}_j)&=\sum_{p=0}^{n}\sigma_p(\mathcal{P}_j)=\frac{\sum_{p=0}^{n}\operatorname{tr}\Lambda^p\mathcal{J}'_{\mathcal{P}_j}(g)}{\det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)}\\ &=\frac{\det\bigl(1+\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)}{\det\bigl(1-\mathcal{J}'_{\mathcal{P}_j}(g)\bigr)}=\prod_{k=1}^{n}\frac{1+e^{2\pi i x_k^{(j)}/p}}{1-e^{2\pi i x_k^{(j)}/p}}.\end{aligned} \tag{14}$$

Here we have again used formula (7). Hnece it follows from theorem 3.2 that the equivariant index is

$$\mathrm{ind}(g,L)=\sum_{j=1}^{q}\sigma_L(\mathcal{P}_j)=\sum_{j=1}^{q}\prod_{k=1}^{n}\frac{1+e^{2\pi i x_k^{(j)}/p}}{1-e^{2\pi i x_k^{(j)}/p}}.$$

As before, $\frac{1}{p}\sum_{l\in\mathbb{Z}/p}\operatorname{ind}(g^l,L)=s$ is the alternating sum of the dimensions of the invariant subspaces for the action of g on the cohomology of the complex L, and we have $\operatorname{ind}(1,L)=\operatorname{ind}(L)=L(M)$. Hence

$$L(M)=\operatorname{ind}(1,L)=-\sum_{l=1}^{p-1}\operatorname{ind}(g^l,L)+ps=-\sum_{j=1}^{q}\sum_{l=1}^{p-1}\prod_{k=1}^{n}\frac{1+e^{2\pi i x_k^{(j)}l/p}}{1-e^{2\pi i x_k^{(j)}l/p}}+ps.$$

Again, we consider the number-theoretical trace $\operatorname{Tr}\colon \mathbb{Q}(\zeta)\to\mathbb{Q}$ and introduce the Atiyah–Bott functions $AB_L(x_1,\dots,x_n)$ as

$$AB_L(x_1,\dots,x_n)=-\operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1+e^{2\pi i x_k/p}}{1-e^{2\pi i x_k/p}}\right). \tag{15}$$

Then we have

$$\sum_{j=1}^{q}AB_L\big(x_1^{(j)},\dots,x_n^{(j)}\big)\equiv L(M)\mod p. \tag{16}$$

Relations (15) and (16) are analogous to relations (11), (12) for the Todd genus. It remains to calculate the number-theoretical trace in the definition of the Atiyah–Bott functions $AB_L(x_1,\dots,x_n)$.

We set $\theta=\frac{1-\zeta}{1+\zeta}$. Then $\zeta=e^{2\pi i/p}=\frac{1-\theta}{1+\theta}$. We must calculate $\operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1+\zeta^{x_k}}{1-\zeta^{x_k}}\right)$. We shall carry out our calculations in the p-adic extension $\mathbb{Q}_p(\zeta)$ of the field $\mathbb{Q}(\zeta)$. Let us write $A\simeq B$ if A and B are equal modulo $p\mathbb{Z}_p\subset\mathbb{Q}_p$. First of all, we prove the following statement.

Lemma 4.2. $\operatorname{Tr}(\theta^k)\simeq 0$ *for any* $k\geq 1$.

Proof. Since $2=\zeta^p+1=(\zeta+1)(\zeta^{p-1}-\zeta^{p-2}+\zeta^{p-3}-\dots-\zeta+1)$, we have

$$\theta=\frac{(1-\zeta)(\zeta^{p-1}-\zeta^{p-2}+\zeta^{p-3}-\dots-\zeta+1)}{2}=\zeta^{p-1}-\zeta^{p-2}+\zeta^{p-3}-\dots-\zeta.$$

Therefore,

$$\operatorname{Tr}\theta^k=\sum_{m=1}^{p-1}\big((\zeta^m)^{p-1}-(\zeta^m)^{p-2}+(\zeta^m)^{p-3}-\dots-\zeta^m\big)^k\simeq 0$$

because

$$\sum_{m=1}^{p-1}\big((\zeta^m)^{p-1}-(\zeta^m)^{p-2}+\dots-\zeta^m\big)^k=\sum_{m=0}^{p-1}\big((\zeta^m)^{p-1}-(\zeta^m)^{p-2}+\dots-\zeta^m\big)^k,$$

and because $\sum_{m=0}^{p-1}(\zeta^m)^r\simeq 0$ for any r. The lemma is proved.

We further deduce that

$$\prod_{k=1}^{n}\frac{1+\zeta^{x_k}}{1-\zeta^{x_k}}=\prod_{k=1}^{n}\frac{1+\left(\frac{1-\theta}{1+\theta}\right)^{x_k}}{1-\left(\frac{1-\theta}{1+\theta}\right)^{x_k}}=\frac{1}{\theta^n}\prod_{k=1}^{n}\frac{\theta\left(1+\left(\frac{1-\theta}{1+\theta}\right)^{x_k}\right)}{1-\left(\frac{1-\theta}{1+\theta}\right)^{x_k}}=:\frac{1}{\theta^n}\sum_{k=0}^{\infty}A_k\theta^k,$$

where the $A_k \in \mathbb{Z}_p$ are p-adic integers (since $1/x_k \in \mathbb{Z}_p$). Therefore, we can write

$$\operatorname{Tr}\Bigl(\frac{1}{\theta^n}\sum_{k=0}^{\infty} A_k\theta^k\Bigr) \simeq \operatorname{Tr}\Bigl(\frac{1}{\theta^n}\sum_{k=0}^{n} A_k\theta^k\Bigr) = \sum_{k=0}^{n}\bigl(A_k \operatorname{Tr}(\theta^{k-n})\bigr).$$

Set $\operatorname{Tr}\theta^{-s} = B_s$ and introduce two formal power series $A(u) = \sum_{k=0}^{\infty} A_k u^k$, $B(u) = \sum_{k=0}^{\infty} B_k u^k$. It follows from (2_L) that $A(u) = \prod_{k=1}^{n} \frac{u}{[u]^L_{x_k}}$. Therefore, we must calculate the coefficient of u^n in the series $A(u)B(u)$. We have

$$\begin{aligned}B(u) &= \operatorname{Tr}\Bigl(1+\sum_{s=1}^{\infty}\theta^{-s}u^s\Bigr) = \operatorname{Tr}\left(\frac{1}{1-\theta^{-1}u}\right)\\ &= \operatorname{Tr}\left(\frac{\theta}{\theta-u}\right) = \operatorname{Tr}\left(1+\frac{u}{\theta-u}\right) = (p-1) + u\operatorname{Tr}\left(\frac{1}{\theta-u}\right).\end{aligned}$$

Observe that if $\varphi_\alpha(u)$ is the minimal polynomial for an element α with respect to the extension $\mathbb{Q}_p(\zeta) \mid \mathbb{Q}_p$, then $\operatorname{Tr}\frac{1}{\alpha-u} = -\frac{\varphi'_\alpha(u)}{\varphi_\alpha(u)}$. Since

$$\begin{aligned}0 &= \frac{\zeta^p-1}{\zeta-1} = \frac{\left(\frac{1-\theta}{1+\theta}\right)^p - 1}{\left(\frac{1-\theta}{1+\theta}\right) - 1}\\ &= \frac{\bigl((1-\theta)^p-(1+\theta)^p\bigr)(1+\theta)}{(1+\theta)^p(-2\theta)} = \frac{1}{(1+\theta)^{p-1}}\frac{(1+\theta)^p-(1-\theta)^p}{2\theta},\end{aligned}$$

we deduce that $\varphi_\theta(u) = \frac{(1+u)^p-(1-u)^p}{2u}$ is the minimal polynomial for $\theta = \frac{1-\zeta}{1+\zeta}$. Hence,

$$\begin{aligned}-\operatorname{Tr}\frac{1}{\theta-u} &= \frac{\varphi'_\theta(u)}{\varphi_\theta(u)} = p\frac{(1+u)^{p-1}+(1-u)^{p-1}}{(1+u)^p-(1-u)^p} - \frac{1}{u},\\ B(u) &= (p-1)+u\operatorname{Tr}\left(\frac{1}{\theta-u}\right) = \frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p-(1-u)^p},\\ \operatorname{Tr}\Bigl(\prod_{k=1}^{n}\frac{1+\zeta^{x_k}}{1-\zeta^{x_k}}\Bigr) &\simeq \bigl\langle A(u)B(u)\bigr\rangle_n = \Bigl\langle p\frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p-(1-u)^p}\prod_{k=1}^{n}\frac{u}{[u]^L_{x_k}}\Bigr\rangle_n,\end{aligned}$$

We deduce that

$$AB_L(x_1,\dots,x_n) \equiv -\Bigl\langle p\frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p-(1-u)^p}\prod_{k=1}^{n}\frac{u}{[u]^L_{x_k}}\Bigr\rangle_n \mod p.$$

This formula is analogous to the first formula in (13). Further,

$$\begin{aligned}p\frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p-(1-u)^p} &= p\frac{(1-u)(1+u)^p-(1+u)(1-u)^p}{(1-u)(1+u)\bigl((1+u)^p-(1-u)^p\bigr)}\\ &= p\frac{(1+u)^p-(1-u)^p-u\bigl((1+u)^p+(1-u)^p\bigr)}{(1-u^2)\bigl((1+u)^p-(1-u)^p\bigr)}\\ &= \frac{p}{1-u^2} - \frac{pu}{(1-u^2)\frac{(1+u)^p-(1-u)^p}{(1+u)^p+(1-u)^p}} = \frac{p}{1-u^2} - \frac{pu}{(1-u^2)[u]^L_p}\\ &\simeq -\frac{pu}{[u]^L_p}(1+u^2+u^4+\cdots).\end{aligned}$$

From this we obtain

$$\begin{aligned}
AB_L(x_1,\dots,x_n) &\equiv -\Big\langle p\frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p-(1-u)^p}\prod_{k=1}^n \frac{u}{[u]^L_{x_k}}\Big\rangle_n \\
&\equiv \Big\langle \frac{pu}{[u]^L_p}\prod_{k=1}^n \frac{u}{[u]^L_{x_k}}(1+u^2+u^4+\cdots)\Big\rangle_n \mod p,
\end{aligned}$$

whence

$$AB_L(x_1,\dots,x_n) \equiv \sum_{i=0}^{[n/2]}\Big\langle \frac{pu}{[u]^L_p}\prod_{k=1}^n \frac{u}{[u]^L_{x_k}}\Big\rangle_{n-2i}.$$

At the same time, $[u]^L_k = \frac{(1+u)^k-(1-u)^k}{(1+u)^k+(1-u)^k} = \tanh(k \operatorname{arctanh} u)$, and the series $\operatorname{arctanh} u$ (as well as $\tanh u$) contains only odd powers of u. Therefore, the series $\frac{pu}{[u]^L_p}\prod_{k=1}^n \frac{u}{[u]^L_{x_k}}$ contains only even powers of u. Thus we can finally write the formulae for the Atiyah–Bott functions for the L-genus which are analogous to formulae (13) for the Todd genus:

$$\begin{aligned}
AB_L(x_1,\dots,x_n) &\equiv -\Big\langle p\frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p-(1-u)^p}\prod_{k=1}^n \frac{u}{[u]^L_{x_k}}\Big\rangle_n \mod p, \\
AB_L(x_1,\dots,x_n) &\equiv \sum_{m=0}^{n}\Big\langle \frac{pu}{[u]^L_p}\prod_{k=1}^n \frac{u}{[u]^L_{x_k}}\Big\rangle_m \mod p.
\end{aligned} \tag{17}$$

By (16), we see that

$$L(M) \equiv \sum_{j=1}^{q}\sum_{m=0}^{n}\Big\langle \frac{pu}{[u]^L_p}\prod_{k=1}^n \frac{u}{[u]^L_{x^{(j)}_k}}\Big\rangle_m \mod p, \tag{18}$$

where $[u]^L_p = \frac{(1+u)^p-(1-u)^p}{(1+u)^p+(1-u)^p}$.

