Torus actions and their applications in topology and combinatorics

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ABSTRACT. Here, the study of torus actions on topological spaces is presented as a bridge connecting combinatorial and convex geometry with commutative and homological algebra, algebraic geometry, and topology. This link helps in understanding the geometry and topology of a space with torus action by studying the combinatorics of the space of orbits. Conversely, the most subtle properties of a combinatorial object can be recovered by realizing it as the orbit structure for a proper manifold or complex acted on by a torus. The latter can be a symplectic manifold with Hamiltonian torus action, a toric variety or manifold, a subspace arrangement complement, etc., while the combinatorial objects include simplicial and cubical complexes, polytopes, and arrangements. This approach also provides a natural topological interpretation in terms of torus actions of many constructions from commutative and homological algebra used in combinatorics.

The exposition centers around the theory of moment-angle complexes, providing an effective way to study triangulations by methods of equivariant topology. The book includes many new and well-known open problems and would be suitable as a textbook. We hope that it will be useful for specialists both in topology and in combinatorics and will help to establish even tighter connections between the subjects involved.

Contents

Introduction

Torus actions on topological spaces is classical and one of the most developed fields in equivariant topology. Specific problems connected with torus actions arise in different areas of mathematics and mathematical physics, which results in permanent interest in the theory, constant source of new applications and penetration of new ideas in topology. Many volumes devoted to particular aspects of this wide field of mathematical knowledge are available. The topological approach is the subject of monograph [**26**] by G. Bredon. Monograph [**9**] by M. Audin deals with torus actions from the symplectic geometry viewpoint. The algebro-geometrical part of the study, known as the geometry of toric varieties or simply "toric geometry", is presented in several texts. These include V. Danilov's original survey article [**46**] and more recent monographs by T. Oda [**105**], W. Fulton [**64**] and G. Ewald [**61**].

The orbit space of a torus action carries a rich combinatorial structure. In many cases studying the combinatorics of the quotient is the easiest and the most efficient way to understand the topology of a toric space. This approach works in the opposite direction as well: the equivariant topology of a torus action sometimes helps to interpret and prove the most subtle combinatorial results topologically. In the most symmetric and regular cases (such as projective toric varieties or Hamiltonian torus actions on symplectic manifolds) the quotient can be identified with a convex polytope. More general toric spaces give rise to other combinatorial structures related with their quotients. Examples here include simplicial spheres, triangulated manifolds, general simplicial complexes, cubical complexes, subspace arrangements, etc.

Combined applications of combinatorial, topological and algebro-geometrical methods stimulated intense development of toric geometry during the last three decades. This remarkable confluence of ideas enriched all the subjects involved with a number of spectacular results. Another source of applications of topological and algebraic methods in combinatorics was provided by the theory of Stanley–Reisner face rings and Cohen–Macaulay complexes, described in R. Stanley's monograph [**128**]. Our motivation was to broaden the existing bridge between torus actions and combinatorics by giving some new constructions of toric spaces, which naturally arise from combinatorial considerations. We also interpret many existing results in such a way that their relationships with combinatorics become more transparent. Traditionally, simplicial complexes, or triangulations, were used in topology as a tool for combinatorial treatment of topological invariants of spaces or manifolds. On the other hand, triangulations themselves can be regarded as particular structures, so the space of triangulations becomes the object of study. The idea of considering the space of triangulations of a given manifold has been also motivated by some physical problems. One gets an effective way of treatment of combinatorial results and problems concerning the number of faces in a triangulation by interpreting them as extremal value problems on the space of triangulations. We implement some of these ideas in our book as well, by constructing and investigating invariants of triangulations using the equivariant topology of toric spaces.

The book is intended to be a systematic but elementary overview for the aspects of torus actions mostly related to combinatorics. However, our level of exposition is not balanced between topology and combinatorics. We do not assume any particular reader's knowledge in combinatorics, but in topology a basic knowledge of characteristic classes and spectral sequences techniques may be very helpful in the last chapters. All necessary information is contained, for instance, in S. Novikov's book [**104**]. We would recommend this book since it is reasonably concise, has a rather broad scope and pays much attention to the combinatorial aspects of topology. Nevertheless, we tried to provide necessary background material in the algebraic topology and hope that our book will be of interest to combinatorialists as well.

A significant part of the text is devoted to the theory of moment-angle complexes, currently being developed by the authors. This study was inspired by paper [**48**] of M. Davis and T. Januszkiewicz, where a topological analogue of toric varieties was introduced. In their work, Davis and Januszkiewicz used a certain universal T^m-space $\mathcal{Z}_K$, assigned to every simplicial complex on the vertex set $[m] = \{1, \dots, m\}$. In its turn, the definition of $\mathcal{Z}_K$ was motivated (see [**47**, §13]) by the construction of the Coxeter complex of a Coxeter group and its generalizations by E. Vinberg [**137**].

Our approach brings the space $\mathcal{Z}_K$ to the center of attention. To each subset $\sigma \subset [m]$ there is assigned a canonical T^m-equivariant embedding $(D^2)^k \times T^{m-k} \subset (D^2)^m$, where $(D^2)^m$ is the standard poly-disc in $\mathbb{C}^m$ and k is the cardinality of σ. This correspondence extends to any simplicial complex K on $[m]$ and produces a canonical *bi*graded cell decomposition of the Davis–Januszkiewicz T^m-space $\mathcal{Z}_K$, which we refer to as the *moment-angle complex*. There is also a more general version of moment-angle complexes, defined for any cubical subcomplex in a unit cube (see section 4.2). The construction of $\mathcal{Z}_K$ gives rise to a functor (see Proposition 7.12) from the category of simplicial complexes and inclusions to the category of T^m-spaces and equivariant maps. This functor induces a homomorphism between the standard simplicial chain complex of a simplicial pair (K_1, K_2) and the bigraded cellular chain complex of $(\mathcal{Z}_{K_1}, \mathcal{Z}_{K_2})$. The remarkable property of the functor is that it takes a simplicial Lefschetz pair (K_1, K_2) (i.e. a pair such that $K_1 \setminus K_2$ is an open manifold) to another Lefschetz pair (of moment-angle complexes) in such a way that the fundamental cycle is mapped to the fundamental cycle. For instance, if K is a triangulated manifold, then the simplicial pair $(K, \varnothing)$ is mapped to the pair $(\mathcal{Z}_K, \mathcal{Z}_\varnothing)$, where $\mathcal{Z}_\varnothing \cong T^m$ and $\mathcal{Z}_K \setminus \mathcal{Z}_\varnothing$ is an open manifold. Studying the functor $K \mapsto \mathcal{Z}_K$, one interprets the combinatorics of simplicial complexes in terms of the bigraded cohomology rings of moment-angle complexes. In the case when K is a triangulated manifold, the important additional information is provided by the bigraded Poincaré duality for the Lefschetz pair $(\mathcal{Z}_K, \mathcal{Z}_\varnothing)$. For instance, the duality implies the generalized Dehn–Sommerville equations for the numbers of k-simplices in a triangulated manifold.

Each chapter and most sections of the book refer to a separate subject and contain necessary introductory remarks. Below we schematically overview the contents. The chapter dependence chart is shown in Figure 0.1.

$$\begin{tikzcd}
 & & 3 \arrow[dr] & & & \\
1 \arrow[r] \arrow[drr] & 2 \arrow[ur] \arrow[r] & 4 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \\
 & & 5 \arrow[ur] & & &
\end{tikzcd}$$

FIGURE 0.1. Chapter dependence scheme.

Chapter 1 contains combinatorial and geometrical background material on convex polytopes. Since a lot of literature is available on this subject (see e.g. recent excellent lectures [**145**] by G. Ziegler), we just give a short overview of constructions used in the book. Although most of these constructions descend from the convex geometry, we tried to emphasize their combinatorial properties. Section 1.1 contains two classical definitions of convex polytopes, examples, the notions of simple and simplicial polytopes, and the construction of connected sum of simple polytopes. In section 1.2 we introduce the f- and the h-vector of a polytope and give a "Morse-theoretical" proof of the Dehn–Sommerville equations. Section 1.3 is devoted to the g-theorem, and in section 1.4 we discuss the Upper Bound and the Lower Bound for the number of faces of a simple (or simplicial) polytope. In section 1.5 we introduce the Stanley–Reisner ring of a simple polytope.

Simplicial complexes appear in the full generality in Chapter 2. In section 2.1 we define abstract and geometrical simplicial complexes (polyhedrons). In section 2.2 we introduce some standard notions from PL-topology and describe basic constructions of simplicial complexes (joins, connected sums etc.). We also discuss the Alexander duality and its simplicial version here. From the early days of topology, triangulations of nice topological spaces such as manifolds or spheres were of particular interest. Triangulations of spheres, or "simplicial spheres", are the subject of section 2.3. Here we also discuss the inter-relations between some particular subclasses of simplicial spheres (such as PL-spheres, polytopal spheres etc.) and one famous combinatorial problem, the so-called g-conjecture for face vectors. Triangulated (or simplicial) manifolds are the subject of section 2.4; some related open problems from low-dimensional and PL topology are also included there. The notion of bistellar moves, as a particularly interesting and useful class of operations on simplicial complexes, is discussed in section 2.5.

In chapter 3 we give an overview of commutative algebra involved in the combinatorics of simplicial complexes. Many of the constructions from this chapter, especially those appearing in the beginning, are taken from Stanley's monograph [**128**]; however, we tried to emphasize their functorial properties and relationships with operations from chapter 2. The Stanley–Reisner face ring of simplicial complex is introduced in section 3.1. The important class of Cohen–Macaulay complexes is the subject of section 3.2; we also give Stanley's argument for the Upper Bound theorem for spheres here. Section 3.3 contains the homological algebra background, including resolutions and the graded functor Tor. Koszul complexes and Tor-algebras associated with simplicial complexes are described in section 3.4 together with their basic properties. Gorenstein algebras and Gorenstein* complexes are the subject of section 3.5. This class of "self-dual" Cohen–Macaulay complexes contains simplicial and homology spheres and, in a sense, provides the best possible algebraic approximation to them. The chapter ends up with a discussion of some generalizations of the Dehn–Sommerville equations.

Cubical complexes are the subject of chapter 4. We give definitions and discuss some interesting related problems from the discrete geometry in section 4.1. Section 4.2 introduces some particular cubical complexes necessary for the construction of moment-angle complexes. These include the cubical subdivisions of simple polytopes and simplicial complexes.

