ON MANIFOLDS DEFINED BY 4-COLOURINGS OF SIMPLE 3-POLYTOPES

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Let $\mathcal{P}$ be the class of combinatorial 3-dimensional simple polytopes P, different from a tetrahedron, without 3- and 4-belts of facets. By the results of Pogorelov [1] and Andreev [2], a polytope P admits a realisation in Lobachevsky space $\mathbb{L}^3$ with right dihedral angles if and only if $P \in \mathcal{P}$. We consider two families of smooth manifolds defined by regular 4-colourings of polytopes $P \in \mathcal{P}$: six-dimensional quasitoric manifolds over P and three-dimensional small covers of P; the latter are also known as three-dimensional hyperbolic manifolds of Löbell type [3]. We prove that two manifolds from either of the families are diffeomorphic if and only if the corresponding 4-colourings are equivalent.

A *quasitoric manifold* (respectively, a *small over*) over a simple n-polytope P is a $2n$-dimensional (n-dimensional) smooth manifold M with a locally standard action of the torus T^n (the group $\mathbb{Z}_2^n$) and a projection $M \to P$ whose fibres are the orbits of the action, see [4], [5].

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be the set of facets of a simple 3-polytope P. A *characteristic function* over $\mathbb{Z}$ (over $\mathbb{Z}_2$) is a map $\lambda\colon \mathcal{F} \to \mathbb{Z}^n$ ($\lambda\colon \mathcal{F} \to \mathbb{Z}_2^n$) satisfying the condition: if $F_{i_1} \cap F_{i_2} \cap F_{i_3}$ is a vertex, then $\lambda(F_{i_1}), \lambda(F_{i_2}), \lambda(F_{i_3})$ is a basis of the lattice $\mathbb{Z}^n$ (of the space $\mathbb{Z}_2^n$). Characteristic functions λ and λ' are *equivalent* ($\lambda \sim \lambda'$) if one is obtained from the other by a change of basis in $\mathbb{Z}^n$ and changing the direction of some of the vectors $\lambda(F_i)$ to the opposite (by a change of basis in $\mathbb{Z}_2^n$). *Characteristic pairs* (P, λ) and (P', λ') are *equivalent* ($(P, \lambda) \sim (P', \lambda')$) if P and P' are combinatorially equivalent ($P \simeq P'$) and $\lambda \sim \lambda'$.

Every quasitoric manifold (small cover) M over P is defined by a characteristic pair (P, λ); with two such manifolds $M = M(P, \lambda)$ and $M' = M'(P', \lambda')$ being equivariantly homeomorphic if and only if $(P, \lambda) \sim (P', \lambda')$ (see [4], [5, Prop. 7.3.11]). In general, there exist non-equivalent pairs (P, λ) and (P', λ') whose corresponding manifolds M and M' are (non-equivariantly) diffeomorphic.

A (regular) 4-*colouring* of a simple polytope P is a map $\chi\colon \mathcal{F} \to \{1, 2, 3, 4\}$ such that $\chi(F_i) \neq \chi(F_j)$ whenever $F_i \cap F_j \neq \varnothing$. Such a 4-colouring always exists by the Four Colour Theorem. Two 4-colourings χ and χ' are *equivalent* ($\chi \sim \chi'$) if $\chi' = \sigma\chi$ for a permutation $\sigma \in S_4$.

Let χ be a 4-colouring, $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ a basis in $\mathbb{Z}^3$, and $\varepsilon_i = \pm 1$, $i = 1, 2, 3$. These data define a characteristic function $\lambda = \lambda(\chi, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ given by

$$\lambda(F) = \begin{cases} \boldsymbol{a}_i & \text{if } \chi(F) = i, \quad i = 1, 2, 3, \\ \varepsilon_1 \boldsymbol{a}_1 + \varepsilon_2 \boldsymbol{a}_2 + \varepsilon_3 \boldsymbol{a}_3 & \text{if } \chi(F) = 4. \end{cases}$$

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Denote by $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ the standard basis $(1,0,0), (0,1,0), (0,0,1)$ in $\mathbb{Z}^3$ or $\mathbb{Z}_2^3$.

Proposition 1. *We have* $\lambda(\chi, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3) \sim \lambda(\chi, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, 1, 1, 1)$.

Proof. We have $\lambda(\chi, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3) \sim \lambda(\chi, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3) \sim \lambda(\chi, \varepsilon_1\boldsymbol{e}_1, \varepsilon_2\boldsymbol{e}_2, \varepsilon_3\boldsymbol{e}_3, 1, 1, 1) \sim \lambda(\chi, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, 1, 1, 1)$, where the first and third equivalences come from a change of basis in $\mathbb{Z}^3$, while the second equivalence comes from a change of the direction of vectors. □

Note that the equivalence classes of characteristic functions are the orbits of the group $\mathrm{GL}_3(\mathbb{Z})$ or $\mathrm{GL}_3(\mathbb{Z}_2)$, while the equivalence classes of 4-colourings are the orbits of the symmetric group S_4. Nevertheless, we have

Proposition 2. $\chi \sim \chi' \Leftrightarrow \lambda_\chi \sim \lambda_{\chi'}$, *where* $\lambda_\chi := \lambda(\chi, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, 1, 1, 1)$

Proof. Assume that $\chi' = \sigma\chi$, $\sigma \in S_4$. Denote $\boldsymbol{e}_4 := \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$. We have $\boldsymbol{e}_{\sigma(4)} = \varepsilon_1\boldsymbol{e}_{\sigma(1)} + \varepsilon_2\boldsymbol{e}_{\sigma(2)} + \varepsilon_3\boldsymbol{e}_{\sigma(3)}$ for some $\varepsilon_i = \pm 1$. Then $\lambda_{\chi'} = \lambda(\chi', \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, 1, 1, 1) = \lambda(\chi, \boldsymbol{e}_{\sigma(1)}, \boldsymbol{e}_{\sigma(2)}, \boldsymbol{e}_{\sigma(3)}, \varepsilon_1, \varepsilon_2, \varepsilon_3) \sim \lambda_\chi$, where the equivalence follows from Proposition 1.