Remark. Relations similar to (12), (16) could also be obtained for the Euler number. In this case, the Atiyah–Bott functions are

$$AB_e(x_1,\dots,x_n) = -\operatorname{Tr}\Big(\prod_{k=1}^n \frac{1-\zeta}{1-\zeta}\Big) = -\operatorname{Tr}(1) = -(p-1) \equiv 1 \mod p.$$

Therefore,

$$q = \sum_{j=1}^{q} AB_e(x^{(j)}_1,\dots,x^{(j)}_n) \equiv e(M) \mod p. \tag{19}$$

This relation is analogous to relations (12) and (16) for the Todd genus and the L-genus.

§ 5. General results on the calculation of Hirzebruch genera via invariants of the $\mathbb{Z}/p$-action

Here we consider another approach to calculating the equivariant index $\operatorname{ind}(g,E)=\sum_{i=0}^{m}(-1)^i\operatorname{tr}(g,H^i)$ of an elliptic complex E. This approach is taken from [3] (see also [7]).

In what follows we adopt somewhat weaker assumptions about the action of an operator g, $g^p=1$, on a stably complex manifold M^{2n}. Namely, we remove the transversality condition. Let $M^g=\{x\in M\mid gx=x\}$ be the fixed point set, and let $M^g=\cup M^g_\nu$ be its decomposition into connected components. Then the equivariant index can be computed as the sum of the contributions $\sigma(M^g_\nu)$ corresponding to the fixed point components M^g_ν (see [7]). These contributions are calculated as follows.

Let $Y=M^g_\nu$ be one of the fixed point components of M^{2n}. For each point $p\in Y$, g acts linearly on the tangent space T_pM. This tangent space decomposes into the direct sum of the eigenspaces $N_{p,\lambda}$ for eigenvalues λ, $|\lambda|=1$. In this way we obtain the eigenbundle N_λ over Y. The N_1 is just the tangent bundle TY to Y. With $d_\lambda=\operatorname{rk}N_\lambda$, we therefore have:

$$TM|_Y=\bigoplus_\lambda N_\lambda,\qquad c(N_\lambda)=\prod_{i=1}^{d_\lambda}(1+x_i^\lambda),\tag{20}$$

that is,

$$c(TM_Y)=\prod_\lambda\prod_{i=1}^{d_\lambda}(1+x_i^\lambda)=\prod_{i=1}^{n}(1+x_i).$$

The recipe for calculating $\sigma(Y)$ is as follows. Consider the index formula (3) in Theorem 2.1:

$$\operatorname{ind}(E)=\left(\left(\sum_{i=0}^{m}(-1)^i\operatorname{ch}(E_i)\right)c_n(M)\prod_{j=1}^{n}\left(\frac{1}{1-e^{-x_j}}\frac{1}{1-e^{x_j}}\right)\right)[M]\tag{21}$$

and replace M by Y and e^{x_i} by $\lambda^{-1}e^{x_i}$ (where x_i belongs to the eigenvalue λ). Apply the same process to the terms $\operatorname{ch}(E_i)$. This can obviously be done if the E_i are associated to the tangent bundle of M. This "recipe" is taken from [7].

In the case of finite number of fixed points we have $Y=\mathbf{pt}$, $c_n(Y)=1$, and so we must replace $x_1x_2\dots x_n$ by 1, and $e^{x_j^\lambda}$ by λ_j^{-1}. Therefore, introducing the "weight" x_i by the formula $\lambda_j=\exp\left(\frac{2\pi i x_j}{p}\right)$, we just have to replace x_j^λ by $-\frac{2\pi i}{p}x_j$. (Here in the first case x_j^λ stands for the first Chern class of "virtual" line subbundle in TM corresponding to the eigenvalue λ_j, while in the second case x_j is the "weight" of the fixed point and is defined only modulo p.)

Example 5.1. Let us consider the χ_y-genus of a manifold M. Applying our recipe to the formula from Theorem 2.2, we obtain the following formula for the contribution of each fixed point $\mathcal{P}$:

$$\sigma(\mathcal{P})=\prod_{k=1}^{n}\frac{1+ye^{2\pi i x_k/p}}{1-e^{2\pi i x_k/p}}.\tag{22}$$

Putting $y=-1,0,1$, we obtain the formulae for the contribution function for the Euler number, the Todd genus and the L-genus:

$$\sigma_e(\mathcal{P})=1,\qquad \sigma_{\mathrm{td}}(\mathcal{P})=\prod_{k=1}^{n}\frac{1}{1-e^{2\pi i x_k/p}},\qquad \sigma_L(\mathcal{P})=\prod_{k=1}^{n}\frac{1+e^{2\pi i x_k/p}}{1-e^{2\pi i x_k/p}}.$$

These formulae coincide with (9), (14) and the formula in [5], which were deduced from the Atiyah–Bott theorem 3.2.

Now consider the general case of an arbitrary Hirzebruch genus φ:

$$\varphi(M)=\left(\prod_{i=1}^{n}\frac{x_i}{f_\varphi(x_i)}\right)[M],$$

where $g_\varphi(u)=f_\varphi^{-1}(u)$ is the logarithm of the corresponding formal group law. Suppose that there is an elliptic complex E_φ associated to TM whose index equals $\varphi(M)$. Applying the above recipe, we see that the contribution functions of the fixed points for the action of g on M are given by the formula

$$\sigma(\mathcal{P})=\prod_{k=1}^{n}\frac{1}{f_\varphi(-2\pi i x_k/p)}. \tag{23}$$

Here the x_k are the “weights” of the fixed point $\mathcal{P}$. They are determined by the formula $\lambda_k=\exp\left(\frac{2\pi i x_k}{p}\right)$, $x_k\neq 0 \mod p$, where the λ_k are the eigenvalues of the Jacobi matrix $\mathcal{J}_{\mathcal{P}}(g)$ of the map g at the point $\mathcal{P}$. The equivariant index of E_φ is then determined by the formula

$$\operatorname{ind}(g,E_\varphi)=\sum_{j=1}^{q}\prod_{k=1}^{n}\frac{1}{f_\varphi\big(-2\pi i x_k^{(j)}/p\big)}.$$

Further, $\frac{1}{p}\sum_{l\in\mathbb{Z}/p}\operatorname{ind}(g^l,E_\varphi)=s$ is the alternating sum of the dimensions of the equivariant subspaces for the action of g on the cohomology of E_φ, and $\operatorname{ind}(1,E_\varphi)=\operatorname{ind}(E_\varphi)=\varphi(M)$. Therefore,

$$\varphi(M)=\operatorname{ind}(1,E_\varphi)=-\sum_{l=1}^{p-1}\operatorname{ind}(g^l,E_\varphi)+ps=-\sum_{j=1}^{q}\sum_{l=1}^{p-1}\prod_{k=1}^{n}\frac{1}{f_\varphi\big(-2\pi i x_k^{(j)}l/p\big)}+ps.$$

Consider the number-theoretical trace $\operatorname{Tr}\colon \mathbb{Q}_p(\zeta)\to\mathbb{Q}_p$, where $\zeta:=\exp\frac{2\pi i}{p}$. Then

$$\sum_{l=1}^{p-1}\prod_{k=1}^{n}\frac{1}{f_\varphi\big(-2\pi i x_k^{(j)}l/p\big)}=\operatorname{Tr}\prod_{k=1}^{n}\frac{1}{f_\varphi\big(-2\pi i x_k^{(j)}/p\big)}.$$

Thus, we obtain the following result.

Theorem 5.1. *Suppose that there is an elliptic complex of bundles associated to TM whose index is equal to the Hirzebruch genus $\varphi(M)$ of the manifold M. Let g be a holomorphic transversal endomorphism acting on M such that $g^p=1$. Then we have the following formula for $\varphi(M)$:*

$$\varphi(M)\equiv-\sum_{j=1}^{q}\operatorname{Tr}\prod_{k=1}^{n}\frac{1}{f_\varphi\big(-2\pi i x_k^{(j)}/p\big)}\mod p.$$

It is now convenient to make the following definition.

Definition 5.2. *The Atiyah–Bott fixed point function* $AB_\varphi(x_1,\dots,x_n)$ *corresponding to the genus* φ is defined to be the following function of the set of weights $x_1,\dots,x_n,\ x_i\in\mathbb{Z}/p$:

$$AB_\varphi(x_1,\dots,x_n)=-\operatorname{Tr}\prod_{k=1}^n\frac{1}{f_\varphi\bigl(-2\pi i x_k/p\bigr)}.$$

Then we immediately get

$$\sum_{j=1}^q AB_\varphi\bigl(x_1^{(j)},\dots,x_n^{(j)}\bigr)\equiv\varphi(M)\mod p. \tag{24}$$

Now we set $\theta=f_\varphi(-2\pi i/p)$. Then $-2\pi i/p=g_\varphi(\theta)$, and $f_\varphi(-2\pi i/px_k)=f_\varphi\bigl(x_kg_\varphi(\theta)\bigr)=[\theta]^\varphi_{x_k}$. Hence the following statement holds:

Proposition 5.3. *The Atiyah–Bott fixed point function* $AB_\varphi(x_1,\dots,x_n)$ *corresponding to the genus* φ *can be computed as*

$$AB_\varphi(x_1,\dots,x_n)=-\operatorname{Tr}\biggl(\prod_{k=1}^n\frac{1}{[\theta]^\varphi_{x_k}}\biggr),\qquad \theta=f_\varphi\left(\frac{-2\pi i}{p}\right).$$

Lemma 5.4. *For any* $k>0$ *we have* $\operatorname{Tr}\theta^k\simeq 0$ (*that is,* $\operatorname{Tr}\theta^k\in p\mathbb{Z}_p$).

Proof. Let $\theta=f_\varphi\left(-\frac{2\pi i}{p}\right)=C_0+C_1\zeta+\dots+C_{p-1}\zeta^{p-1}\in\mathbb{Q}_p(\zeta),\quad C_i\in\mathbb{Z}_p$. Then $C_0+C_1\zeta^m+C_2\zeta^{2m}+\dots+C_{p-1}\zeta^{(p-1)m}=f_\varphi\left(-\frac{2\pi i}{p}m\right)$. In particular, $C_0+C_1+\dots+C_{p-1}=0$. Therefore,

$$\begin{aligned}\operatorname{Tr}\theta^k&=\sum_{m=1}^{p-1}f_\varphi\left(-\frac{2\pi i}{p}m\right)=\sum_{m=1}^{p-1}\bigl(C_0+C_1\zeta^m+C_2\zeta^{2m}+\dots+C_{p-1}\zeta^{(p-1)m}\bigr)\\&=\sum_{m=0}^{p-1}\bigl(C_0+C_1\zeta^m+C_2\zeta^{2m}+\dots+C_{p-1}\zeta^{(p-1)m}\bigr)\simeq 0,\end{aligned}$$

since $\sum_{m=0}^{p-1}\zeta^{rm}\simeq 0$ for any r. The lemma is proved.