Different aspects of torus actions is the main theme of the second part of the book. Chapter 5 starts with a brief review of the algebraic geometry of toric varieties in section 5.1. We stress upon those features of toric varieties which can be taken as a starting point for their subsequent topological generalizations. We also give Stanley's famous argument for the necessity part of the g-theorem, one of the first and most known applications of the algebraic geometry in the combinatorics of polytopes. In section 5.2 we give the definition and basic properties of quasitoric manifolds, the notion introduced by Davis and Januszkiewicz (under the name "toric manifolds") as a topological generalization of toric varieties. The topology of quasitoric manifolds is the subject of sections 5.3 and 5.4 (this includes the discussion of their cohomology, cobordisms, characteristic classes, Hirzebruch genera, etc.). Quasitoric manifolds work particularly well in the cobordism theory and may serve as a convenient framework for different cobordism calculations. Evidences for this are provided by some recent results of V. Buchstaber and N. Ray. It is proved that a certain class of quasitoric manifolds provides an alternative additive basis for the complex cobordism ring. (Note that the standard basis consists of Milnor hypersurfaces, which are not quasitoric.) Moreover, using the combinatorial construction of connected sum of polytopes, it is proved that each complex cobordism class contains a quasitoric manifold with a canonical stably almost complex structure respected by the torus action. Since quasitoric manifolds are necessarily connected, the nature of this result resembles the famous Hirzebruch problem about connected algebraic representatives in complex cobordisms. All these arguments, presented in section 5.3, open the way to evaluation of global cobordism invariants on manifolds by choosing a quasitoric representative and studying the local invariants of the action. As an application, in section 5.4 we give combinatorial formulae, due to the second author, for Hirzebruch genera of quasitoric manifolds. Section 5.5 is a discussion of several known results on the classification of toric and quasitoric manifolds over a given simple polytope.

The theory of moment-angle complexes is the subject of chapters 6 and 7. We start in section 6.1 with the definition of the moment-angle manifold $\mathcal{Z}_P$ corresponding to a simple polytope P. The general moment-angle complexes $\mathcal{Z}_K$ are introduced in section 6.2, using special cubical subdivisions from section 4.2. Here we prove that $\mathcal{Z}_K$ is a manifold provided that K is a simplicial sphere. Two types of bigraded cell decompositions of moment-angle complexes are introduced in section 6.3. In section 6.4 we discuss different functorial properties of moment-angle complexes with respect to simplicial maps and constructions from section 2.2. A basic homotopy theory of moment-angle complexes is the subject of section 6.5. Concluding section 6.6 aims for a more broad view on the constructions of quasitoric manifolds and moment-angle complexes. We discuss different inter-relations, similar constructions and possible generalizations there.

The cohomology of moment-angle complexes, and its rôle in investigating combinatorial invariants of triangulations, is studied in chapter 7. In section 7.1 we review the Eilenberg–Moore spectral sequence, our main computational tool. The bigraded cellular structure and the Eilenberg–Moore spectral sequence are the main ingredients in the calculation of cohomology of a general moment-angle complex $\mathcal{Z}_K$, carried out in section 7.2. Additional results on the cohomology in the case when K is a simplicial sphere are given in section 7.4. These calculations reveal some new connections with well-known constructions from homological algebra and open the way to some further combinatorial applications. In particular, the cohomology of the Koszul complex for a Stanley–Reisner ring and its Betti numbers now get a topological interpretation. In section 7.5 we study the quotients of moment-angle manifolds $\mathcal{Z}_P$ by subtori $H \subset T^m$ of rank $< m$. Quasitoric manifolds arise in this scheme as quotients for freely acting subtori of the maximal possible rank. Moment-angle complexes corresponding to triangulated manifolds are considered in section 7.6. In this situation all singular points of $\mathcal{Z}_K$ form a single orbit of the torus action, and the complement of an equivariant neighborhood of this orbit is a manifold with boundary.

In chapter 8 we apply the theory of moment-angle complexes to the topology of subspace arrangement complements. Section 8.1 is a brief review of general arrangements. Then we restrict to the cases of coordinate subspace arrangements and diagonal subspace arrangements (sections 8.2 and 8.3 respectively). In particular, we calculate the cohomology ring of the complement of a coordinate subspace arrangement by reducing it to the cohomology of a moment-angle complex. This also reveals some remarkable connections between certain results from commutative algebra of monomial ideals (such as the famous Hochster's theorem) and topological results on subspace arrangements (e.g. the Goresky–Macpherson formula for the cohomology of complement). In the diagonal subspace arrangement case the cohomology of complement is included as a canonical subspace into the cohomology of the loop space on a certain moment angle complex $\mathcal{Z}_K$.

Almost all new concepts in our book are accompanied with explanatory examples. We also give many examples of particular computations, illustrating general theorems. Throughout the text the reader will encounter a number of open problems. Some of these problems and conjectures are widely known, while others are new. In most cases we tried to give a topological interpretation for the question under consideration, which might provide an alternative approach to its solution.

Many of those results in the book which are due to the authors have already appeared in their papers [**30**]–[**34**], [**111**], [**112**], or papers [**37**], [**38**] by N. Ray and the first author. We sometimes omit the corresponding quotations in the text. The whole book has grown up from our survey article [**35**].

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CHAPTER 1

Polytopes

1.1. Definitions and main constructions

Both combinatorial and geometrical aspects of the theory of convex polytopes are exposed in a vast number of textbooks, monographs and papers. Among them are the classical monograph [**69**] by Grünbaum and more recent Ziegler's lectures [**145**]. Face vectors and other combinatorial questions are discussed in books by McMullen–Shephard [**99**], Brønsted [**29**], Yemelichev–Kovalev–Kravtsov [**141**] and survey article [**87**] by Klee and Kleinschmidt. These sources contain a host of further references. In this section we review some basic concepts and constructions used in the rest of the book.

There are two algorithmically different ways to define a convex polytope in n-dimensional affine Euclidean space $\mathbb{R}^n$.

Definition 1.1. A *convex polytope* is the convex hull of a finite set of points in some $\mathbb{R}^n$.

Definition 1.2. A *convex polyhedron* P is an intersection of finitely many half-spaces in some $\mathbb{R}^n$:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \ : \ \langle \boldsymbol{l}_i, \boldsymbol{x} \rangle \geqslant -a_i, \ i = 1, \ldots, m\}, \tag{1.1}$$

where $\boldsymbol{l}_i \in (\mathbb{R}^n)^*$ are some linear functions and $a_i \in \mathbb{R}$, $i = 1, \ldots, m$. A *(convex) polytope* is a bounded convex polyhedron.

Nevertheless, the above two definitions produce the same geometrical object, i.e. the subset of $\mathbb{R}^n$ is a convex hull of a finite point set if and only if it is a bounded intersection of finitely many half-spaces. This classical fact is proved in many textbooks on polytopes and convex geometry, see e.g. [**145**, Theorem 1.1].

Definition 1.3. The *dimension* of a polytope is the dimension of its affine hull. Unless otherwise stated we assume that any n-dimensional polytope, or simply *n-polytope*, P^n is a subset in n-dimensional ambient space $\mathbb{R}^n$. A *supporting hyperplane* of P^n is an affine hyperplane H which intersects P^n and for which the polytope is contained in one of the two closed half-spaces determined by the hyperplane. The intersection $P^n \cap H$ is then called a *face* of the polytope. We also regard the polytope P^n itself as a face; other faces are called *proper faces*. The *boundary* ∂P^n is the union of all proper faces of P^n. Each face of an n-polytope is itself a polytope of dimension $\leqslant n$. 0-dimensional faces are called *vertices*, 1-dimensional faces are *edges*, and codimension one faces are *facets*.

Two polytopes $P_1 \subset \mathbb{R}^{n_1}$ and $P_2 \subset \mathbb{R}^{n_2}$ of the same dimension are said to be *affinely equivalent* (or *affinely isomorphic*) if there is an affine map $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ that is a bijection between the points of the two polytopes. Two polytopes are *combinatorially equivalent* if there is a bijection between their sets of faces that preserves the inclusion relation.

Note that two affinely isomorphic polytopes are combinatorially equivalent, but the opposite is not true. A more consistent definition of combinatorial equivalence uses the combinatorial notions of poset and lattice.

DEFINITION 1.4. A *poset* (or finite partially ordered set) $(\mathcal{S}, \leqslant)$ is a finite set $\mathcal{S}$ equipped with a relation "$\leqslant$" which is reflexive ($x \leqslant x$ for all $x \in \mathcal{S}$), transitive ($x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$), and antisymmetric ($x \leqslant y$ and $y \leqslant x$ imply $x = y$). When the partial order is clear we denote the poset by just $\mathcal{S}$. A *chain* in $\mathcal{S}$ is a totally ordered subset of S.

DEFINITION 1.5. The faces of a polytope P of all dimensions form a poset with respect to inclusion, called the *face poset*.

Now we observe that two polytopes are combinatorially equivalent if and only if their face posets are isomorphic. More information about face posets of polytopes can be found in [**145**, §2.2].

DEFINITION 1.6. A *combinatorial polytope* is a class of combinatorial equivalent convex (or *geometrical*) polytopes. Equivalently, a combinatorial polytope is the face poset of a geometrical polytope.

AGREEMENT. Suppose that a polytope P^n is represented as an intersection of half-spaces as in (1.1). In the sequel we assume that there are no redundant inequalities $\langle \boldsymbol{l}_i, \boldsymbol{x} \rangle \geqslant -a_i$ in such a representation. That is, no inequality can be removed from (1.1) without changing the polytope P^n. In this case P^n has exactly m facets which are the intersections of hyperplanes $\langle \boldsymbol{l}_i, \boldsymbol{x} \rangle = -a_i$, $i = 1, \dots, m$, with P^n. The vector $\boldsymbol{l}_i$ is orthogonal to the corresponding facet and points towards the interior of the polytope.

EXAMPLE 1.7 (simplex and cube). An n-dimensional *simplex* Δ^n is the convex hull of $(n+1)$ points in $\mathbb{R}^n$ that do not lie on a common affine hyperplane. All faces of an n-simplex are simplices of dimension $\leqslant n$. Any two n-simplices are affinely equivalent. The *standard* n-simplex is the convex hull of points $(1, 0, \dots, 0)$, $(0, 1, \dots, 0), \dots, (0, \dots, 0, 1)$, and $(0, \dots, 0)$ in $\mathbb{R}^n$. Alternatively, the standard n-simplex is defined by $(n+1)$ inequalities

$$x_i \geqslant 0,\ i = 1, \dots, n, \quad \text{and} \quad -x_1 - \ldots - x_n \geqslant -1. \tag{1.2}$$

The *regular* n-simplex is the convex hull of $n+1$ points $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$ in $\mathbb{R}^{n+1}$.