Conversely, assume $\lambda_\chi \sim \lambda_{\chi'}$. By the definition of characteristic functions, we have $\lambda_{\chi'} = \lambda(\chi, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ for some basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and $\varepsilon_i = \pm 1$. The image of $\lambda_{\chi'}$ is the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\}$, while the image of $\lambda(\chi, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the set $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \varepsilon_1\boldsymbol{a}_1 + \varepsilon_2\boldsymbol{a}_2 + \varepsilon_3\boldsymbol{a}_3\}$. Therefore, these two sets of 4 vectors coincide, that is, $\boldsymbol{a}_1 = \boldsymbol{e}_{\sigma(1)}$, $\boldsymbol{a}_2 = \boldsymbol{e}_{\sigma(2)}$, $\boldsymbol{a}_3 = \boldsymbol{e}_{\sigma(3)}$ and $\varepsilon_1\boldsymbol{a}_1 + \varepsilon_2\boldsymbol{a}_2 + \varepsilon_3\boldsymbol{a}_3 = \boldsymbol{e}_{\sigma(4)}$ for some $\sigma \in S_4$. Thus, $\chi' = \sigma\chi$ and $\chi \sim \chi'$. □

Theorem 1 ([6])**.** *Let $M = M(P, \lambda)$ and $M' = M'(P', \lambda')$ be quasitoric manifolds (or small covers), an assume that P belongs to the class $\mathcal{P}$. Then M and M' are diffeomorphic if and only if the characteristic pairs (P, λ) and (P', λ') are equivalent.*

Theorem 2 (main result)**.** *Assume given a polytope $P \in \mathcal{P}$ with a 4-colouring χ, and let P' be another simple 3-polytope with a 4-colouring χ'. Then the 6-dimensional quasitoric manifolds (or 3-dimensional hyperbolic manifolds of Löbell type) $M = M(P, \lambda_\chi)$ and $M' = M'(P', \lambda_{\chi'})$ are diffeomorphic if and only if $P \simeq P'$ and $\chi \sim \chi'$.*

Proof. If $P \simeq P'$ and $\chi \sim \chi'$, then $\lambda_\chi \sim \lambda_{\chi'}$, by Proposition 2. Then the pairs (P, λ_χ) and $(P', \lambda_{\chi'})$ are equivalent, so M and M' are diffeomorphic.

Conversely, if M and M' are diffeomorphic, then $P \simeq P'$ and $\lambda_\chi \sim \lambda_{\chi'}$, by Theorem 1. Therefore, $\chi \sim \chi'$, by Proposition 2. □

We say that a characteristic function $\lambda\colon \mathcal{F} \to \mathbb{Z}^3$ *is defined by a 4-colouring* χ if $\lambda(F) = \lambda(F')$ whenever $\chi(F) = \chi(F')$. The image of such a characteristic function consists of 4 vectors. Examples are $\lambda(\chi, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ and λ_χ. A regular 4-colouring of a simple 3-polytope is *complete* if for any set of three different colours there exists a vertex whose incident facets have these colours.

Proposition 3. *Let χ be a complete 4-colouring. Then any characteristic function $\lambda\colon \mathcal{F} \to \mathbb{Z}^3$ defined by χ is equivalent to λ_χ.*

Note that it is necessary that the 4-colouring is complete. For example, assume that there is no vertex with the combination of colours $\{1, 2, 4\}$ for the incident facets. Then for each $k \in \mathbb{Z}$ consider the characteristic function $\lambda_{\chi,k}$ given by

$$\lambda_{\chi,k}(F) = \begin{cases} \boldsymbol{e}_i & \text{if } \chi(F) = i, \quad i = 1, 2, 3, \\ \boldsymbol{e}_1 + \boldsymbol{e}_2 + k\boldsymbol{e}_3 & \text{if } \chi(F) = 4. \end{cases}$$

Then $\lambda_{\chi,k} \not\sim \lambda_\chi$ for $k \neq \pm 1$ (and $\lambda_{\chi,0} \not\sim \lambda_{\chi,1} = \lambda_\chi$ over $\mathbb{Z}_2$).

The class $\mathcal{P}$ contains all *fullerenes*, that is, simple 3-polytopes with only pentagonal and hexagonal facets [7]. If a fullerene has two adjacent pentagons, then all four combinations of three colours are realised in the vertices of these two pentagons, so any 4-colouring of such a fullerene is complete. For fullerenes without adjacent pentagons (*IPR-fullerenes*) there exist non-complete 4-colourings χ. It is easy to see that the corresponding hyperbolic 3-manifolds $M(P, \lambda_{\chi,0})$ are non-orientable, unlike $M(P, \lambda_\chi)$.

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