Example 5.2. For the Todd genus, $\theta=f_{\mathrm{td}}\left(-\frac{2\pi i}{p}\right)=1-e^{2\pi i/p}=1-\zeta$ (see formula (2_{td})). Hence, $C_0=1,\ \ C_1=-1, C_i=0$ for $i>1$.

Example 5.3. For the L-genus, $\theta=\hat f_L\left(-\frac{2\pi i}{p}\right)=\frac{1-e^{2\pi i/p}}{1+e^{2\pi i/p}}=\frac{1-\zeta}{1+\zeta}=\zeta^{p-1}-\zeta^{p-2}+\zeta^{p-3}-\dots-\zeta$ (see Lemma 4.2). Hence, $C_0=0,\ \ C_{2i+1}=-1,\ \ C_{2i}=1$ for $i>1$.

§ 6. Calculations for the $\hat A$-genus and the χ_y-genus

6.1. Calculations for the $\hat A$-genus. We consider the $\hat A$-genus

$$\hat A(M)=\biggl(\prod_{j=1}^n\frac{x_je^{-x_j/2}}{1-e^{-x_j}}\biggr)[M]$$

for a stably complex manifold M such that $c_1(M) \equiv 0 \mod 2$. In Theorem 2.4, we gave an elliptic complex whose index is $\hat{A}(M)$. The Atiyah–Bott functions for the $\hat{A}$-genus are as follows (see Definition 5.2):

$$AB_{\hat{A}}(x_1, \ldots, x_n) = -\operatorname{Tr}\left(\prod_{k=1}^{n} \frac{\exp\left(\frac{2\pi i x_k}{2p}\right)}{1 - \exp\left(\frac{2\pi i x_k}{p}\right)} \right) = -\operatorname{Tr}\left(\prod_{k=1}^{n} \frac{\zeta^{x_k/2}}{1 - \zeta^{x_k}} \right), \tag{25}$$

$$\sum_{j=1}^{q} AB_{\hat{A}}\left(x_1^{(j)}, \ldots, x_n^{(j)}\right) \equiv \hat{A}(M) \mod p. \tag{26}$$

Now we have to calculate the number-theoretical trace in the definition of the Atiyah–Bott functions $AB_{\hat{A}}(x_1, \ldots, x_n)$. By Proposition 5.3, $\frac{\zeta^{x_k/2}}{1-\zeta^{x_k}} = \frac{1}{[\theta]^A_{x_k}}$, where $[u]^A_m = 2\sinh\left(m \operatorname{arcsinh}\left(\frac{u}{2}\right)\right)$ is the mth power in the formal group law corresponding to the $\hat{A}$-genus (see formula (2_A)), and $\theta = 2\sinh\left(-\frac{2\pi i}{2p}\right) = -2\sinh\left(\frac{\pi i}{p}\right) = e^{-\pi i/p} - e^{\pi i/p} = \zeta^{-1/2} - \zeta^{1/2} = \zeta^{(p+1)/2} - \zeta^{(p-1)/2}$.

We claim that the minimal polynomial for the element $\theta = -2\sinh\left(\frac{\pi i}{p}\right)$ is

$$\varphi_\theta(u) = \frac{2\sinh\left(p \operatorname{arcsinh}\left(\frac{u}{2}\right)\right)}{u}.$$

Indeed,

$$\varphi_\theta(\theta) = \frac{2\sinh\left(p \operatorname{arcsinh}\left(\left(-2\sinh\left(\frac{\pi i}{p}\right)\right)\big/2\right)\right)}{\theta} = \frac{-2\sinh\left(p \cdot \frac{\pi i}{p}\right)}{\theta} = 0$$

and $\varphi_\theta(u)$ is a polynomial of degree $p-1$ with leading term u^{p-1}. For instance, for $p = 3$ we get $\varphi_\theta(u) = u^2 + 3$, for $p = 5$ we get $\varphi_\theta(u) = u^4 + 5u^2 + 5$, and so on.

Further, we have

$$\begin{aligned} \prod_{k=1}^{n} \frac{\zeta^{x_k/2}}{1 - \zeta^{x_k}} &= \prod_{k=1}^{n} \frac{1}{f_{\hat{A}}\left(x_k g_{\hat{A}}(\theta)\right)} = \prod_{k=1}^{n} \frac{1}{[\theta]^{\hat{A}}_{x_k}} \\ &= \frac{1}{\theta^n} \prod_{k=1}^{n} \frac{\theta}{[\theta]^{\hat{A}}_{x_k}} =: \frac{1}{\theta^n} \sum_{i=0}^{\infty} A_i \theta^i = \frac{1}{\theta^n} A(\theta), \end{aligned}$$

where $A(u) := \prod_{k=1}^{n} \frac{u}{[u]^{\hat{A}}_{x_k}} = \sum_{i=0}^{\infty} A_i u^i$, $A_i \in \mathbb{Z}_p$ are p-adic integers. It follows from lemma 5.4 that $\operatorname{Tr}(\theta^k) \simeq 0$ for any $k > 0$. (Here $\theta = \zeta^{(p+1)/2} - \zeta^{(p-1)/2}$.) So the required trace is

$$\begin{aligned}\operatorname{Tr}\biggl(\prod_{k=1}^{n} \frac{\zeta^{x_k/2}}{1-\zeta^{x_k}}\biggr) &= \operatorname{Tr}\biggl(\frac{1}{\theta^n}\sum_{i=0}^{\infty} A_i\theta^i\biggr) \simeq \operatorname{Tr}\biggl(\frac{1}{\theta^n}\sum_{i=0}^{n} A_i\theta^i\biggr)\\ &= \sum_{i=0}^{n}\bigl(A_i \operatorname{Tr}(\theta^{i-n})\bigr) = \bigl\langle A(u)B(u)\bigr\rangle_n,\end{aligned}$$

where $B_s := \operatorname{Tr}\theta^{-s}$, $B(u) := \sum_{i=0}^{\infty} B_i u^i$. As before, $B(u) = (p-1) + u\operatorname{Tr}\left(\frac{1}{\theta-u}\right)$, $\operatorname{Tr}\left(\frac{1}{\theta-u}\right) = -\frac{\varphi_\theta'(u)}{\varphi_\theta(u)}$. Therefore,

$$\begin{aligned}\varphi_\theta(u) &= \frac{2\sinh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{u},\\ \varphi_\theta'(u) &= \frac{p\cosh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{u\sqrt{1+\frac{u^2}{4}}} - \frac{2\sinh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{u^2},\\ \frac{\varphi_\theta'(u)}{\varphi_\theta(u)} &= \frac{p}{2\sqrt{1+\frac{u^2}{4}}}\frac{\cosh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{\sinh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)} - \frac{1}{u},\end{aligned}$$

$$\begin{aligned}B(u) &= p-1-u\frac{\varphi_\theta'(u)}{\varphi_\theta(u)} = p - \frac{p\frac{u}{2}}{\sqrt{1+\frac{u^2}{4}}}\frac{\cosh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{\sinh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}\\ &= \frac{p\sinh\bigl((p-1)\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{\cosh\bigl(\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)\sinh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)} = \frac{p\sinh\bigl((p-1)\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{\sqrt{1+\frac{u^2}{4}}\,\sinh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}.\end{aligned}$$

Here we have used the following formulae: $\cosh\bigl(\operatorname{arcsinh}(u/2)\bigr) = \sqrt{1+u^2/4}$ and $\sinh(x-y) = \sinh x\cosh y - \sinh y\cosh x$. We further deduce that

$$\begin{aligned}B(u) &= \frac{p\sinh\bigl((p-1)\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{\cosh\bigl(\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)\sinh\bigl(p\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}\\ &= \frac{2p\sinh\bigl((p-1)\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}{\sinh\bigl((p-1)\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)+\sinh\bigl((p+1)\operatorname{arcsinh}\bigl(\frac{u}{2}\bigr)\bigr)}\\ &= p\frac{2[u]^A_{p-1}}{[u]^A_{p-1}+[u]^A_{p+1}} \simeq p\frac{[u]^A_{p-1}-[u]^A_{p+1}}{[u]^A_{p-1}+[u]^A_{p+1}}.\end{aligned}$$

Here $[u]^A_m = 2\sinh\bigl(m\operatorname{arcsinh}(u/2)\bigr)$, and we have used the formula $2\cosh x\sinh y = \sinh(y+x)+\sinh(y-x)$. Thus we obtain the following formulae for the Atiyah–Bott functions $AB_{\hat{A}}(x_1,\ldots,x_n) = -\operatorname{Tr}\left(\prod_{k=1}^n \frac{\zeta^{x_k/2}}{1-\zeta^{x_k}}\right)$ for the $\hat{A}$-genus:

$$AB_{\hat{A}}(x_1,\ldots,x_n) = -\left\langle p\frac{\sinh\left((p-1)\operatorname{arcsinh}\left(\frac{u}{2}\right)\right)}{\sqrt{1+\frac{u^2}{4}}\,\sinh\left(p\operatorname{arcsinh}\left(\frac{u}{2}\right)\right)}\prod_{k=1}^n \frac{u}{[u]^{\hat{A}}_{x_k}}\right\rangle_n,$$

$$AB_{\hat{A}}(x_1,\ldots,x_n) = -\left\langle p\frac{2[u]^A_{p-1}}{[u]^A_{p-1}+[u]^A_{p+1}}\prod_{k=1}^n \frac{u}{[u]^{\hat{A}}_{x_k}}\right\rangle_n, \tag{27}$$

$$AB_{\hat{A}}(x_1,\ldots,x_n) \simeq \left\langle p\frac{[u]^A_{p+1}-[u]^A_{p-1}}{[u]^A_{p-1}+[u]^A_{p+1}}\prod_{k=1}^n \frac{u}{[u]^{\hat{A}}_{x_k}}\right\rangle_n.$$

Then the $\hat{A}$-genus itself is calculated as $\hat{A}(M) \equiv \sum_{j=1}^q AB_{\hat{A}}\big(x_1^{(j)},\ldots,x_n^{(j)}\big)$.

6.2. Calculations for the χ_y-genus. Let us consider the χ_y-genus

$$\chi_y(M) = \left(\prod_{j=1}^n \frac{x_j(1+ye^{-x_j})}{1-e^{-x_j}}\right)[M]$$

of a manifold M. An elliptic complex whose index is $\chi_y(M)$ is given by Theorem 2.2. The Atiyah–Bott functions for the $\chi_y(M)$-genus are as follows (see Definition 5.2):

$$\begin{aligned} AB_{\chi_y}(x_1,\ldots,x_n) &= -\operatorname{Tr}\left(\prod_{k=1}^n \frac{1+y\exp\left(\frac{2\pi i x_k}{p}\right)}{1-\exp\left(\frac{2\pi i x_k}{p}\right)}\right) \\ &= -\operatorname{Tr}\Bigg(\prod_{k=1}^n \frac{1+y\zeta^{x_k}}{1-\zeta^{x_k}}\Bigg), \end{aligned} \tag{28}$$

$$\sum_{j=1}^q AB_{\chi_y}\big(x_1^{(j)},\ldots,x_n^{(j)}\big) \equiv \chi_y(M) \mod p. \tag{29}$$

These formulae could be also deduced from the Atiyah–Bott theorem 3.2 as was done for the L-genus in § 4.3.