The *standard q-cube* is the convex polytope $I^q \subset \mathbb{R}^q$ defined by

$$I^q = \{(y_1, \dots, y_q) \in \mathbb{R}^q \ :\ 0 \leqslant y_i \leqslant 1,\ i = 1, \dots, q\}. \tag{1.3}$$

Alternatively, the standard q-cube is the convex hull of the 2^q points in $\mathbb{R}^q$ that have only zero or unit coordinates.

The following construction shows that any convex n-polytope with m facets is affinely equivalent to the intersection of the *positive cone*

$$\mathbb{R}^m_+ = \bigl\{(y_1, \dots, y_m) \in \mathbb{R}^m \ :\ y_i \geqslant 0,\ i = 1, \dots, m\bigr\} \subset \mathbb{R}^m \tag{1.4}$$

with a certain n-dimensional plane.

CONSTRUCTION 1.8. Let $P \subset \mathbb{R}^n$ be a convex n-polytope given by (1.1) with some $\boldsymbol{l}_i \in (\mathbb{R}^n)^*$, $a_i \in \mathbb{R}$, $i = 1, \dots, m$. Form the $n \times m$-matrix L whose columns are the vectors $\boldsymbol{l}_i$ written in the standard basis of $(\mathbb{R}^n)^*$, i.e. $L_{ji} = (\boldsymbol{l}_i)_j$. Note that

L is of rank n. Likewise, let $\boldsymbol{a} = (a_1, \dots, a_m)^t \in \mathbb{R}^m$ be the column vector with entries a_i. Then we can rewrite (1.1) as

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \; : \; (L^t \boldsymbol{x} + \boldsymbol{a})_i \geqslant 0, \; i = 1, \dots, m\}, \tag{1.5}$$

where L^t is the transposed matrix and $\boldsymbol{x} = (x_1, \dots, x_n)^t$ is the column vector. Consider the affine map

$$A_P : \mathbb{R}^n \to \mathbb{R}^m, \quad A_P(\boldsymbol{x}) = L^t \boldsymbol{x} + \boldsymbol{a} \in \mathbb{R}^m. \tag{1.6}$$

Its image is an n-dimensional plane in $\mathbb{R}^m$, and $A_P(P)$ is the intersection of this plane with the positive cone $\mathbb{R}^m_+$, see (1.5). Let W be an $m \times (m-n)$-matrix whose columns form a basis of linear dependencies between the vectors $\boldsymbol{l}_i$. That is, W is a rank $(m-n)$ matrix satisfying $LW = 0$. Then it is easy to see that

$$A_P(P) = \{\boldsymbol{y} \in \mathbb{R}^m \; : \; W^t \boldsymbol{y} = W^t \boldsymbol{a}, \; y_i \geqslant 0, \quad i = 1, \dots, m\}.$$

By definition, the polytopes P and $A_P(P)$ are affinely equivalent.

EXAMPLE 1.9. Consider the standard n-simplex $\Delta^n \subset \mathbb{R}^n$ defined by inequalities (1.2). It has $m = n+1$ facets and is given by (1.1) with $\boldsymbol{l}_1 = (1, 0, \dots, 0)^t$, $\dots$, $\boldsymbol{l}_n = (0, \dots, 0, 1)^t$, $\boldsymbol{l}_{n+1} = (-1, \dots, -1)^t$, $a_1 = \dots = a_n = 0$, $a_{n+1} = 1$. One can take $W = (1, \dots, 1)^t$ in Construction 1.8. Hence, $W^t \boldsymbol{y} = y_1 + \dots + y_m$, $W^t \boldsymbol{a} = 1$, and we have

$$A_{\Delta^n}(\Delta^n) = \{\boldsymbol{y} \in \mathbb{R}^{n+1} \; : \; y_1 + \dots + y_{n+1} = 1, \; y_i \geqslant 0, \quad i = 1, \dots, n\}.$$

This is the regular n-simplex in $\mathbb{R}^{n+1}$.

The notion of *generic* polytope depends on the choice of definition of convex polytope. Below we describe the two possibilities.

A set of $m > n$ points in $\mathbb{R}^n$ is in *general position* if no $(n+1)$ of them lie on a common affine hyperplane. Now Definition 1.1 implies that a convex polytope is generic if it is the convex hull of a set of general positioned points. This implies that all proper faces of the polytope are simplices, i.e. every facet has the minimal number of vertices (namely, n). Such polytopes are called *simplicial*.

On the other hand, a set of $m > n$ hyperplanes $\langle \boldsymbol{l}_i, \boldsymbol{x} \rangle = -a_i$, $\boldsymbol{l}_i \in (\mathbb{R}^n)^*$, $\boldsymbol{x} \in \mathbb{R}^n$, $a_i \in \mathbb{R}$, $i = 1, \dots, m$, is in *general position* if no point belongs to more than n hyperplanes. From the viewpoint of Definition 1.2, a convex polytope P^n is generic if its bounding hyperplanes (see (1.1)) are in general position. That is, there are exactly n facets meeting at each vertex of P^n. Such polytopes are called *simple*. Note that each face of a simple polytope is again a simple polytope.

DEFINITION 1.10. For any convex polytope $P \subset \mathbb{R}^n$ define its *polar set* $P^* \subset (\mathbb{R}^n)^*$ by

$$P^* = \{\boldsymbol{x}' \in (\mathbb{R}^n)^* \; : \; \langle \boldsymbol{x}', \boldsymbol{x} \rangle \geqslant -1 \text{ for all } \boldsymbol{x} \in P\}.$$

REMARK. We adopt the definition of the polar set from the algebraic geometry of toric varieties, not the classical one from the convex geometry. The latter is obtained by replacing the inequality "$\geqslant -1$" above by "$\leqslant 1$". Obviously, the toric geometers polar set is taken into the convex geometers one by the central symmetry with respect to 0.

It is well known in convex geometry that the polar set P^* is convex in the dual space $(\mathbb{R}^n)^*$ and 0 is contained in the interior of P^*. Moreover, if P itself contains 0 in its interior then P^* is a convex polytope (i.e. is bounded) and $(P^*)^* = P$, see e.g. [**145**, §2.3]. The polytope P^* is called the *polar* (or *dual*) of P. There is a one-to-one order reversing correspondence between the face posets of P and P^*. In other words, the face poset of P^* is the opposite of the face poset of P. In particular, if P is simple then P^* is simplicial, and vice versa.

EXAMPLE 1.11. Any polygon (2-polytope) is simple and simplicial at the same time. In dimensions $\geqslant 3$ the simplex is the only polytope that is simultaneously simple and simplicial. The cube is a simple polytope. The polar of simplex is again the simplex. The polar of cube is called the *cross-polytope*. The 3-dimensional cross-polytope is known as the octahedron.

CONSTRUCTION 1.12 (Product of simple polytopes). The *product* $P_1 \times P_2$ of two simple polytopes P_1 and P_2 is a simple polytope as well. The dual operation on simplicial polytopes can be described as follows. Let $S_1 \subset \mathbb{R}^{n_1}$ and $S_2 \subset \mathbb{R}^{n_2}$ be two simplicial polytopes. Suppose that both S_1 and S_2 contain 0 in their interiors. Now define

$$S_1 \circ S_2 := \operatorname{conv}\bigl(S_1 \times 0 \cup 0 \times S_2\bigr) \subset \mathbb{R}^{n_1+n_2}$$

(here conv means the convex hull). It is easy to see that $S_1 \circ S_2$ is a simplicial polytope, and for any two simple polytopes P_1, P_2 containing 0 in their interiors the following holds:

$$P_1^* \circ P_2^* = (P_1 \times P_2)^*.$$

Obviously, both product and $\circ$ operations are also defined on combinatorial polytopes; in this case the above formula holds without any restrictions.

CONSTRUCTION 1.13 (Connected sum of simple polytopes). Suppose we are given two simple polytopes P^n and Q^n, both of dimension n, with distinguished vertices v and w respectively. The informal way to get the *connected sum* $P^n \#_{v,w} Q^n$ of P^n at v and Q^n at w is as follows. We "cut off" v from P^n and w from Q^n; then, after a projective transformation, we can "glue" the rest of P^n to the rest of Q^n along the new simplex facets to obtain $P^n \#_{v,w} Q^n$. Below we give the formal definition, following [**38**, §6].

First, we introduce an n-polyhedron Γ^n, which will be used as a template for the construction; it arises by considering the standard $(n-1)$-simplex Δ^{n-1} in the subspace $\{x : x_1 = 0\}$ of $\mathbb{R}^n$, and taking its cartesian product with the first coordinate axis. The facets G_r of Γ^n therefore have the form $\mathbb{R} \times D_r$, where D_r, $1 \le r \le n$, are the facets of Δ^{n-1}. Both Γ^n and the G_r are divided into positive and negative halves, determined by the sign of the coordinate x_1.

We order the facets of P^n meeting in v as $E_1, \ldots, E_n$, and the facets of Q^n meeting in w as $F_1, \ldots, F_n$. Denote the complementary sets of facets by $\mathcal{C}_v$ and $\mathcal{C}_w$; those in $\mathcal{C}_v$ avoid v, and those in $\mathcal{C}_w$ avoid w.

We now choose projective transformations ϕ_P and ϕ_Q of $\mathbb{R}^n$, whose purpose is to map v and w to $x_1 = \pm\infty$ respectively. We insist that ϕ_P embeds P^n in Γ^n so as to satisfy two conditions; firstly, that the hyperplane defining E_r is identified with the hyperplane defining G_r, for each $1 \le r \le n$, and secondly, that the images of the hyperplanes defining $\mathcal{C}_v$ meet Γ^n in its negative half. Similarly, ϕ_Q identifies the hyperplane defining F_r with that defining G_r, for each $1 \le r \le n$, but the images of the hyperplanes defining $\mathcal{C}_w$ meet Γ^n in its positive half. We define the *connected sum* $P^n \#_{v,w} Q^n$ of P^n at v and Q^n at w to be the simple convex n-polytope determined by the images of the hyperplanes defining $\mathcal{C}_v$ and $\mathcal{C}_w$ and hyperplanes defining G_r, $r = 1, \ldots, n$. It is defined only up to combinatorial equivalence;

moreover, different choices for either of v and w, or either of the orderings for E_r and F_r, are likely to affect the combinatorial type. When the choices are clear, or their effect on the result irrelevant, we use the abbreviation $P^n \# Q^n$.

The related construction of connected sum $P \# S$ of a simple polytope P and a *simplicial* polytope S is described in [**145**, Example 8.41].

EXAMPLE 1.14. 1. If P^2 is an m_1-gon and Q^2 is an m_2-gon then $P \# Q$ is an $(m_1 + m_2 - 2)$-gon.

2. If both P and Q are n-simplices then $P \# Q = \Delta^{n-1} \times I^1$ (the product of $(n-1)$-simplex and segment). In particular, for $n = 3$ we get a triangular prism.