We shall now calculate the number-theoretical trace in the definition of the Atiyah–Bott functions $AB_{\chi_y}(x_1,\ldots,x_n)$. By Proposition 5.3, $\frac{1+y\zeta^{x_k}}{1-\zeta^{x_k}} = \frac{1}{[\theta]^{\chi_y}_{x_k}}$, where $[u]^{\chi_y}_m = \frac{(1+yu)^m-(1-u)^m}{(1+yu)^m+y(1-u)^m}$ (for $y \neq -1$) is the mth power in the formal group law corresponding to the χ_y-genus (see formula (2_{χ_y})) and

$$\theta = \hat{f}_{\chi_y}\left(-\frac{2\pi i}{p}\right) = \frac{1-e^{2\pi i/p}}{1+ye^{2\pi i/p}} = \frac{1-\zeta}{1+y\zeta}.$$

Then $\zeta = \frac{1-\theta}{1+y\theta}$. In what follows we assume that $y \in \mathbb{Z}_p$ and $y \neq -1 \mod p$. The case $y = -1 \mod p$ corresponds to the Euler number which has already been considered above. Note that for $y = 0, 1$ we get $\theta = 1-\zeta$ and $\theta = \frac{1-\zeta}{1+\zeta}$ respectively. This coincides with the corresponding values for the Todd genus and the L-genus obtained above.

It is easy to see that the minimal polynomial for the element $\theta=\frac{1-\zeta}{1+y\zeta}$ is

$$\varphi_\theta(u)=\frac{(1+yu)^p-(1-u)^p}{(1+y)u}.$$

Indeed,

$$0=\frac{\zeta^p-1}{\zeta-1}=\frac{\left(\frac{1-\theta}{1+y\theta}\right)^p-1}{\frac{1-\theta}{1+y\theta}-1}=\frac{1}{(1+y\theta)^{p-1}}\varphi_\theta(\theta).$$

Hence $\varphi_\theta(\theta)=0$, and $\varphi_\theta(u)$ is a polynomial of degree $p-1$ with leading term u^{p-1}.

Further, we have

$$\begin{aligned}\prod_{k=1}^n\frac{1+y\zeta^{x_k}}{1-\zeta^{x_k}}&=\prod_{k=1}^n\frac{1}{\hat f_{\chi_y}\big(x_k\hat g_{\chi_y}(\theta)\big)}=\prod_{k=1}^n\frac{1}{[\theta]_{x_k}^{\chi_y}}\\&=\frac{1}{\theta^n}\prod_{k=1}^n\frac{\theta}{[\theta]_{x_k}^{\chi_y}}=:\frac{1}{\theta^n}\sum_{i=0}^\infty A_i\theta^i=\frac{1}{\theta^n}A(\theta),\end{aligned}$$

where $A(u):=\prod_{k=1}^n\frac{u}{[u]_{x_k}^{\chi_y}}=\sum_{i=0}^\infty A_iu^i$ with $A_i\in\mathbb{Z}_p$ being p-adic integers. It follows from Lemma 5.4 that $\mathrm{Tr}(\theta^k)\simeq 0$ for any $k>0$.

Remark. The presentation of θ as $\theta=C_0+C_1\zeta+\dots+C_{p-1}\zeta^{p-1}$, $C_i\in\mathbb{Z}_p$, which is used in the proof of Lemma 5.4, is obtained as follows. Since

$$1+y^p=1+(y\zeta)^p=(1+y\zeta)\big((y\zeta)^{p-1}-(y\zeta)^{p-2}+(y\zeta)^{p-3}-\dots-y\zeta+1\big),$$

we have

$$\begin{aligned}\theta=\frac{1-\zeta}{1+y\zeta}&=\frac{(1-\zeta)\big((y\zeta)^{p-1}-(y\zeta)^{p-2}+(y\zeta)^{p-3}-\dots-y\zeta+1\big)}{1+y^p}=\frac{1}{1+y^p}\\&\times\Big(\big(y^{p-1}+y^{p-2}\big)\zeta^{p-1}-(y^{p-2}+y^{p-3})\zeta^{p-2}+\dots-(y+1)\zeta+1-y^{p-1}\Big).\end{aligned}$$

Since $y^p+1\equiv y+1\neq 0 \mod p$, we obtain $C_i=(-1)^i\frac{y^i+y^{i-1}}{1+y^p}\in\mathbb{Z}_p \quad (i>0)$, $C_0=\frac{1-y^{p-1}}{1+y^p}\in\mathbb{Z}_p$.

So, the number-theoretical trace we are interested in is

$$\mathrm{Tr}\Big(\prod_{k=1}^n\frac{1+y\zeta^{x_k}}{1-\zeta^{x_k}}\Big)\simeq\mathrm{Tr}\Big(\frac{1}{\theta^n}\sum_{i=0}^nA_i\theta^i\Big)=\sum_{i=0}^n\big(A_i\,\mathrm{Tr}(\theta^{i-n})\big)=\big\langle A(u)B(u)\big\rangle_n,$$

where $B_s := \operatorname{Tr}\theta^{-s}$, $B(u) := \sum_{i=0}^{\infty} B_i u^i$. As before, $B(u) = (p-1) + u\operatorname{Tr}\left(\frac{1}{\theta-u}\right)$, $\operatorname{Tr}\left(\frac{1}{\theta-u}\right) = -\frac{\varphi_\theta'(u)}{\varphi_\theta(u)}$. Hence, the following formulae hold:

$$\begin{aligned}
\varphi_\theta(u) &= \frac{(1+yu)^p-(1-u)^p}{(1+y)u},\\
\varphi_\theta'(u) &= \frac{py(1+yu)^{p-1}+p(1-u)^{p-1}}{(1+y)u} - \frac{(1+yu)^p-(1-u)^p}{(1+y)u^2},\\
\frac{\varphi_\theta'(u)}{\varphi_\theta(u)} &= p\frac{y(1+yu)^{p-1}+(1-u)^{p-1}}{(1+yu)^p-(1-u)^p} - \frac{1}{u},\\
B(u) &= (p-1) - u\frac{\varphi_\theta'(u)}{\varphi_\theta(u)}\\
&= p - up\frac{y(1+yu)^{p-1}+(1-u)^{p-1}}{(1+yu)^p-(1-u)^p} = p\frac{(1+yu)^{p-1}-(1-u)^{p-1}}{(1+yu)^p-(1-u)^p}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
AB_{\chi_y}(x_1,\dots,x_n) &= -\operatorname{Tr}\biggl(\prod_{k=1}^n \frac{1+y\zeta^{x_k}}{1-\zeta^{x_k}}\biggr)\\
&= -\left\langle p\frac{(1+yu)^{p-1}-(1-u)^{p-1}}{(1+yu)^p-(1-u)^p}\prod_{k=1}^n \frac{u}{[u]_{x_k}^{\chi_y}}\right\rangle_n.
\end{aligned}$$

We deduce that

$$\begin{aligned}
B(u) &= p\frac{(1+yu)^{p-1}-(1-u)^{p-1}}{(1+yu)^p-(1-u)^p} = p\frac{(1-u)(1+yu)^p-(1+yu)(1-u)^p}{(1-u)(1+yu)\bigl((1+yu)^p-(1-u)^p\bigr)}\\
&= \frac{p}{(1-u)(1+yu)} - \frac{pu}{(1-u)(1+yu)\frac{(1+yu)^p-(1-u)^p}{(1+yu)^p+y(1-u)^p}}\\
&\simeq -\frac{pu}{[u]_p^{\chi_y}}\frac{1}{(1-u)(1+yu)},\\
&\qquad \frac{1}{(1-u)(1+yu)} = \sum_{m=0}^{\infty}\frac{1+(-1)^m y^{m+1}}{1+y}u^m,\\
AB_{\chi_y}(x_1,\dots,x_n) &\simeq \left\langle \frac{pu}{[u]_p^{\chi_y}}\frac{1}{(1-u)(1+yu)}\prod_{k=1}^n\frac{u}{[u]_{x_k}^{\chi_y}}\right\rangle_n\\
&= \left\langle \frac{pu}{[u]_p^{\chi_y}}\prod_{k=1}^n\frac{u}{[u]_{x_k}^{\chi_y}}\sum_{m=0}^{\infty}\frac{1+(-1)^m y^{m+1}}{1+y}u^m\right\rangle_n.
\end{aligned}$$

We finally obtain the following formulae for the Atiyah–Bott functions $AB_{\chi_y}(x_1,\dots,x_n) = -\operatorname{Tr}\left(\prod_{k=1}^n \frac{1+y\zeta^{x_k}}{1-\zeta^{x_k}}\right)$ of the χ_y-genus ($y \neq -1 \mod p$):

$$\begin{aligned}
AB_{\chi_y}(x_1,\dots,x_n) &= -\left\langle p\frac{(1+yu)^{p-1}-(1-u)^{p-1}}{(1+yu)^p-(1-u)^p}\prod_{k=1}^n \frac{u}{[u]_{x_k}^{\chi_y}}\right\rangle_n,\\
AB_{\chi_y}(x_1,\dots,x_n) &\simeq \sum_{m=0}^{n}\frac{1+(-1)^m y^{m+1}}{1+y}\left\langle\frac{pu}{[u]_p^{\chi_y}}\prod_{k=1}^n\frac{u}{[u]_{x_k}^{\chi_y}}\right\rangle_{n-m}.
\end{aligned} \tag{30}$$

Here $[u]^{\chi_y}_m = \frac{(1+yu)^m-(1-u)^m}{(1+yu)^m+y(1-u)^m}$ is the mth power in the formal group law corresponding to the χ_y-genus. The χ_y-genus itself is the calculated as follows:

$$\begin{aligned}\chi_y(M) &\equiv \sum_{j=1}^{q} AB_{\chi_y}\bigl(x_1^{(j)},\dots,x_n^{(j)}\bigr)\\ &\equiv \sum_{j=1}^{q}\sum_{m=0}^{n}\frac{1+(-1)^m y^{m+1}}{1+y}\biggl\langle \frac{pu}{[u]^{\chi_y}_p}\prod_{k=1}^{n}\frac{u}{[u]^{\chi_y}_{x_k^{(j)}}}\biggr\rangle_{n-m} \mod p.\end{aligned}$$

Formulae (13), (17) for the Todd genus and the L-genus are obtained from this formula by substituting $y=0$ and $y=1$ respectively.

§ 7. The Conner–Floyd equations and the calculation of Hirzebruch genera in terms of invariants of the action

Here we consider the connection of the results in § 5 with the so-called Conner–Floyd equations, which were introduced by Novikov in [12], [13]. (Similar relations were also obtained in [8], [11].) Namely, it was shown there that the sets $x_1^{(j)},\dots,x_n^{(j)}$, $x_k^{(j)}\in\mathbb{Z}/p$ are the sets of weights for some action of $\mathbb{Z}/p$ on a manifold M^{2n} if and only if they satisfy the following Conner–Floyd equations:

$$\sum_{j=1}^{q}\biggl\langle \frac{pu}{[u]_p}\prod_{k=1}^{n}\frac{u}{[u]_{x_k^{(j)}}}\biggr\rangle_m \simeq 0, \qquad m=0,\dots,n-1. \tag{31}$$

Here $[u]_m$ is the mth power in the universal formal group law of geometric cobordisms (cf. [4], [12]). Applying a Hirzebruch genus $\varphi\colon \Omega_U\to\Lambda$, we obtain the Conner–Floyd equations corresponding to φ:

$$\sum_{j=1}^{q}\biggl\langle \frac{pu}{[u]^{\varphi}_p}\prod_{k=1}^{n}\frac{u}{[u]^{\varphi}_{x_k^{(j)}}}\biggr\rangle_m \simeq 0, \qquad m=0,\dots,n-1, \tag{32}$$

where $[u]^{\varphi}_m$ is the mth power in the formal group law corresponding to the genus φ.

The following formula for the Todd genus was deduced from cobordism theory in [5]:

$$\operatorname{td}(M)\simeq\sum_{j=1}^{q}\biggl\langle \frac{pu}{[u]^{\mathrm{td}}_p}\prod_{k=1}^{n}\frac{u}{[u]^{\mathrm{td}}_{x_k^{(j)}}}\biggr\rangle_n. \tag{33}$$

Furthermore, it was shown there that formula (33) is exactly the difference between the formula

$$\operatorname{td}(M)\simeq-\sum_{j=1}^{q}\operatorname{Tr}\biggl(\prod_{k=1}^{n}\frac{1}{1-\zeta^{x_k^{(j)}}}\biggr)\simeq\sum_{j=1}^{q}\sum_{m=0}^{n}\biggl\langle \frac{pu}{[u]^{\mathrm{td}}_p}\prod_{k=1}^{n}\frac{u}{[u]^{\mathrm{td}}_{x_k^{(j)}}}\biggr\rangle_m,$$

deduced from the Atiyah–Bott theorem (see § 4.2) and the sum of the Conner–Floyd equations (32) for the Todd genus. (Here $\zeta=e^{2\pi i/p}$, $[u]^{\mathrm{td}}_m=1-(1-u)^m$.)

Below we generalize this result considering the case of an arbitrary genus φ that satisfies the hypotheses of Theorem 5.1.

heorem 7.1. *The difference between the formula of theorem* 5.1*,*

$$\varphi(M) \simeq -\sum_{j=1}^{q} \mathrm{Tr}\biggl(\prod_{k=1}^{n} \frac{1}{[\theta]^{\varphi}_{x_k^{(j)}}}\biggr), \qquad \theta = f_{\varphi}\left(-\frac{2\pi i}{p}\right)$$

and the sum of the Conner-Floyd equations (32) *for the genus φ with some p-adic integer coefficients gives the following formula for the genus φ:*

$$\varphi(M) \simeq \sum_{j=1}^{q} \biggl\langle \frac{pu}{[u]^{\varphi}_{p}} \prod_{k=1}^{n} \frac{u}{[u]^{\varphi}_{x_k^{(j)}}} \biggr\rangle_n .$$

Proof. We have

$$\prod_{k=1}^{n} \frac{1}{[\theta]^{\varphi}_{x_k}} = \frac{1}{\theta^n} \prod_{k=1}^{n} \frac{\theta}{[\theta]^{\varphi}_{x_k}} =: \frac{1}{\theta^n} \sum_{i=0}^{\infty} A_i \theta^i = \frac{1}{\theta^n} A(\theta),$$

where $A(u) := \prod_{k=1}^{n} \frac{u}{[u]^{\varphi}_{x_k}} = \sum_{i=0}^{\infty} A_i u^i,\ \ A_i \in \mathbb{Z}_p$ are p-adic integers. Therefore,

$$\begin{aligned}\mathrm{Tr}\biggl(\prod_{k=1}^{n} \frac{1}{[\theta]^{\varphi}_{x_k}}\biggr) &= \mathrm{Tr}\biggl(\frac{1}{\theta^n}\sum_{i=0}^{\infty} A_i\theta^i\biggr) \simeq \mathrm{Tr}\biggl(\frac{1}{\theta^n}\sum_{i=0}^{n} A_i\theta^i\biggr) \\ &= \sum_{i=0}^{n} A_i \,\mathrm{Tr}\,\theta^{i-n} = \biggl\langle \prod_{k=1}^{n} \frac{u}{[u]^{\varphi}_{x_k}} B(u) \biggr\rangle_n ,\end{aligned}$$

where $B_s := \mathrm{Tr}\,\theta^{-s},\ \ B(u) := \sum_{i=0}^{\infty} B_i u^i$.

Now we introduce a power series $h(u)$ as follows:

$$h(u) = p\frac{[u]^{\varphi}_{p} - u}{B(u)[u]^{\varphi}_{p}}. \tag{34}$$

Then $h(u)$ is a series with p-adic integer coefficients beginning with 1. Indeed,

$$h(0) = p\frac{[u]^{\varphi}_{p} - u}{B(u)[u]^{\varphi}_{p}}\bigg|_{u=0} = p\frac{1 - \frac{u}{[u]^{\varphi}_{p}}}{B(u)}\bigg|_{u=0} = \frac{p\left(1 - \frac{1}{p}\right)}{p-1} = 1,$$

since $B(0) = B_0 = \mathrm{Tr}\,\theta^0 = p - 1,\ \ [u]^{\varphi}_{p} = pu + \cdots$. It follows from (34) that

$$B(u) = \frac{p}{h(u)} - \frac{pu}{[u]^{\varphi}_{p}} \frac{1}{h(u)}.$$

Hence,

$$B(u) \simeq -\frac{pu}{[u]^{\varphi}_{p}} \frac{1}{h(u)} = -\frac{pu}{[u]^{\varphi}_{p}} \biggl(1 + \sum_{i=1}^{\infty} H_i u^i\biggr),$$

where the H_i are the coefficients of the series $\frac{1}{h(u)}$. Thus,

$$\begin{aligned}\mathrm{Tr}\Big(\prod_{k=1}^{n}\frac{1}{[\theta]^{\varphi}_{x_k}}\Big) &\simeq -\Big\langle \frac{pu}{[u]^{\varphi}_{p}}\Big(1+\sum_{i=1}^{\infty}H_iu^i\Big)\prod_{k=1}^{n}\frac{u}{[u]^{\varphi}_{x_k}}\Big\rangle_n\\ &= -\Big\langle \frac{pu}{[u]^{\varphi}_{p}}\prod_{k=1}^{n}\frac{u}{[u]^{\varphi}_{x_k}}\Big\rangle_n - \sum_{m=0}^{n-1}H_{n-m}\Big\langle \frac{pu}{[u]^{\varphi}_{p}}\prod_{k=1}^{n}\frac{u}{[u]^{\varphi}_{x_k}}\Big\rangle_m,\end{aligned}$$

$$\begin{aligned}\varphi(M) &\simeq -\sum_{j=1}^{q}\mathrm{Tr}\Big(\prod_{k=1}^{n}\frac{1}{[\theta]^{\varphi}_{x_k^{(j)}}}\Big)\\ &\simeq \sum_{j=1}^{q}\Big\langle \frac{pu}{[u]^{\varphi}_{p}}\prod_{k=1}^{n}\frac{u}{[u]^{\varphi}_{x_k^{(j)}}}\Big\rangle_n + \sum_{m=0}^{n-1}H_{n-m}\Big(\sum_{j=1}^{q}\Big\langle \frac{pu}{[u]^{\varphi}_{p}}\prod_{k=1}^{n}\frac{u}{[u]^{\varphi}_{x_k^{(j)}}}\Big\rangle_m\Big).\end{aligned}$$

This proves the theorem.

Example 7.1. Consider the Todd genus $\mathrm{td}(M)$. Then $[u]^{\mathrm{td}}_p = 1-(1-u)^p$, $B(u) = p\frac{1-(1-u)^{p-1}}{1-(1-u)^p}$ (see § 4.2). Therefore,

$$h(u) = \frac{1-(1-u)^p-u}{1-(1-u)^{p-1}} = 1-u$$

(see formula (34)).

Example 7.2. Consider the L-genus $L(M)$. Then

$$[u]^{L}_p = \frac{(1+u)^p-(1-u)^p}{(1+u)^p+(1-u)^p}, \quad B(u) = p\frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p-(1-u)^p}$$

(see § 4.3). Therefore,

$$\begin{aligned}h(u) &= \frac{\frac{(1+u)^p-(1-u)^p}{(1+u)^p+(1-u)^p}-u}{\frac{(1+u)^{p-1}-(1-u)^{p-1}}{(1+u)^p+(1-u)^p}}\\ &= \frac{(1+u)^p-(1-u)^p-u(1+u)^p-u(1-u)^p}{(1+u)^{p-1}-(1-u)^{p-1}} = (1+u)(1-u),\end{aligned}$$

which is in accordance with the calculations from § 4.2, 4.3.

Example 7.3. Consider the $\hat{A}$-genus $\hat{A}(M)$. Then

$$[u]^{\hat{A}}_p = 2\sinh\Big(p\,\mathrm{arcsinh}\,\frac{u}{2}\Big), \quad B(u) = p\frac{\sinh\big((p-1)\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}{\cosh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)\sinh\big(p\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}$$

(see § 6.1). Therefore,

$$\begin{aligned}h(u) &= \frac{\big(2\sinh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)-u\big)\cosh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}{2\sinh\big((p-1)\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}\\ &= \frac{\big(\sinh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)-\sinh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)\big)\cosh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}{\sinh\big((p-1)\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}\\ &= \frac{2\sinh\big(\frac{p-1}{2}\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)\cosh\big(\frac{p+1}{2}\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)\cosh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}{2\sinh\big(\frac{p-1}{2}\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)\cosh\big(\frac{p-1}{2}\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}\\ &= \frac{\cosh\big(\frac{p+1}{2}\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)\cosh\big(\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}{\cosh\big(\frac{p-1}{2}\,\mathrm{arcsinh}\big(\frac{u}{2}\big)\big)}.\end{aligned}$$

§ 8. The elliptic genus for manifolds with $\mathbb{Z}/p$-action

Let M^{2n} be a $2n$-dimensional real orientable manifold with a complex structure in its stable tangent bundle.

Definition 8.1 (see [7]). A Hirzebruch genus $\varphi(M^{2n}) = \left(\prod_{i=1}^n \frac{x_i}{f(x_i)}[M^{2n}]\right)$ is called the *elliptic genus* if f satisfies one of the following equivalent conditions:

1) $f'^2 = 1 - 2\delta f^2 + \varepsilon f^4, \ \ f(0) = 0;$
2) $f(u+v) = \dfrac{f(u)f'(v) + f'(u)f(v)}{1 - \varepsilon f(u)^2 f(v)^2}.$

Let us consider the lattice $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ in $\mathbb{C}$, with $\operatorname{Im}\tau > 0$, and put $L' = L\backslash\{0\}$. The Weierstraß $\wp$-function is $\wp(z) = \frac{1}{z^2} + \sum_{\omega\in L'}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$. It satisfies the following differential equation:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4\bigl(\wp(z) - e_1\bigr)\bigl(\wp(z) - e_2\bigr)\bigl(\wp(z) - e_3\bigr),$$

where $e_1 = \wp(\pi i), \ e_2 = \wp(\pi i\tau), \ e_3 = \wp\bigl(\pi i(\tau+1)\bigr)$ are the zeros of the derivative $\wp'$. The function $f(z) = 1\big/\sqrt{\wp(z) - e_1}$ (that is, $f(z) = \operatorname{sn}(z)$, the elliptic sine) satisfies the conditions of Definition 8.1 for $\delta = -\frac{3}{2}e_1, \ \ \varepsilon = (e_1 - e_2)(e_1 - e_3)$ (see [7]). Thus it gives rise to the elliptic genus. However, $\varepsilon = (e_1 - e_2)(e_1 - e_3) \neq 0$ and $\delta^2 - \varepsilon = \frac{1}{4}(e_2 - e_3)^2 \neq 0$, since e_1, e_2, e_3 are all different. The degenerate cases $e_1 = e_2$ and $e_1 = e_3$ ($\varepsilon = 0$) correspond to the $\hat{A}$-genus. (Putting $\delta = -1/8$, we obtain $f'^2 = 1 + \frac{1}{4}f^2$, that is, $f(x) = 2\sinh(x/2)$.) The degenerate case $e_2 = e_3$ ($\delta^2 - \varepsilon = 0$) corresponds to the L-genus. (Putting $\delta = \varepsilon = 1$, we obtain $f'^2 = (1 - f^2)^2$, that is, $f(x) = \tanh x$.)