3. More generally, if P is an n-simplex then $P \#_{v,w} Q$ is the result of "cutting" the vertex w from Q by a hyperplane that isolate w from other vertices. For more relations between connected sums and hyperplane cuts see [**38**, §6].

DEFINITION 1.15. A simplicial polytope S is called *k-neighborly* if any k vertices span a face of S. Likewise, a simple polytope P is called k-neighborly if any k facets of P have non-empty intersection (i.e. share a common codimension-k face). Obviously, every simplicial (or simple) polytope is 1-neighborly. It can be shown ([**29**, Corollary 14.5], see also Example 1.24 below) that if S is a k-neighborly simplicial n-polytope and $k > \left[\frac{n}{2}\right]$, then S is an n-simplex. This implies that any 2-neighborly simplicial 3-polytope is a simplex. However, there exist simplicial n-polytopes with an arbitrary number of vertices which are $\left[\frac{n}{2}\right]$-neighborly. Such polytopes are called *neighborly*. In particular, there is a simplicial 4-polytope (different from the 4-simplex) any two vertices of which are connected by an edge.

EXAMPLE 1.16 (neighborly 4-polytope). Let $P = \Delta^2 \times \Delta^2$ be the product of two triangles. Then P is a simple polytope, and it is easy to see that any two facets of P share a common 2-face. Hence, P is 2-neighborly. The polar P^* is a neighborly simplicial 4-polytope.

More generally, if a simple polytope P_1 is k_1-neighborly and a simple polytope P_2 is k_2-neighborly, then the product $P_1 \times P_2$ is a $\min(k_1, k_2)$-neighborly simple polytope. It follows that $(\Delta^n \times \Delta^n)^*$ and $(\Delta^n \times \Delta^{n+1})^*$ provide examples of neighborly simplicial $2n$- and $(2n+1)$-polytopes. The following example gives a neighborly polytope with an arbitrary number of vertices.

EXAMPLE 1.17 (cyclic polytopes). Define the *moment curve* in $\mathbb{R}^n$ by

$$\boldsymbol{x} : \mathbb{R} \longrightarrow \mathbb{R}^n, \qquad t \mapsto \boldsymbol{x}(t) = (t, t^2, \ldots, t^n) \in \mathbb{R}^n.$$

For any $m > n$ define the *cyclic polytope* $C^n(t_1, \ldots, t_m)$ as the convex hull of m distinct points $\boldsymbol{x}(t_i)$, $t_1 < t_2 < \ldots < t_m$, on the moment curve. It follows from the Vandermonde determinant identity that no $(n+1)$ points on the moment curve belong to a common affine hyperplane. Hence, $C^n(t_1, \ldots, t_m)$ is a simplicial n-polytope. It can be shown (see [**145**, Theorem 0.7]) that $C^n(t_1, \ldots, t_m)$ has exactly m vertices $\boldsymbol{x}(t_i)$, the combinatorial type of cyclic polytope does not depend on the specific choice of the parameters $t_1, \ldots, t_m$, and $C^n(t_1, \ldots, t_m)$ is a neighborly simplicial n-polytope. We will denote the combinatorial cyclic n-polytope with m vertices by $C^n(m)$.

1.2. Face vectors and Dehn–Sommerville equations

The notion of the f-vector (or face vector) is a central concept in the combinatorial theory of polytopes. It has been studied there since the time of Euler.

DEFINITION 1.18. Let S be a simplicial n-polytope. Denote by f_i the number of i-dimensional faces of S. The integer vector $\boldsymbol{f}(S) = (f_0, \dots, f_{n-1})$ is called the *f-vector* of S. We also put $f_{-1} = 1$. The *h-vector* of S is the integer vector $(h_0, h_1, \dots, h_n)$ defined from the equation

$$h_0 t^n + \ldots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \ldots + f_{n-1}. \tag{1.7}$$

Finally, the sequence $(g_0, g_1, \dots, g_{[\frac{n}{2}]})$ where $g_0 = 1$, $g_i = h_i - h_{i-1}$, $i > 0$, is called the *g-vector* of S.

The f-vector of a simple n-polytope P^n is defined as the f-vector of its polar: $\boldsymbol{f}(P) := \boldsymbol{f}(P^*)$, and similarly for the h- and the g-vector of P. More explicitly, $\boldsymbol{f}(P) = (f_0, \dots, f_{n-1})$, where f_i is the number of faces of P of codimension $(i+1)$ (i.e. of dimension $(n-i-1)$). In particular, f_0 is the number of facets of P, which we usually denote $m(P)$ or just m. The agreement $f_{-1} = 1$ is now justified by the fact that P itself is a face of codimension 0.

REMARK. The definition of h-vector may seem to be unnatural at first glance. However, as we will see later, the h-vector has a number of combinatorial-geometrical and algebraic interpretations and in some situations is more convenient to work with than the f-vector.

Obviously, the f-vector is a *combinatorial invariant* of P^n, that is, it depends only on the face poset. For convenience we assume all polytopes in this section to be combinatorial, unless otherwise stated.

EXAMPLE 1.19. Two different combinatorial simple polytopes may have same f-vectors. For instance, let P_1^3 be the 3-cube and P_2^3 be the simple 3-polytope with 2 triangular, 2 quadrangular and 2 pentagonal facets, see Figure 1.1. (Note that P_2^3 is dual to the cyclic polytope $C^3(6)$ from Definition 1.17.) Then $\boldsymbol{f}(P_1^3) = \boldsymbol{f}(P_2^3) = (6, 12, 8)$.

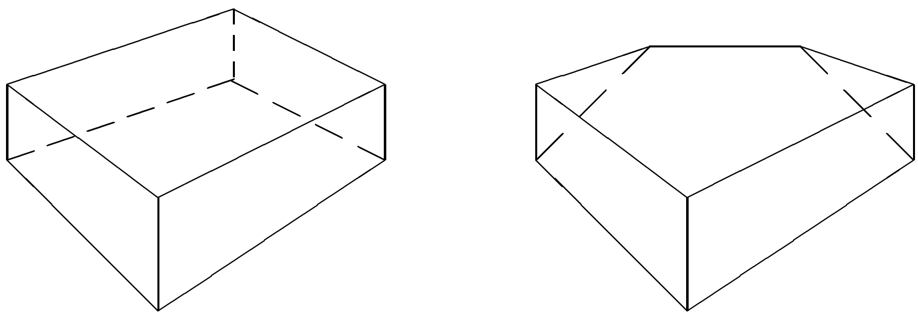


FIGURE 1.1. Two combinatorially non-equivalent simple polytopes with the same f-vectors.

The f-vector and the h-vector carry the same information about the polytope and determine each other by means of linear relations, namely

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{n-k} f_{i-1}, \quad f_{n-1-k} = \sum_{q=k}^{n} \binom{q}{k} h_{n-q}, \quad k = 0, \dots, n. \tag{1.8}$$

In particular, $h_0 = 1$ and $h_n = (-1)^n\big(1 - f_0 + f_1 + \ldots + (-1)^n f_{n-1}\big)$. By Euler's theorem,

$$f_0 - f_1 + \cdots + (-1)^{n-1} f_{n-1} = 1 + (-1)^{n-1}, \tag{1.9}$$

which is equivalent to $h_n = h_0 (= 1)$. In the case of simple polytopes Euler's theorem admits the following generalization.

THEOREM 1.20 (Dehn–Sommerville relations). *The h-vector of any simple or simplicial n-polytope is symmetric, i.e.*

$$h_i = h_{n-i}, \quad i = 0, 1, \ldots, n.$$

The Dehn–Sommerville equations can be proved in many different ways. We give a proof which uses a Morse-theoretical argument, which first appeared in [**29**]. We will return to this argument in chapter 5.

PROOF OF THEOREM 1.20. Let $P^n \subset \mathbb{R}^n$ be a simple polytope. Choose a linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ which is generic in the sense that it distinguishes the vertices of P^n. For this φ there is a vector $\boldsymbol{\nu}$ in $\mathbb{R}^n$ such that $\varphi(\boldsymbol{x}) = \langle \boldsymbol{\nu}, \boldsymbol{x} \rangle$. The assumption on φ implies that $\boldsymbol{\nu}$ is parallel to no edge of P^n. Now we can view φ as a height function on P^n. Using φ, we make the 1-skeleton of P^n a directed graph by orienting each edge in such a way that φ increases along it (this can be done since φ is generic), see Figure 1.2. For each vertex v of P^n define its index,

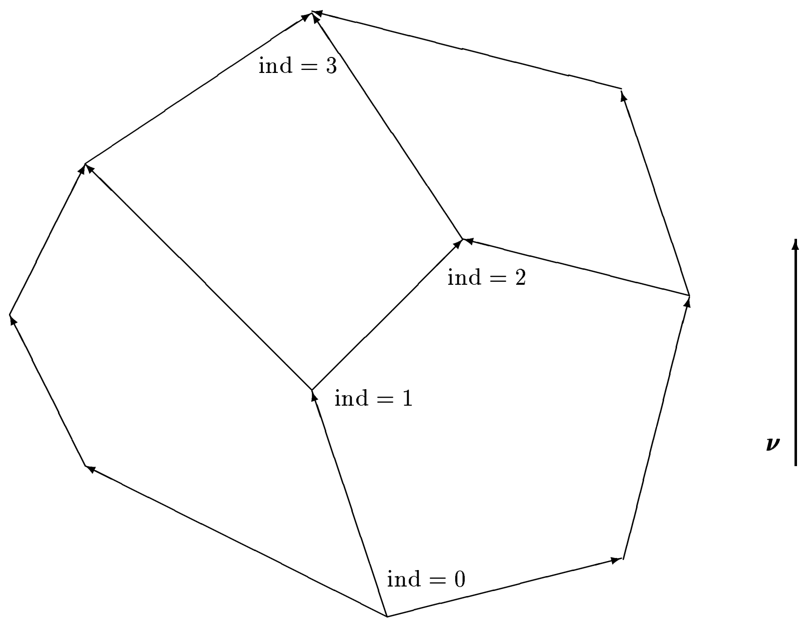


FIGURE 1.2. Oriented 1-skeleton of P and index of vertex.