The differential equation defining the function $f(x)$ for the elliptic genus implies that if we put $\deg\varepsilon = 4$ and $\deg\delta = 2$, then the coefficient of x^{2k} in the series $f(x)$ becomes a weighted homogeneous polynomial of degree $2k$ in δ and ε with coefficients in the ring $\mathbb{Z}\left[\frac{1}{2}\right]$. Therefore, the elliptic genus $\varphi(M^{2n}) = \left(\prod_{i=1}^n \frac{x_i}{f(x_i)}\right)[M^{2n}]$ is a homogeneous polynomial of degree $2n$ in δ and ε. So φ can be regarded as a homomorphism from the complex cobordism ring Ω_U to the ring $\mathbb{Z}\left[\frac{1}{2}\right][\delta, \varepsilon]$ (or to the ring $\mathbb{Z}_p[\delta, \varepsilon], \ \ p > 2$).

The function $f(\tau, x)$ is an elliptic function for the sublattice $\tilde{L} = 2\pi i(\mathbb{Z}\cdot 2\tau + \mathbb{Z}) \subset L$ of index 2. The divisor of this function is $(0) + (\pi i \cdot 2\tau) - (\pi i) - \bigl(\pi i(1 + 2\tau)\bigr)$. We have the following decomposition of $f(\tau, x)$ into an infinite product (see [7]):

$$f(\tau, x) = 2\frac{1 - e^{-x}}{1 + e^{-x}} \prod_{k=1}^{\infty} \frac{(1 - q^k e^x)(1 - q^k e^{-x})(1 + q^k)^2}{(1 + q^k e^x)(1 + q^k e^{-x})(1 - q^k)^2}, \quad q = e^{2\pi i\tau}. \tag{35}$$

Now we consider the following object:

$$\left\{ L_i^{(q)} = \sum_{p=0}^{n} \Lambda^{p,i} \otimes \left(\bigotimes_{k=1}^{\infty} S_{q^k} T_{\mathbb{C}} M \otimes \bigotimes_{k=1}^{\infty} \Lambda_{q^k} T_{\mathbb{C}} M \right) \right\}, \tag{36}$$

where $T_{\mathbb{C}}M$ is the complexification of the tangent bundle to M, $\Lambda_t E := \sum_{k=0}^{\infty}(\Lambda^k E)t^k, \ \ S_t E := \sum_{k=0}^{\infty}(S^k E)t^k$. Then $L^{(q)}$ is a power series in q whose coefficients are elliptic complexes associated to TM. Its index is a well-known power series in q, namely, the so-called twisted signature (see [7]): $\mathrm{ind}(L^{(q)}) = \mathrm{sign}\left(M, \bigotimes_{k=1}^{\infty} S_{q^k} T_{\mathbb{C}}M \otimes \bigotimes_{k=1}^{\infty} \Lambda_{q^k} T_{\mathbb{C}}M\right)$. It follows from the Atiyah–Singer theorem 2.1 and Lemma 2.3 that this index is

$$\mathrm{ind}(L^{(q)}) = \left(\prod_{j=1}^{n}\left(x_j \frac{1+e^{-x_j}}{1-e^{-x_j}} \prod_{k=1}^{\infty} \frac{(1+q^k e^{x_j})(1+q^k e^{-x_j})}{(1-q^k e^{x_j})(1-q^k e^{-x_j})}\right)\right)[M^{2n}]. \tag{37}$$

It is clear from this formula that $\mathrm{ind}(L^{(q)})$ is a power series in q with integer coefficients and with constant term (the coefficient of q^0) $L(M) = \mathrm{sign}\, M$. Comparing this expression with (35) and taking into account that

$$\frac{1}{2}\prod_{k=1}^{\infty} \frac{(1-q^k)^2}{(1+q^k)^2} = \varepsilon^{1/4} \tag{38}$$

(see [7]), we obtain

$$\varphi(M) = \mathrm{sign}\left(M, \bigotimes_{k=1}^{\infty} S_{q^k} T_{\mathbb{C}}M \otimes \bigotimes_{k=1}^{\infty} \Lambda_{q^k} T_{\mathbb{C}}M\right)\varepsilon^{n/4}. \tag{39}$$

Suppose that an operator g, $g^p = 1$, acts on a manifold M^{2n} with finitely many fixed points $\mathcal{P}_1, \ldots, \mathcal{P}_r$. According to the recipe from § 5, the equivariant index $\mathrm{ind}\big(g, L^{(q)}\big)$ of the complex $L^{(q)}$ equals the sum of the contribution functions $\sigma_{L^{(q)}}(\mathcal{P}_j)$ of the fixed points. These contributions are obtained from formula (37) by replacing M^{2n} by $\mathcal{P}_j$, $x_1 x_2 \ldots x_n$ by 1 and x_k by $-\frac{2\pi i}{p} x_k^{(j)}$:

$$\sigma_{L^{(q)}}(\mathcal{P}_j) = \prod_{k=1}^{n}\left(\frac{1+\zeta^{x_k^{(j)}}}{1-\zeta^{x_k^{(j)}}} \prod_{i=1}^{\infty} \frac{\left(1+q^i\zeta^{-x_k^{(j)}}\right)\left(1+q^i\zeta^{x_k^{(j)}}\right)}{\left(1-q^i\zeta^{-x_k^{(j)}}\right)\left(1-q^i\zeta^{x_k^{(j)}}\right)}\right), \quad \zeta = e^{\frac{2\pi i}{p}}.$$

We note that $\frac{1}{p}\sum_{l\in\mathbb{Z}/p}\operatorname{ind}\bigl(g^l, L^{(q)}\bigr) = s(q)$ is a power series in q whose coefficient at q^k is the alternating sum of the dimensions of the invariant subspaces for the action of g on the cohomology of a certain complex. Namely, this complex is the coefficient of q^k in the series $L^{(q)}$:

$$L^{(q)} = L + 2L\otimes T_{\mathbb{C}}Mq + L\otimes(2T_{\mathbb{C}}M + T_{\mathbb{C}}M\otimes T_{\mathbb{C}}M + S^2T_{\mathbb{C}}M + \Lambda^2T_{\mathbb{C}}M)q^2 + \cdots,$$

where $L = \sum_{p=0}^{n}\Lambda^{p,i}$. Therefore, $s(q)$ is a series with integer coefficients. Furthermore, $\operatorname{ind}\bigl(1, L^{(q)}\bigr) = \operatorname{sign}\bigl(M, \bigotimes_{k=1}^{\infty} S_{q^k}T_{\mathbb{C}}M\otimes\bigotimes_{k=1}^{\infty}\Lambda_{q^k}T_{\mathbb{C}}M\bigr)$. Hence,

$$\begin{aligned}
&\operatorname{sign}\Bigl(M, \bigotimes_{k=1}^{\infty} S_{q^k}T_{\mathbb{C}}M\otimes\bigotimes_{k=1}^{\infty}\Lambda_{q^k}T_{\mathbb{C}}M\Bigr)\\
&\quad = \operatorname{ind}(1, L^{(q)}) = -\sum_{l=1}^{p-1}\operatorname{ind}(g^l, L^{(q)}) + ps(q)\\
&\quad = -\sum_{j=1}^{r}\sum_{l=1}^{p-1}\prod_{k=1}^{n}\left(\frac{1+\zeta^{lx_k^{(j)}}}{1-\zeta^{lx_k^{(j)}}}\prod_{i=1}^{\infty}\frac{\Bigl(1+q^i\zeta^{-lx_k^{(j)}}\Bigr)\Bigl(1+q^i\zeta^{lx_k^{(j)}}\Bigr)}{\Bigl(1-q^i\zeta^{-lx_k^{(j)}}\Bigr)\Bigl(1-q^i\zeta^{lx_k^{(j)}}\Bigr)}\right) + ps(q).
\end{aligned}$$

The left-hand side of this relation belongs to the ring $\mathbb{Z}\bigl[[q]\bigr]$ of power series with integer coefficients, while its right-hand side *a priori* belongs to an extension of $\mathbb{Z}\bigl[[q]\bigr]$, namely, to the ring $\mathbb{Z}\bigl[[q]\bigr](\zeta)$, $\zeta^p = 1$. We embed both rings in the p-adic extensions of the corresponding fields and consider the number-theoretical trace $\operatorname{Tr}\colon \mathbb{Q}_p\{q\}(\zeta)\to\mathbb{Q}_p\{q\}$, where $\mathbb{Q}_p\{q\} = \mathbb{Q}_p[[q]][q^{-1}]$ is the field of Laurent series in q with rational p-adic coefficients and $\mathbb{Q}_p\{q\}(\zeta)$ is the algebraic extension of $\mathbb{Q}_p\{q\}$ by the element ζ, $\zeta^p = 1$. Then

$$\begin{aligned}
&\sum_{l=1}^{p-1}\prod_{k=1}^{n}\left(\frac{1+\zeta^{lx_k^{(j)}}}{1-\zeta^{lx_k^{(j)}}}\prod_{i=1}^{\infty}\frac{\Bigl(1+q^i\zeta^{-lx_k^{(j)}}\Bigr)\Bigl(1+q^i\zeta^{lx_k^{(j)}}\Bigr)}{\Bigl(1-q^i\zeta^{-lx_k^{(j)}}\Bigr)\Bigl(1-q^i\zeta^{lx_k^{(j)}}\Bigr)}\right)\\
&\qquad = \operatorname{Tr}\left(\prod_{k=1}^{n}\left(\frac{1+\zeta^{x_k^{(j)}}}{1-\zeta^{x_k^{(j)}}}\prod_{i=1}^{\infty}\frac{\Bigl(1+q^i\zeta^{-x_k^{(j)}}\Bigr)\Bigl(1+q^i\zeta^{x_k^{(j)}}\Bigr)}{\Bigl(1-q^i\zeta^{-x_k^{(j)}}\Bigr)\Bigl(1-q^i\zeta^{x_k^{(j)}}\Bigr)}\right)\right).
\end{aligned}$$

Thus the following proposition holds:

Proposition 8.2. *We have the following formula for the index of the complex $L^{(q)}$ in* (36)*:*

$$\begin{aligned}
&\operatorname{ind}(L^{(q)}) = \operatorname{sign}\Bigl(M, \bigotimes_{k=1}^{\infty} S_{q^k}T_{\mathbb{C}}M\otimes\bigotimes_{k=1}^{\infty}\Lambda_{q^k}T_{\mathbb{C}}M\Bigr)\\
&\equiv -\sum_{j=1}^{r}\operatorname{Tr}\left(\prod_{k=1}^{n}\left(\frac{1+\zeta^{x_k^{(j)}}}{1-\zeta^{x_k^{(j)}}}\prod_{i=1}^{\infty}\frac{\Bigl(1+q^i\zeta^{-x_k^{(j)}}\Bigr)\Bigl(1+q^i\zeta^{x_k^{(j)}}\Bigr)}{\Bigl(1-q^i\zeta^{-x_k^{(j)}}\Bigr)\Bigl(1-q^i\zeta^{x_k^{(j)}}\Bigr)}\right)\right) \bmod p\mathbb{Z}\bigl[[q]\bigr].
\end{aligned}\tag{40}$$

Using relations (38) and (39), we thus obtain the following formula for the elliptic genus $\varphi(M)$:

$$\varphi(M) \equiv -\sum_{j=1}^{r} \operatorname{Tr}\left(\prod_{k=1}^{n}\left(\frac{1}{2}\cdot\frac{1+\zeta^{x_k^{(j)}}}{1-\zeta^{x_k^{(j)}}}\right.\right.$$
$$\left.\left.\times\prod_{i=1}^{\infty}\frac{\left(1+q^i\zeta^{-x_k^{(j)}}\right)\left(1+q^i\zeta^{x_k^{(j)}}\right)(1-q^i)^2}{\left(1-q^i\zeta^{-x_k^{(j)}}\right)\left(1-q^i\zeta^{x_k^{(j)}}\right)(1+q^i)^2}\right)\right) \mod p\mathbb{Z}_p\big[[q]\big].$$

Taking into account the decomposition (35) of $f(\tau,x)=\operatorname{sn}x$, we rewrite this formula as

$$\varphi(M) \equiv -\sum_{j=1}^{r} \operatorname{Tr}\left(\prod_{k=1}^{n}\frac{1}{f\left(\tau,-\frac{2\pi i x_k^{(j)}}{p}\right)}\right) \mod p\mathbb{Z}_p\big[[q]\big], \quad q=e^{2\pi i\tau}.$$

However, $\varphi(M)$ is a homogeneous polynomial in δ, ε with coefficients in the ring $\mathbb{Z}\left[\frac{1}{2}\right]\subset\mathbb{Z}_p$. So we want to obtain the expression for $\varphi(M)$ in the ring $\mathbb{Z}_p[\delta,\varepsilon]$, not in the ring $\mathbb{Z}_p\big[[q]\big]$. ($\mathbb{Z}_p[\delta,\varepsilon]$ is the subring of $\mathbb{Z}_p\big[[q]\big]$.) As before, we put $\theta=f\left(\tau,-\frac{2\pi i}{p}\right)$. Then $f\left(\tau,-\frac{2\pi i x_k}{p}\right)=[\theta]_{x_k}$, where $[\theta]_m=f\big(mf^{-1}(\theta)\big)$ is the mth power in the formal group law corresponding to the elliptic genus. We note that θ is an algebraic element over the field $\mathbb{Q}(\delta,\varepsilon)$. Indeed,

$$[\theta]_p=f\left(\tau,-\frac{2\pi i p}{p}\right)=f(\tau,-2\pi i)=0,$$

and $[u]_p$ is a rational function of u when p is odd. (For example, $[u]_3=u\frac{3-8\delta u^2+6\varepsilon u^4-\varepsilon^2u^8}{1-6\varepsilon u^4+8\delta\varepsilon u^6-3\varepsilon^2u^8}$. This follows from the differential equation and the addition theorem for $f(\tau,x)=\operatorname{sn}x$.) The numerator of $[u]_p$ is a polynomial in u with coefficients δ, ε, and θ is a zero of this polynomial. Obviously, this polynomial is of degree p^2 in u, and its zeros are the values of f at all p-division points of the lattice $2\pi i(\mathbb{Z}\cdot 2\tau+\mathbb{Z})$, that is, they equal $f\left(\tau,2\pi i\frac{2k\tau+l}{p}\right)$, $k,l\in\mathbb{Z}$. In particular, $\theta,[\theta]_2,\dots,[\theta]_{p-1},[\theta]_p=0$ are among zeros.

Finally, we have

$$\varphi(M)\equiv-\sum_{j=1}^{r}\operatorname{Tr}\Bigg(\prod_{k=1}^{n}\frac{1}{[\theta]_{x_k}}\Bigg) \mod p\mathbb{Z}_p\big[[q]\big], \quad \theta=f\left(\tau,-\frac{2\pi i}{p}\right), \tag{41}$$

where both right- and left-hand sides are polynomials in δ, ε.

Lemma 8.3. *Let $P(\delta,\varepsilon)$ be a homogeneous polynomial in δ, ε such that $P(\delta,\varepsilon)\equiv 0 \mod p\mathbb{Z}_p\big[[q]\big]$. Then $P(\delta,\varepsilon)\equiv 0 \mod p\mathbb{Z}_p[\delta,\varepsilon]$, that is, all coefficients of $P(\delta,\varepsilon)$ belong to $p\mathbb{Z}_p$.*

Proof. The functions δ, ε are modular forms of weights 2 and 4 respectively on the subgroup $\Gamma_0(2)\subset\mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0(2):=\left\{A\in\mathrm{SL}_2(\mathbb{Z})\;\middle|\;A\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\pmod 2\right\}$.

They have the following Fourier expansions near infinity (that is, for small $q = e^{2\pi i\tau}$):

$$\begin{aligned}
\delta &= -\frac{3}{2}e_1 = \frac{1}{4} + \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d\equiv 1 \mod 2}} dq^n \\
&= \frac{1}{4} + 6(q + q^2 + 4q^3 + q^4 + 6q^5 + 4q^6 + \cdots), \\
\varepsilon &= (e_1 - e_2)(e_1 - e_3) \\
&= \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d|n} (-1)^d d^3 q^n = \frac{1}{16} - q + 7q^2 - 28q^3 + 71q^4 - \cdots
\end{aligned}$$

Furthermore, $\delta(0) = -\frac{1}{8}$, $\varepsilon(0) = 0$, $\delta\left(\frac{1+i}{2}\right) = 0$, $\varepsilon\left(\frac{1+i}{2}\right) \neq 0$ (see [7]).

The proof is by induction on the degree $\deg P = 2n$ of the polynomial P. If $\deg P(\delta,\varepsilon) = 2$, then $P(\delta,\varepsilon) = b\delta$. Hence, $b \in p\mathbb{Z}_p$, because $\delta \notin p\mathbb{Z}_p$. Now assume that the lemma holds for all polynomials of degree $< 2k$ and let $\deg P(\delta,\varepsilon) = 2k$. Write $P(\delta,\varepsilon) = a\varepsilon^{k/2} + Q(\delta,\varepsilon)\varepsilon\delta + b\delta^k$, where $\deg Q(\delta,\varepsilon) < 2k$, and therefore $Q(\delta,\varepsilon) \equiv 0 \mod p\mathbb{Z}_p[\delta,\varepsilon]$. Evaluation of $P(\delta,\varepsilon)$ at the point $\tau = \frac{1+i}{2}$ shows that $a\varepsilon\left(\frac{1+i}{2}\right)^{k/2}$ is divisible by p, whence a is divisible by p. Evaluation of $P(\delta,\varepsilon)$ at $\tau = 0$ shows that $\left(-\frac{1}{8}\right)^k b$ is divisible by p, whence b is divisible by p. Thus all the coefficients of $P(\delta,\varepsilon)$ belong to $p\mathbb{Z}_p$. The lemma is proved.

From this lemma and (41) we obtain the following theorem.

Theorem 8.4. *Let g be a transversal endomorphism acting on M^{2n} such that $g^p = 1$. Then we have the following formula for the elliptic genus $\varphi(M^{2n})$:*

$$\varphi(M) \equiv -\sum_{j=1}^{r} \operatorname{Tr}\left(\prod_{k=1}^{n} \frac{1}{f\left(\tau, -\frac{2\pi i x_k^{(j)}}{p}\right)}\right) \mod p\mathbb{Z}_p[\delta,\varepsilon], \tag{42}$$

where Tr *in the right-hand side denotes the number-theoretical trace $\mathbb{Q}_p(\delta,\varepsilon)(\zeta) \to \mathbb{Q}_p(\delta,\varepsilon)$.*

All constructions in § 5 and § 7 (in particular, Lemma 5.4 and Theorem 7.1) can be repeated for the elliptic genus (we just replace the ring $\mathbb{Z}_p$ by $\mathbb{Z}_p[\delta,\varepsilon]$ in all formulae). Thus, the difference between formula (42) and a certain weighted sum of the following Conner–Floyd equations for the elliptic genus:

$$\sum_{j=1}^{r} \left\langle \frac{pu}{[u]_p} \prod_{k=1}^{n} \frac{u}{[u]_{x_k^{(j)}}} \right\rangle_m \equiv 0 \mod p\mathbb{Z}_p[\delta,\varepsilon], \quad m = 0,\dots,n-1,$$

gives the following formula for the elliptic genus:

$$\varphi(M^{2n}) \equiv \sum_{j=1}^{r} \left\langle \frac{pu}{[u]_p} \prod_{k=1}^{n} \frac{u}{[u]_{x_k^{(j)}}} \right\rangle_n \mod p\mathbb{Z}_p[\delta,\varepsilon]. \tag{43}$$

In the last two formulae, $[u]_m$ denotes the mth power in the formal group law corresponding to the elliptic genus.

To give an example, we apply the formulae for the elliptic genus to a particular action of the group $\mathbb{Z}/p$. Namely, we consider the action of $\mathbb{Z}/p$ on $\mathbb{C}P^n$ such that the generator g acts in homogeneous coordinates as $(z_0 : z_1 : \dots : z_n) \to (\lambda_0 z_0 : \lambda_1 z_1 : \dots : \lambda_n z_n)$, $\lambda_i = e^{\frac{2\pi i}{p} y_i}$. If $y_0, \dots, y_n$ are distinct as residues $\bmod p$, which implies that $n < p$, then this action has only finitely many fixed points on $\mathbb{C}P^n$. Namely, the fixed points are $\mathcal{P}_j = \left(0, \dots, \overset{j}{1}, \dots, 0\right)$, $j = 0, \dots, n$. In the local coordinates $\left(\frac{z_0}{z_j}, \frac{z_1}{z_j}, \dots, \widehat{\frac{z_j}{z_j}}, \dots, \frac{z_n}{z_j}\right)$ near $\mathcal{P}_j$, the operator g acts linearly: $\frac{z_i}{z_j} \to \frac{\lambda_i z_i}{\lambda_j z_j} = \exp\left(\frac{2\pi i}{p}(y_i - y_j)\right) \frac{z_i}{z_j}$. Therefore, the eigenvalues of the map g at the point $\mathcal{P}_j$ are $\exp\left(\frac{2\pi i}{p}(y_i - y_j)\right)$, $i \neq j$, and the corresponding weights are $x_i^{(j)} = y_i - y_j$.