$\mathrm{ind}(v)$, as the number of incident edges that point towards v. Denote the number of vertices of index i by $I_\nu(i)$. We claim that $I_\nu(i) = h_{n-i}$. Indeed, each face of P^n has a unique top vertex (the maximum of the height function φ restricted to the face) and a unique bottom vertex (the minimum of φ). Let F^k be a k-face of P^n, and v_F its top vertex. Since P^n is simple, there are exactly k edges of F^k meeting at v_F, whence $\operatorname{ind}(v_F) \geqslant k$. On the other hand, each vertex of index $q \geqslant k$ is the top vertex for exactly $\binom{q}{k}$ faces of dimension k. It follows that f_{n-1-k} (the number of k-faces) can be calculated as

$$f_{n-1-k} = \sum_{q \geqslant k} \tbinom{q}{k} I_\nu(q).$$

Now, the second identity from (1.8) shows that $I_\nu(q) = h_{n-q}$, as claimed. In particular, the number $I_\nu(q)$ does not depend on $\boldsymbol{\nu}$. At the same time, since $\operatorname{ind}_\nu(v) = n - \operatorname{ind}_{-\nu}(v)$ for any vertex v, one has

$$h_{n-q} = I_\nu(q) = I_{-\nu}(n-q) = h_q.$$

□

Using (1.8), we can rewrite the Dehn–Sommerville equations in terms of the f-vector as follows:

$$f_{k-1} = \sum_{j=k}^{n} (-1)^{n-j} \tbinom{j}{k} f_{j-1}, \quad k = 0, 1, \ldots, n. \tag{1.10}$$

The Dehn–Sommerville equations were established by Dehn [**52**] for $n \leqslant 5$ in 1905, and by Sommerville in the general case in 1927 [**122**] in the form similar to (1.10).

EXAMPLE 1.21. Let $P_1^{n_1}$ and $P_2^{n_2}$ be simple polytopes. Each face of $P_1 \times P_2$ is the product of a face of P_1 and a face of P_2, whence

$$f_k(P_1 \times P_2) = \sum_{i=-1}^{n_1-1} f_i(P_1) f_{k-i-1}(P_2), \quad k = -1, 0, \ldots, n_1 + n_2 - 1.$$

Set

$$h(P; t) = h_0 + h_1 t + \cdots + h_n t^n.$$

Then it follows from the above formula and (1.7) that

$$h(P_1 \times P_2; t) = h(P_1; t) h(P_2; t). \tag{1.11}$$

EXAMPLE 1.22. Let us express the f-vector and the h-vector of the connected sum $P^n \# Q^n$ in terms of that of P^n and Q^n. From Construction 1.13 we deduce that

$$f_i(P^n \# Q^n) = f_i(P^n) + f_i(Q^n) - \tbinom{n}{i+1}, \quad i = 0, 1, \ldots, n-2;$$
$$f_{n-1}(P^n \# Q^n) = f_{n-1}(P^n) + f_{n-1}(Q^n) - 2.$$

Then it follows from (1.8) that

$$h_0(P^n \# Q^n) = h_n(P^n \# Q^n) = 1;$$
$$h_i(P^n \# Q^n) = h_i(P^n) + h_i(Q^n), \quad i = 1, 2, \ldots, n-1.$$

This property raises the following question.

PROBLEM 1.23. *Describe all integer-valued functions on the set of simple polytopes which are linear with respect to the connected sum operation.*

The previous identities show that examples of such functions are provided by h_i for $i = 1, \ldots, n-1$.

EXAMPLE 1.24. Suppose S is a q-neighborly simplicial n-polytope (see Definition 1.15) different from the n-simplex. Then $f_{k-1}(S)=\binom{m}{k},\ \ k\leqslant q$. From (1.8) we get

$$h_k(S)=\sum_{i=0}^{k}(-1)^{k-i}\binom{n-i}{k-i}\binom{m}{i}=\binom{m-n+k-1}{k},\quad k\leqslant q. \tag{1.12}$$

The latter equality is obtained by calculating the coefficient of t^k from two sides of the identity

$$\frac{1}{(1+t)^{n-k+1}}(1+t)^m=(1+t)^{m-n+k-1}.$$

Since S is not a simplex, we have $m>n+1$, which together with (1.12) gives $h_0<h_1<\cdots<h_q$. These inequalities together with the Dehn–Sommerville equations imply the upper bound $q\leqslant\left[\frac{n}{2}\right]$ mentioned in Definition 1.15.

1.3. The g-theorem

The g-theorem gives answer to the following natural question: which integer vectors may appear as the f-vectors of simple (or, equivalently, simplicial) polytopes? The Dehn–Sommerville relations provide a necessary condition. As far as only linear equations are concerned, there are no further restrictions.

PROPOSITION 1.25 (Klee [**86**]). *The Dehn–Sommerville relations are the most general linear equations satisfied by the f-vectors of all simple (or simplicial) polytopes.*

PROOF. In [**86**] the statement was proved directly, in terms of f-vectors. However, the usage of h-vectors significantly simplifies the proof. It is sufficient to prove that the affine hull of the h-vectors $(h_0,h_1,\dots,h_n)$ of simple n-polytopes is an $\left[\frac{n}{2}\right]$-dimensional plane. This can be done, for instance, by providing $\left[\frac{n}{2}\right]+1$ simple polytopes with affinely independent h-vectors. Set $Q_k:=\Delta^k\times\Delta^{n-k},\ \ k=0,1\dots,\left[\frac{n}{2}\right]$. Since $h(\Delta^k)=1+t+\cdots+t^k$, the formula (1.11) gives

$$h(Q_k)=\frac{1-t^{k+1}}{1-t}\cdot\frac{1-t^{n-k+1}}{1-t}.$$

It follows that $h(Q_{k+1})-h(Q_k)=t^{k+1}+\cdots+t^{n-k-1},\ \ k=0,1,\dots,\left[\frac{n}{2}\right]-1$. Therefore, the vectors $\boldsymbol{h}(Q_k),\ \ k=0,1,\dots,\left[\frac{n}{2}\right]$, are affinely independent. □

EXAMPLE 1.26. Each vertex of a simple n-polytope P^n is contained in exactly n edges and each edge connects two vertices. This implies the following "obvious" linear equation for the components of the f-vector of P^n:

$$2f_{n-2}=nf_{n-1}. \tag{1.13}$$

Proposition 1.25 shows that this equation is a corollary of the Dehn–Sommerville equations. (One may observe that it is equation (1.10) for $k=n-1$.) It follows from (1.13) and Euler identity (1.9) that for simple (or simplicial) 3-polytopes the f-vector is completely determined by the number of facets, namely,

$$\boldsymbol{f}(P^3)=(f_0,3f_0-6,2f_0-4).$$

We mention also that Euler identity (1.9) is the only linear relation satisfied by the face vectors of *general* convex polytopes. (This can be proved in a similar way as Proposition 1.25, by specifying sufficiently many polytopes with affinely independent face vectors.)

The conditions characterizing the f-vectors of simple (or simplicial) polytopes, now know as the *g-theorem*, were conjectured by P. McMullen [**96**] in 1970 and proved by R. Stanley [**125**] (necessity) and Billera and Lee [**18**] (sufficiency) in 1980. Besides the Dehn–Sommerville equations, the g-theorem contains two groups of inequalities, one linear and one non-linear. To formulate the g-theorem we need the following construction.

DEFINITION 1.27. For any two positive integers a, i there exists a unique *binomial i-expansion* of a of the form

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j},$$

where $a_i > a_{i-1} > \cdots > a_j \geqslant j \geqslant 1$. Define

$$a^{\langle i\rangle} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j+1}{j+1}, \quad 0^{\langle i\rangle} = 0.$$

EXAMPLE 1.28. 1. For $a > 0$, $a^{\langle 1\rangle} = \binom{a+1}{2}$.

2. If $i \geqslant a$ then the binomial expansion has the form

$$a = \binom{i}{i} + \binom{i-1}{i-1} + \cdots + \binom{i-a+1}{i-a+1} = 1 + \cdots + 1,$$

and therefore $a^{\langle i\rangle} = a$.

3. Let $a = 28$, $i = 4$. The corresponding binomial expansion is

$$28 = \binom{6}{4} + \binom{5}{3} + \binom{3}{2}.$$

Hence,

$$28^{\langle 4\rangle} = \binom{7}{5} + \binom{6}{4} + \binom{4}{3} = 40.$$

THEOREM 1.29 (g-theorem). *An integer vector $(f_0, f_1, \dots, f_{n-1})$ is the f-vector of a simple n-polytope if and only if the corresponding sequence $(h_0, \dots, h_n)$ determined by* (1.7) *satisfies the following three conditions:*

(a) $h_i = h_{n-i}$, $i = 0, \dots, n$ *(the Dehn–Sommerville equations);*

(b) $h_0 \leqslant h_1 \leqslant \dots \leqslant h_{\left[\frac{n}{2}\right]}$, $i = 0, 1, \dots, \left[\frac{n}{2}\right]$;

(c) $h_0 = 1$, $h_{i+1} - h_i \leqslant (h_i - h_{i-1})^{\langle i\rangle}$, $i = 1, \dots, \left[\frac{n}{2}\right] - 1$.

REMARK. Obviously, the same conditions characterize the f-vectors of *simplicial* polytopes.

EXAMPLE 1.30. 1. The first inequality $h_0 \leqslant h_1$ from part (b) of g-theorem is equivalent to $f_0 = m \geqslant n + 1$. This just expresses the fact that it takes at least $n + 1$ hyperplanes to bound a polytope in $\mathbb{R}^n$.

2. Taking into account that

$$h_2 = \binom{n}{2} - (n-1)f_0 + f_1$$

(see (1.8)), we rewrite the first inequality $h_2 - h_1 \leqslant (h_1 - h_0)^{\langle 1\rangle}$ from part (c) of g-theorem as

$$\binom{n+1}{2} - nf_0 + f_1 \leqslant \binom{f_0-n}{2}.$$

(see Example 1.28.1). This is equivalent to the upper bound

$$f_1 \leqslant \binom{f_0}{2},$$

which says that two facets share at most one face of codimension two. In the dual "simplicial" notations, two vertices are joined by at most one edge.

3. The second inequality $h_1 \leqslant h_2$ (for $n \geqslant 4$) from part (b) of g-theorem is equivalent to

$$f_1 \geqslant nf_0 - \binom{n+1}{2}.$$

This is the first (and most significant) inequality from the famous *Lower Bound Conjecture* for simple polytopes (see Theorem 1.37 below).

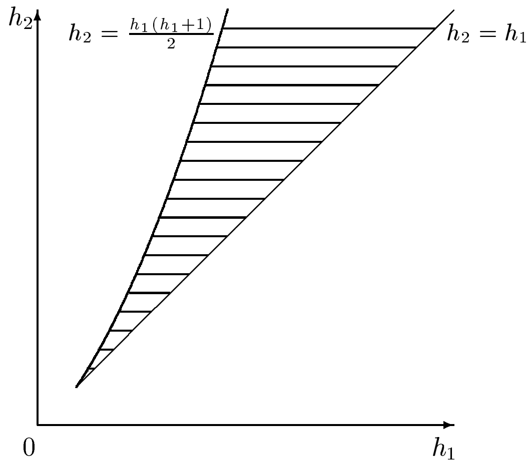


FIGURE 1.3. (h_1, h_2)-domain, $n \geqslant 4$.

Thus, the first two coordinates of the h-vectors of simple polytopes P^n, $n \geqslant 4$, fall between the two curves $h_2 = \frac{h_1(h_1+1)}{2}$ and $h_2 = h_1$ in the (h_1, h_2)-plane (see Figure 1.3). Note that the most general *linear* inequalities satisfied by points from this domain are $h_1 \geqslant 1$ and $h_2 \geqslant h_1$.

DEFINITION 1.31. An integral sequence $(k_0, k_1, \dots, k_r)$ satisfying $k_0 = 1$ and $0 \leqslant k_{i+1} \leqslant k_i^{\langle i \rangle}$ for $i = 1, \dots, r-1$ is called an *M-vector* (after M. Macaulay).

Conditions (b) and (c) from g-theorem are equivalent to that the g-vector $(g_0, g_1, \dots, g_{[\frac{n}{2}]})$ of a simple n-polytope is an M-vector. The notion of M-vector arises in the following classification result of commutative algebra.

THEOREM 1.32. *An integral sequence $(k_0, k_1, \dots, k_r)$ is an M-vector if and only if there exists a commutative graded algebra $A = A^0 \oplus A^2 \oplus \dots \oplus A^{2r}$ over a field $\mathbf{k} = A^0$ such that*

(a) *A is generated (as an algebra) by degree-two elements;*
(b) *the dimension of $2i$-th graded component of A equals k_i:*

$$\dim_{\mathbf{k}} A^{2i} = k_i, \quad i = 1, \dots, r.$$

This theorem is essentially due to Macaulay, but the above explicit formulation is that of [**124**]. The proof can be also found there.

The proof of the sufficiency part of g-theorem, due to Billera and Lee, is quite elementary and relies upon a remarkable combinatorial-geometrical construction of a simplicial polytope with any prescribed M-sequence as its g-vector. On the other hand, Stanley's proof of the necessity part of g-theorem (i.e. that the g-vector of a simple polytope is an M-vector) used deep results from algebraic geometry, in particular, the Hard Lefschetz theorem for the cohomology of toric varieties. We outline Stanley's proof in section 5.1. After 1993 several more elementary combinatorial proofs of the g-theorem appeared. The first such proof is due to McMullen [**97**]. It builds up on the notion of *polytope algebra*, which substitutes the cohomology algebra of toric variety. Despite being elementary, it was a complicated proof. Later, McMullen simplified his approach in [**98**]. Yet another elementary proof of the g-theorem has been recently found by Timorin [**133**]. It relies on the interpretation of McMullen's polytope algebra as the algebra of differential operators (with constant coefficients) vanishing on the *volume polynomial* of the polytope.

1.4. Upper Bound and Lower Bound theorems

The following statement, now know as the *Upper Bound Conjecture* (UBC), was suggested by Motzkin in 1957 and proved by P. McMullen [**95**] in 1970.

THEOREM 1.33 (UBC for simplicial polytopes). *From all simplicial n-polytopes S with m vertices the cyclic polytope $C^n(m)$ (Example 1.17) has the maximal number of i-faces, $2 \leqslant i \leqslant n-1$. That is, if $f_0(S) = m$, then*

$$f_i(S) \leqslant f_i\bigl(C^n(m)\bigr) \quad \text{for } i = 2, \dots, n-1.$$

The equality in the above formula holds if and only if S is a neighborly polytope (see Definition 1.15).

Note that since $C^n(m)$ is neighborly,

$$f_i\bigl(C^n(m)\bigr) = \tbinom{m}{i+1} \quad \text{for } i = 0, \dots, \left[\tfrac{n}{2}\right] - 1.$$

Due to the Dehn–Sommerville equations this determines the full f-vector of $C^n(m)$. The exact values are given by the following lemma.

LEMMA 1.34. *The number of i-faces of cyclic polytope $C^n(m)$ (or any neighborly n-polytope with m vertices) is given by*

$$f_i\bigl(C^n(m)\bigr) = \sum_{q=0}^{\left[\frac{n}{2}\right]} \tbinom{q}{n-1-i}\tbinom{m-n+q-1}{q} + \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \tbinom{n-p}{i+1-p}\tbinom{m-n+p-1}{p}, \quad i = -1, \dots, n-1,$$

where we assume $\binom{p}{q} = 0$ for $p < q$.

PROOF. Using the second identity from (1.8), identity $\left[\frac{n}{2}\right] + 1 = n - \left[\frac{n-1}{2}\right]$, the Dehn–Sommerville equations, and (1.12), we calculate

$$\begin{aligned} f_i &= \sum_{q=0}^{n} \tbinom{q}{n-1-i} h_{n-q} = \sum_{q=0}^{\left[\frac{n}{2}\right]} \tbinom{q}{n-1-i} h_q + \sum_{q=\left[\frac{n}{2}\right]+1}^{n} \tbinom{q}{n-1-i} h_{n-q} \\ &= \sum_{q=0}^{\left[\frac{n}{2}\right]} \tbinom{q}{n-1-i}\tbinom{m-n+q-1}{q} + \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \tbinom{n-p}{i+1-p}\tbinom{m-n+p-1}{p}. \end{aligned}$$

□

The above proof justifies the following statement.

COROLLARY 1.35. *The UBC for simplicial polytopes (Theorem* 1.33*) is implied by the following inequalities for the h-vector of a simplicial polytope S with m vertices*

$$h_i(S) \leqslant \binom{m-n+i-1}{i}, \qquad i = 0, \dots, \left[\tfrac{n}{2}\right].$$

This was one of the key observations in McMullen's original proof of the UBC for simplicial polytopes (see also [**29**, §18] and [**145**, §8.4]). The above corollary is also useful for a different generalization of UBC (we will return to this in section 3.2). We note also that due to the argument of Klee and McMullen (see [**145**, Lemma 8.24]) the UBT holds for *all* convex polytopes, not necessarily simplicial. That is, the cyclic polytope $C^n(m)$ has the maximal number of i-faces from all convex n-polytopes with m vertices.

Another fundamental fact from the theory of convex polytopes is the *Lower Bound Conjecture* (LBC) for simplicial polytopes.

DEFINITION 1.36. A simplicial n-polytope S is called *stacked* if there is a sequence $S_0, S_1, \dots, S_k = S$ of n-polytopes such that S_0 is an n-simplex and S_{i+1} is obtained from S_i by adding a pyramid over some facet of S_i. In the combinatorial language, stacked polytopes are those obtained from a simplex by applying several subsequent stellar subdivisions of facets.

REMARK. Adding a pyramid (or stellar subdivision of a facet) is dual to "cutting a vertex" of a simple polytope (see Example 1.14.3).

THEOREM 1.37 (LBC for simplicial polytopes). *For any simplicial n-polytope S ($n \geqslant 3$) with $m = f_0$ vertices hold*

$$\begin{aligned} f_i(S) &\geqslant \binom{n}{i} f_0 - \binom{n+1}{i+1} i \quad \textit{for } i = 1, \dots, n-2; \\ f_{n-1}(S) &\geqslant (n-1) f_0 - (n+1)(n-2). \end{aligned}$$

The equality is achieved if and only if S is a stacked polytope.

The argument by McMullen, Perles and Walkup [**100**] reduces the LBC to the case $i = 1$, namely, the inequality $f_1 \geqslant f_0 - \binom{n+1}{2}$. The LBC was first proved by Barnette [**13**], [**15**]. The "only if" part in the statement about the equality was proved in [**19**] using g-theorem. Unlike the UBC, little is know about generalizations of the LBC theorem to non-simplicial convex polytopes. Some results in this direction were obtained in [**83**] along with generalizations of the LBC theorem to simplicial spheres and manifolds (see also sections 2.3–2.4 in this book).

In dual notations, the UBC and the LBC provide upper and lower bounds for the number of faces of simple polytopes with given number of facets. Both theorems were proved approximately at the same time (in 1970) and motivated P. McMullen to conjecture the g-theorem [**96**]. On the other hand, both UBC and LBC are corollaries of the g-theorem (see e.g. [**29**, §20]). In fact the LBC follows only from parts (a) and (b) of Theorem 1.29, while the UBC follows from parts (a) and (c).

Part (b) of g-theorem, namely the inequalities

$$h_0 \leqslant h_1 \leqslant \dots \leqslant h_{\left[\frac{n}{2}\right]}, \tag{1.14}$$

where suggested in [**100**] as a generalization of the LBC for simplicial polytopes. The second inequality $h_1 \leqslant h_2$ is equivalent to the $i = 1$ case of LBC (see Example 1.30.3). It follows from the results of [**100**] and [**19**] that (1.14) are the strongest possible linear inequalities satisfied by the f-vectors of simple (or simplicial) polytopes (compare with the comment after Example 1.30). These inequalities are now known as the *Generalized Lower Bound Conjecture* (GLBC).

During the last two decades a lot of work was done in extending the Dehn–Sommerville equations, the GLBC and the g-theorem to objects more general than simplicial polytopes. However, there are still many intriguing open problems here. For more information see the first section of survey article [**129**] by Stanley and section 2.3 in this book.

1.5. Stanley–Reisner face rings of simple polytopes

The only aim of this short section is to define the Stanley–Reisner ring of a simple polytope. This fundamental combinatorial invariant will be one of the main characters in the next chapter. However, it is convenient for us to give it an independent treatment in the polytopal case.

Let P be a simple n-polytope with m facets $F_1, \dots, F_m$. Fix a commutative ring $\mathbf{k}$ with unit. Let $\mathbf{k}[v_1, \dots, v_m]$ be the polynomial algebra over $\mathbf{k}$ on m generators. We make it a graded algebra by setting $\deg(v_i) = 2$.

DEFINITION 1.38. The *face ring* (or the *Stanley–Reisner ring*) of a simple polytope P is the quotient ring

$$\mathbf{k}(P) = \mathbf{k}[v_1, \dots, v_m]/\mathcal{I}_P,$$

where $\mathcal{I}_P$ is the ideal generated by all square-free monomials $v_{i_1} v_{i_2} \cdots v_{i_s}$ such that $F_{i_1} \cap \dots \cap F_{i_s} = \varnothing$ in P, $i_1 < \dots < i_s$.

Since $\mathcal{I}_P$ is a homogeneous ideal, $\mathbf{k}(P)$ is a graded $\mathbf{k}$-algebra.

REMARK. In certain circumstances it is convenient to choose a different grading in $\mathbf{k}[v_1, \dots, v_m]$ and correspondingly $\mathbf{k}(K)$. These cases will be particularly mentioned.