Now we consider the elliptic genus φ. It is well known that

$$\varphi(\mathbb{C}P^n) = \left\langle \frac{1}{\sqrt{1 - 2\delta u^2 + \varepsilon u^4}} \right\rangle_n . \tag{44}$$

Indeed, it follows from formula (1), that $\varphi(\mathbb{C}P^n) = \left\langle g'_\varphi(u) \right\rangle_n$, where $g_\varphi(u) = f_\varphi^{-1}(u)$ is the logarithm of the corresponding formal group law. But $g'_\varphi(u) = \frac{1}{\sqrt{1-2\delta u^2 + \varepsilon u^4}}$, since $(f')^2 = 1 - 2\delta f^2 + \varepsilon f^4$.

At the same time, it follows from (43) that

$$\varphi(\mathbb{C}P^n) \equiv \sum_{j=0}^{n} \left\langle \frac{pu}{[u]_p} \prod_{i \neq j} \frac{u}{[u]_{y_i - y_j}} \right\rangle_n \mod p\mathbb{Z}_p[\delta, \varepsilon].$$

Hence for any set $y_0, y_1, \dots, y_n$ of distinct residues modulo p, where $n < p$, we have

$$\sum_{j=0}^{n} \left\langle \frac{pu}{[u]_p} \prod_{i \neq j} \frac{u}{[u]_{y_i - y_j}} \right\rangle_n \equiv \left\langle \frac{1}{\sqrt{1 - 2\delta u^2 + \varepsilon u^4}} \right\rangle_n \mod p\mathbb{Z}_p[\delta, \varepsilon]$$

The function $(1 - 2\delta u + u^2)^{-1/2}$ is the generating function for the Legendre polynomials $P_n(\delta)$, that is, $\frac{1}{\sqrt{1-2\delta u + u^2}} = 1 + \sum_{n>0} P_n(\delta) u^n$. Hence,

$$\frac{1}{\sqrt{1 - 2\delta u^2 + \varepsilon u^4}} = 1 + \sum_{n>0} P_n\left(\frac{\delta}{k}\right)(ku^2)^n,$$

with $k^2 = \varepsilon$. Therefore,

$$\begin{aligned} &\left\langle \frac{1}{\sqrt{1 - 2\delta u^2 + \varepsilon u^4}} \right\rangle_{2m} = k^m P_m\left(\frac{\delta}{k}\right), \\ &\sum_{j=0}^{2m} \left\langle \frac{pu}{[u]_p} \prod_{i \neq j} \frac{u}{[u]_{y_i - y_j}} \right\rangle_{2m} \equiv k^m P_m\left(\frac{\delta}{k}\right) \mod p\mathbb{Z}_p[\delta, \varepsilon]. \end{aligned} \tag{45}$$

In the simplest case $2m = n = p-1$, the set $y_0 - y_j, y_1 - y_j, \dots, \widehat{y_j - y_j}, \dots, y_{p-1} - y_j$ forms the complete set $1, 2, \dots, p-1$ of non-zero residues modulo p (since the y_i are distinct). Hence, all the summands in the left-hand side of the last formula are equal, and we obtain

$$\left\langle \frac{p^2 u^p}{u[u]_2[u]_3\cdots[u]_{p-1}[u]_p} \right\rangle_{p-1} \equiv \left\langle \frac{1}{\sqrt{1-2\delta u^2+\varepsilon u^4}} \right\rangle_{p-1}$$
$$= k^{\frac{p-1}{2}} P_{\frac{p-1}{2}}\left(\frac{\delta}{k}\right) \mod p\mathbb{Z}_p[\delta,\varepsilon]. \tag{46}$$

This formula could be also deduced from the well-known relation

$$[u]_p \equiv P_{(p-1)/2}(\delta)u^p + \cdots \mod p\mathbb{Z}_p[\delta]$$

(see [10]). Here we put $k=1$ for simplicity, and the dots stand for the higher order terms. Indeed, we could rewrite the above relation as

$$[u]_p = pu(1+b_1u+\cdots+b_{p-1}u^{p-1}) + P_{(p-1)/2}(\delta)u^p + \cdots,$$

with $b_1,\dots,b_{p-1}\in\mathbb{Z}[\delta]$. Hence,

$$\begin{aligned}
&\left\langle \frac{p^2u^p}{u[u]_2\cdots[u]_{p-1}[u]_p}\right\rangle_{p-1} \\
&\quad= \left\langle \frac{p^2}{u(2u+\cdots)\dots\big((p-1)u+\cdots\big)} \right. \\
&\qquad\quad \left. \times \frac{u^p}{\big(pu(1+b_1u+\cdots+b_{p-1}u^{p-1})+P_{(p-1)/2}(\delta)u^p+\cdots\big)}\right\rangle_{p-1} \\
&\quad= \left\langle \frac{1}{(p-1)!}\frac{p^2}{p(1+c_1u+\cdots+c_{p-1}u^{p-1})+P_{(p-1)/2}(\delta)u^{p-1}+\cdots}\right\rangle_{p-1} \\
&\quad= \frac{1}{(p-1)!}\left\langle \frac{p}{1+c_1u+\cdots+c_{p-2}u^{p-2}+\big(c_{p-1}+P_{(p-1)/2}(\delta)/p\big)u^{p-1}}\right\rangle_{p-1},
\end{aligned}$$

where $c_1,\dots,c_{p-1}$ belong to $\mathbb{Z}[\delta]$. Calculating the latter expression modulo p, we get

$$\left\langle \frac{p^2u^p}{u[u]_2[u]_3\cdots[u]_{p-1}[u]_p}\right\rangle_{p-1} \equiv -\frac{1}{(p-1)!}P_{(p-1)/2}(\delta) \equiv P_{(p-1)/2}(\delta) \mod p\mathbb{Z}_p[\delta],$$

since $(p-1)! \equiv -1 \mod p$, as required.

§ 9. Generalization to $\mathbb{Z}/p$-actions having fixed submanifolds with trivial normal bundle

Suppose that an operator g, $g^p=1$, acts on a stably complex manifold M^{2n}. Let the fixed point set M^g be written as the union of connected fixed submanifolds: $M^g=\bigcup_\nu M^g_\nu$. Also suppose that all the M^g_ν have trivial normal bundle in M.

Let φ be a Hirzebruch genus that can be calculated as the index of an elliptic complex E_φ of bundles associated to the tangent bundle TM. To avoid misunderstanding, we shall denote the first Chern classes of the "virtual" line subbundles of TM by z_i. Then $c(TM)=(1+z_1)\dots(1+z_n)$.

Let $Y = M_k^g$ be one of the fixed submanifolds for the action of g. According to formulae (20) from § 5, we have:

$$TM|_Y = \bigoplus_j N_{\lambda_j}, \qquad c(N_{\lambda_j}) = \prod_{i=1}^{d_{\lambda_j}} \Bigl(1 + z_i^{(\lambda_j)}\Bigr),$$

$$c(TM_Y) = \prod_j \prod_{i=1}^{d_{\lambda_j}} \Bigl(1 + z_i^{(\lambda_j)}\Bigr) = \prod_{i=1}^{n} (1 + z_i),$$

where N_{λ_j} is the subbundle of $TM|_Y$ corresponding to the eigenvalue λ_j of the differential of g. Here $\lambda_j^p = 1$ (hence, $\lambda_j = e^{2\pi i x_j/p}$) and $d_{\lambda_j} = \dim N_{\lambda_j}$. Obviously, $N_1 = TY$ is the tangent bundle to Y. According to the results of Atiyah and Singer [3] described in § 5, the equivariant index $\operatorname{ind}(g, E_\varphi)$ of the complex E_φ can be computed as $\operatorname{ind}(g, E_\varphi) = \sum_\nu \sigma(M_\nu^g)$. We also have the following "recipe" for calculating the fixed point contribution functions $\sigma(Y)$. In the formula (21) for the index of E_φ, replace M by Y and $e^{z_i^{(\lambda_j)}}$ by $\lambda_j^{-1} e^{z_i^{(\lambda_j)}}$. Equivalently, since $\lambda_j = e^{2\pi i x_j/p}$, one should replace $z_i^{(\lambda_j)}$ by $z_i^{(\lambda_j)} - 2\pi i x_j/p$.

Since $\operatorname{ind}(E_\varphi) = \varphi(M)$, we have

$$\operatorname{ind}(E_\varphi) = \left(\prod_{i=1}^{n} \frac{z_i}{f(z_i)}\right)[M] = \left(\prod_j \prod_{i=1}^{d_{\lambda_j}} \frac{z_i^{(\lambda_j)}}{f\bigl(z_i^{(\lambda_j)}\bigr)}\right)[M].$$

Hence, our "recipe" shows that

$$\sigma(Y) = \left(\prod_{\lambda_j \neq 1} \prod_{i=1}^{d_{\lambda_j}} \frac{1}{f\bigl(z_i^{(\lambda_j)} - 2\pi i x_j/p\bigr)} \prod_{i=1}^{d_1} \frac{z_i^{(1)}}{f\bigl(z_i^{(1)}\bigr)}\right)[Y],$$

because $c_n(M) = \prod_j \prod_{i=1}^{d_{\lambda_j}} z_i^{(\lambda_j)}$ reduces to $c_n(Y) = \prod_{i=1}^{d_1} z_i^{(1)}$ and all the weights x_k corresponding to the eigenvalue $\lambda = 1$ are zero. Furthermore, since Y has trivial normal bundle in M, we have $z_i^{(\lambda_j)} = 0$ for $\lambda_j \neq 1$. Therefore, $\sigma(Y) = \prod_j \frac{1}{f(-2\pi i x_j/p)} \varphi(Y)$, where the x_j are the weights corresponding to the eigenvalues $\lambda_j = e^{2\pi i x_j/p} \neq 1$, that is, the x_j are non-zero modulo p. Finally, we have the following formula for the equivariant index:

$$\operatorname{ind}(g, E_\varphi) = \sum_\nu \left(\prod_j \frac{1}{f\bigl(-2\pi i x_j^{(\nu)}\bigr)} \varphi(M_\nu^g)\right).$$

Theorem 5.1 and Proposition 5.3 can naturally be extended to the case of actions having fixed submanifolds with trivial normal bundle. We get the following statement.

Theorem 9.1. *Let φ be a Hirzebruch genussuch that there is an elliptic complex of bundles associated to TM whose index is equal to $\varphi(M)$. Then we have the* *following formula for $\varphi(M)$:*

$$\begin{aligned}\varphi(M) &\equiv -\sum_{\nu} \mathrm{Tr}\Biggl(\prod_j \frac{1}{f\bigl(-2\pi i x_j^{(\nu)}/p\bigr)}\Biggr)\varphi(M_\nu^g) \\ &= -\sum_{\nu} \mathrm{Tr}\Biggl(\prod_j \frac{1}{[\theta]^{\varphi}_{x_j^\nu}}\Biggr)\varphi(M_\nu^g) \mod p,\end{aligned}$$

where $M^g = \bigcup_\nu M_\nu^g$ is the fixed point set, $\theta = f_\varphi(-2\pi i/p)$, $[\theta]_k$ is the k-th power in the corresponding formal group, and $\mathrm{Tr}\colon \mathbb{Q}(e^{2\pi i/p}) \to \mathbb{Q}$ is the number-theoretical trace. (*If the genus φ takes its values in some ring Λ instead of $\mathbb{Z}$, replace $\mathbb{Q}$ by the corresponding quotient field*).

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