EXAMPLE 1.39. 1. Let P^n be the n-simplex (regarded as a simple polytope). Then

$$\mathbf{k}(P^n) = \mathbf{k}[v_1, \dots, v_{n+1}]/(v_1 v_2 \cdots v_{n+1}).$$

2. Let P be the 3-cube I^3. Then

$$\mathbf{k}(P) = \mathbf{k}[v_1, v_2 \dots, v_6]/(v_1 v_4, v_2 v_5, v_3 v_6).$$

3. Let P^2 be the m-gon, $m \geqslant 4$. Then

$$\mathcal{I}_{P^2} = (v_i v_j \ : \ i - j \neq 0, \pm 1 \mod m).$$

CHAPTER 2

Topology and combinatorics of simplicial complexes

Simplicial complexes or triangulations (first introduced by Poincaré) provide an elegant, rigorous and convenient tool for studying topological invariants by combinatorial methods. The algebraic topology itself evolved from studying triangulations of topological spaces. With the appearance of cellular (or CW) complexes algebraic tools gradually replaced the combinatorial ones in topology. However, simplicial complexes have always played a significant role in PL topology, discrete and combinatorial geometry. The convex geometry provides an important class of sphere triangulations which are the boundary complexes of simplicial polytopes. The emergence of computers resulted in regaining the interest to "Combinatorial Topology", since simplicial complexes provide the most effective way to translate topological structures into machine language. So, it seems to be the proper time for topologists to make use of remarkable achievements in discrete and combinatorial geometry of the last decades, which we started to review in the previous chapter.

2.1. Abstract simplicial complexes and polyhedrons

Let $\mathcal{S}$ be a finite set. Given a subset $\sigma \subset \mathcal{S}$, we denote its cardinality by $|\sigma|$.

DEFINITION 2.1. An (abstract) *simplicial complex* on the set $\mathcal{S}$ is a collection $K = \{\sigma\}$ of subsets of $\mathcal{S}$ such that for each $\sigma \in K$ all subsets of σ (including $\varnothing$) also belong to K. A subset $\sigma \in K$ is called an (abstract) *simplex* of K. One-element subsets are called *vertices* of K. If K contains all one-element subsets of $\mathcal{S}$, then we say that K is a simplicial complex *on the vertex set* $\mathcal{S}$. The *dimension* of an abstract simplex $\sigma \in K$ is its cardinality minus one: $\dim \sigma = |\sigma| - 1$. The dimension of an abstract simplicial complex is the maximal dimension of its simplices. A simplicial complex K is *pure* if all its maximal simplices have the same dimension. A subcollection $K' \subset K$ which is also a simplicial complex is called a *subcomplex* of K.

In most of our constructions it is safe to fix an ordering in $\mathcal{S}$ and identify $\mathcal{S}$ with the index set $[m] = \{1, \ldots, m\}$. This makes the notation more clear; however, in some cases it is more convenient to keep unordered sets.

To distinguish from abstract simplices, the convex polytopes introduced in Example 1.7 (i.e. the convex hulls of affinely independent points) will be referred to as *geometrical simplices*.

DEFINITION 2.2. A *geometrical simplicial complex* (or a *polyhedron*) is a subset $\mathcal{P} \subset \mathbb{R}^n$ represented as a finite union U of geometrical simplices of any dimensions in such a way that the following two conditions are satisfied:

(a) each face of a simplex in U belongs to U;

(b) the intersection of any two simplices in U is a face of each.

A geometrical simplex from U is called a *face* of $\mathcal{P}$; as usual, one-dimensional faces are *vertices*. The dimension of $\mathcal{P}$ is the maximal dimension of its faces.

AGREEMENT. The notion of polyhedron from Definition 1.2 is not the same as that from Definition 2.2. The first meaning of the term "polyhedron" (i.e. the "unbounded polytope") is adopted in the convex geometry, while the second one (i.e. the "geometrical simplicial complex") is used in the combinatorial topology. Since both terms have become standard in the appropriate science, we cannot change their names completely. We will use "polyhedron" for a geometrical simplicial complex and "convex polyhedron" for an "unbounded polytope". Anyway, it will be always clear from the context which "polyhedron" is under consideration.

In the sequel both abstract and geometrical simplicial complexes are assumed to be finite.

AGREEMENT. Depending on the context, we will denote by Δ^{m-1} three different objects: the abstract simplicial complex $2^{[m]}$ consisting of *all* subsets of $[m]$, the convex polytope from Example 1.7 (i.e. the geometrical simplex), and the geometrical simplicial complex which is the union of all faces of the geometrical simplex.

DEFINITION 2.3. Given a simplicial complex K on the vertex set $[m]$, say that a polyhedron $\mathcal{P}$ is a *geometrical realization* of K if there is a bijection between the set $[m]$ and the vertex set of $\mathcal{P}$ that takes simplices of K to vertex sets of faces of $\mathcal{P}$.

If we do not care about the dimension of the ambient space, then there is the following quite obvious way to construct a geometrical realization for any simplicial complex K.

CONSTRUCTION 2.4. Suppose K is a simplicial complex on the set $[m]$. Let $\boldsymbol{e}_i$ denote the i-th unit coordinate vector in $\mathbb{R}^m$. For each subset $\sigma \subset [m]$ denote by Δ_σ the convex hull of vectors $\boldsymbol{e}_i$ with $i \in \sigma$. Then Δ_σ is a (regular, geometrical) simplex. The polyhedron

$$\bigcup_{\sigma \in K} \Delta_\sigma \subset \mathbb{R}^m$$

is a geometrical realization of K.

The above construction is just a geometrical interpretation of the fact that any simplicial complex on $[m]$ is a subcomplex of the simplex Δ^{m-1}. At the same time it is a classical result [**115**] that any n-dimensional abstract simplicial complex K^n admits a geometrical realization in $(2n+1)$-dimensional space.

EXAMPLE 2.5. Let S be a simplicial n-polytope. Then its boundary ∂S is a (geometrical) simplicial complex homeomorphic to an $(n-1)$-sphere. This example will be important in section 2.3.

DEFINITION 2.6. The f-vector, the h-vector and the g-vector of an $(n-1)$-dimensional simplicial complex K^{n-1} are defined in the same way as for simplicial polytopes. Namely, $\boldsymbol{f}(K^{n-1}) = (f_0, f_1, \dots, f_{n-1})$, where f_i is the number of i-dimensional simplices of K^{n-1}, and $\boldsymbol{h}(K^{n-1}) = (h_0, h_1, \dots, h_n)$, where h_i are determined by (1.7). Here we also assume $f_{-1} = 1$. If $K^{n-1} = \partial S$, the boundary of a simplicial n-polytope S, then one obviously has $\boldsymbol{f}(K^{n-1}) = \boldsymbol{f}(S)$.

2.2. Basic PL topology, and operations with simplicial complexes

For a detailed exposition of PL (piecewise linear) topology we refer to the classical monographs [**77**] by Hudson and [**118**] by Rourke and Sanderson. The role of PL category in the modern topology is described, for instance, in the more recent book [**104**] by Novikov.

DEFINITION 2.7. Let K_1, K_2 be simplicial complexes on the sets $[m_1]$, $[m_2]$ respectively, and $\mathcal{P}_1$, $\mathcal{P}_2$ their geometrical realizations. A map $\phi : [m_1] \to [m_2]$ is said to be a *simplicial map* between K_1 and K_2 if $\phi(\sigma) \in K_2$ for any $\sigma \in K_1$. A simplicial map ϕ is said to be *non-degenerate* if $|\phi(\sigma)| = |\sigma|$ for any $\sigma \in K_1$. On the geometrical level, a simplicial map extends linearly on the faces of $\mathcal{P}_1$ to a map $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ (denoted by the same letter for simplicity). We refer to the latter map as a *simplicial map of polyhedrons*. A *simplicial isomorphism* of polyhedrons is a simplicial map for which there exists a simplicial inverse. A polyhedron $\mathcal{P}'$ is called a *subdivision* of polyhedron $\mathcal{P}$ if each simplex of $\mathcal{P}'$ is contained in a simplex of $\mathcal{P}$ and each simplex of $\mathcal{P}$ is a union of finitely many simplices of $\mathcal{P}'$. A *PL map* $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ is a map that is simplicial between some subdivisions of $\mathcal{P}_1$ and $\mathcal{P}_2$. A *PL homeomorphism* is a PL map for which there exists a PL inverse. Two PL homeomorphic polyhedrons sometimes are also called *combinatorially equivalent*. In other words, two polyhedrons $\mathcal{P}_1, \mathcal{P}_2$ are PL homeomorphic if and only if there exists a polyhedron $\mathcal{P}$ isomorphic to a subdivision of each of them.

EXAMPLE 2.8. For any simplicial complex K on $[m]$ there exists a simplicial map (inclusion) $K \hookrightarrow \Delta^{m-1}$.

There is an obvious simplicial homeomorphism between any two geometrical realizations of a given simplicial complex K. This justifies our single notation $|K|$ for any geometrical realization of K. Whenever it is safe, we do not distinguish between abstract simplicial complexes and their geometrical realizations. For example, we would say "simplicial complex K is PL homeomorphic to X" instead of "the geometrical realization of K is PL homeomorphic to X".

CONSTRUCTION 2.9 (join of simplicial complexes). Let K_1, K_2 be simplicial complexes on sets $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. The *join* of K_1 and K_2 is the simplicial complex

$$K_1 * K_2 := \{\sigma \subset \mathcal{S}_1 \cup \mathcal{S}_2 \ : \ \sigma = \sigma_1 \cup \sigma_2, \ \sigma_1 \in K_1, \sigma_2 \in K_2\}$$

on the set $\mathcal{S}_1 \cup \mathcal{S}_2$. If K_1 is realized in $\mathbb{R}^{n_1}$ and K_2 in $\mathbb{R}^{n_2}$, then there is obvious canonical geometrical realization of $K_1 * K_2$ in $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

EXAMPLE 2.10. 1. If $K_1 = \Delta^{m_1-1}$, $K_2 = \Delta^{m_2-1}$, then $K_1 * K_2 = \Delta^{m_1+m_2-1}$.

2. The simplicial complex $\Delta^0 * K$ (the join of K and a point) is called the *cone* over K and denoted $\operatorname{cone}(K)$.

3. Let S^0 be a pair of disjoint points (a 0-sphere). Then $S^0 * K$ is called the *suspension* of K and denoted ΣK. The geometric realization of $\operatorname{cone}(K)$ (of ΣK) is the topological cone (suspension) over $|K|$.

4. Let P_1 and P_2 be simple polytopes. Then

$$\partial\big((P_1 \times P_2)^*\big) = \partial(P_1^* \circ P_2^*) = (\partial P_1^*) * (\partial P_2^*).$$

(see Construction 1.12).

CONSTRUCTION 2.11. The fact that the product of two simplices is not a simplex causes some problems with triangulating the products of spaces. However, there is a canonical triangulation of the product of two polyhedra for each choice of orderings of their vertices. So suppose K_1, K_2 are simplicial complexes on $[m_1]$ and $[m_2]$ respectively (this is one of the few constructions where the ordering of vertices is significant). Then we construct a new simplicial complex on $[m_1]\times[m_2]$, which we call the *Cartesian product* of K_1 and K_2 and denote $K_1 \times K_2$, as follows. By definition, a simplex of $K_1 \times K_2$ is a subset of some $\sigma_1 \times \sigma_2$ (with $\sigma_1 \in K_1$, $\sigma_2 \in K_2$) that establishes a non-decreasing correspondence between σ_1 and σ_2. More formally,

$$\begin{aligned} K_1 \times K_2 := \bigl\{\sigma \subset \sigma_1 \times \sigma_2 \,:\, &\sigma_1 \in K_1,\, \sigma_2 \in K_2, \\ &\text{and } i \leqslant i' \text{ implies } j \leqslant j' \text{ for any two pairs } (i,j),(i',j') \in \sigma\bigr\}. \end{aligned}$$

The polyhedron $|K_1 \times K_2|$ defines a canonical triangulation of $|K_1| \times |K_2|$.

CONSTRUCTION 2.12 (connected sum of simplicial complexes). Let K_1, K_2 be two pure $(n-1)$-dimensional simplicial complexes on sets $\mathcal{S}_1, \mathcal{S}_2$ respectively, $|\mathcal{S}_1| = m_1$, $|\mathcal{S}_2| = m_2$. Suppose we are given two maximal simplices $\sigma_1 \in K_1$, $\sigma_2 \in K_2$. Fix an identification of σ_1 and σ_2, and denote by $\mathcal{S}_1 \cup_\sigma \mathcal{S}_2$ the union of $\mathcal{S}_1$ and $\mathcal{S}_2$ with σ_1 and σ_2 identified (the subset created by the identification is denoted σ). We have $|\mathcal{S}_1 \cup_\sigma \mathcal{S}_2| = m_1 + m_2 - n$. Both K_1 and K_2 now can be viewed as collections of subsets of $\mathcal{S}_1 \cup_\sigma \mathcal{S}_2$. We define the *connected sum* of K_1 at σ_1 and K_2 at σ_2 to be the simplicial complex

$$K_1 \#_{\sigma_1,\sigma_2} K_2 := (K_1 \cup K_2) \setminus \{\sigma\}$$

on the set $\mathcal{S}_1 \cup_\sigma \mathcal{S}_2$. When the choices of σ_1, σ_2 and identification of σ_1 and σ_2 are clear we use the abbreviation $K_1 \# K_2$. Geometrically, the connected sum of $|K_1|$ and $|K_2|$ at σ_1 and σ_2 is produced by attaching $|K_1|$ to $|K_2|$ along the faces σ_1, σ_2 and then removing the face σ obtained from the identification of σ_1 with σ_2.

EXAMPLE 2.13. 1. Let K_1 be an $(n-1)$-simplex, and K_2 a pure $(n-1)$-dimensional complex with a fixed maximal simplex σ_2. Then $K_1 \# K_2 = K_2 \setminus \{\sigma_2\}$, i.e. $K_1 \# K_2$ is obtained by deleting the simplex σ_2 from K_2.

2. Let P_1 and P_2 be simple polytopes. Set $K_1 = \partial(P_1^*)$, $K_2 = \partial(P_2^*)$. Then

$$K_1 \# K_2 = \partial\bigl((P_1 \# P_2)^*\bigr)$$

(see Construction 1.13).

DEFINITION 2.14. The *barycentric subdivision* of an abstract simplicial complex K is the simplicial complex K' on the set $\{\sigma \in K\}$ of simplices of K whose simplices are chains of embedded simplices of K. That is, $\{\sigma_1, \ldots, \sigma_r\} \in K'$ if and only if $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_r$ in K (after possible re-ordering).

The *barycenter* of a (polytopal) simplex $\Delta^n \subset \mathbb{R}^n$ with vertices $v_1, \ldots, v_{n+1}$ is the point $\mathrm{bc}(\Delta^n) = \frac{1}{n+1}(v_1 + \cdots + v_{n+1}) \in \Delta^n$. The *barycentric subdivision* $\mathcal{P}'$ of a polyhedron $\mathcal{P}$ is defined as follows. The vertex set of $\mathcal{P}'$ is formed by the barycenters of simplices of $\mathcal{P}$. A collection of barycenters $\{\mathrm{bc}(\Delta_1^{i_1}), \ldots, \mathrm{bc}(\Delta_r^{i_r})\}$ spans a simplex of $\mathcal{P}'$ if and only if $\Delta_1^{i_1} \subset \cdots \subset \Delta_r^{i_r}$ in $\mathcal{P}$. Obviously $|K'| = |K|'$ for any abstract simplicial complex K.

EXAMPLE 2.15. For any $(n-1)$-dimensional simplicial complex K^{n-1} on $[m]$ there is a non-degenerate simplicial map $K' \to \Delta^{n-1}$ defined on the vertices by

$\sigma \to |\sigma|, \quad \sigma \in K$. (Here σ is regarded as a vertex of K' and $|\sigma|$ as a vertex of Δ^{n-1}.)

EXAMPLE 2.16. Let K be a simplicial complex on a set $\mathcal{S}$, and suppose we are given a choice function $f : K \to \mathcal{S}$ assigning to each simplex $\sigma \in K$ a point in σ. For instance, if $\mathcal{S} = [m]$ we can take $f = \min$, that is, assign to each simplex its minimal vertex. For every such map f there is a canonical simplicial map $\nabla_f : K' \to K$ constructed as follows. By the definition of K', the vertices of K' are in one-to-one correspondence with the simplices of K. For each $\sigma \in K$ (regarded as a vertex of K') set $\nabla_f(\sigma) = f(\sigma)$. This extends to the simplices of K' by

$$\nabla_f(\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_r) = \{f(\sigma_1), f(\sigma_2), \ldots, f(\sigma_r)\}.$$

The latter is a subset of σ_r, whence it is a simplex of K. Thus, ∇_f is indeed a simplicial map.

EXAMPLE 2.17 (order complex of a poset). Let $\mathcal{S}$ be any poset. Define $\operatorname{ord}(\mathcal{S})$ to be the collection of all chains $x_1 < x_2 < \cdots < x_k, \ x_i \in \mathcal{S}$. Then $\operatorname{ord}(\mathcal{S})$ is clearly a simplicial complex. It is called the *order complex* of the poset $(\mathcal{S}, <)$. The order complex of the inclusion poset of non-empty simplices of a simplicial complex K is its barycentric subdivision K'. If we add the empty simplex to the poset, then the resulting order complex will be $\operatorname{cone} K'$.

DEFINITION 2.18. A simplicial complex K is called a *flag complex* if any set of vertices which are pairwise connected spans a simplex of K.

PROPOSITION 2.19. *For each simplicial graph (1-dimensional simplicial complex) ? there exists a unique flag complex K_Γ on the same vertex set whose 1-skeleton is ?.*

PROOF. The simplices of K_Γ are the vertex sets of complete subgraphs in ?. $\square$

DEFINITION 2.20. The minimal simplicial complex that contains a given complex K and is flag is called the *flagification* of K and denoted $\operatorname{fla}(K)$.

DEFINITION 2.21. Given a simplicial complex K on $\mathcal{S}$, a *missing face* of K is a subset $\sigma \subset [m]$ such that $\sigma \notin K$, but every proper subset of σ is a simplex of K.

The following statement is straightforward.

PROPOSITION 2.22. *K is a flag complex if and only if every missing face has two vertices.*

EXAMPLE 2.23. 1. Order complexes of posets (in particular, barycentric subdivisions) are examples of flag complexes. On the other hand, the boundary of a 5-gon is flag complex, but not an order complex of poset.

2. Let $K = K_1 \#_{\sigma_1,\sigma_2} K_2$ (see Construction 2.12). Then σ is a missing face of K provided that at least one of K_1 and K_2 is not a simplex.

DEFINITION 2.24. The *link* and the *star* of a simplex $\sigma \in K$ are the subcomplexes

$$\operatorname{link}_K \sigma = \{\tau \in K \,:\, \sigma \cup \tau \in K, \ \sigma \cap \tau = \varnothing\};$$
$$\operatorname{star}_K \sigma = \{\tau \in K \,:\, \sigma \cup \tau \in K\}.$$

For any vertex $v \in K$ the subcomplex $\operatorname{star}_K v$ can be identified with the cone over $\operatorname{link}_K v$. The polyhedron $|\operatorname{star}_K v|$ consists of all faces of $|K|$ that contain v. We omit the subscripts K whenever the context allows.

For any subcomplex $L \subset K$ define the (closed) *combinatorial neighborhood* $U_K(L)$ of L in K by

$$U_K(L) = \bigcup_{\sigma \in L} \operatorname{star}_K \sigma.$$

Equivalently, the combinatorial neighborhood $U_K(L)$ consists of all simplices of K, together with all their faces, having some simplex of L as a face. Define also the *open combinatorial neighborhood* $\overset{\circ}{U}_K(L)$ of $|L|$ in $|K|$ as the union of relative interiors of faces of $|K|$ having some simplex of $|L|$ as their face.

For any subset $\sigma \subset \mathcal{S}$ define the *full subcomplex* K_σ by

$$K_\sigma = \{\tau \in K \, : \, \tau \subset \sigma\}. \tag{2.1}$$

Set $\operatorname{core} \mathcal{S} = \{v \in \mathcal{S} : \operatorname{star} v \neq K\}$. The *core* of K is the subcomplex $\operatorname{core} K = K_{\operatorname{core} \mathcal{S}}$. Thus, the core is the maximal subcomplex containing all vertices whose stars do not coincide with K.

EXAMPLE 2.25. 1. $\operatorname{link}_K \varnothing = K$.

2. Let $K = \partial\Delta^3$ be the boundary of the tetrahedron on four vertices $1, 2, 3, 4$, and $\sigma = \{1, 2\}$. Then $\operatorname{link} \sigma$ is the subcomplex consisting of two disjoint points 3 and 4.

3. Let K be the cone over L with vertex v. Then $\operatorname{link} v = L$, $\operatorname{star} v = K$, and $\operatorname{core} K \subset L$.

EXAMPLE 2.26 (dual simplicial complex). Let K be a simplicial complex on $\mathcal{S}$. Suppose that K is not the full simplex on $\mathcal{S}$. Define

$$\widehat{K} := \{\sigma \subset \mathcal{S} \, : \, \mathcal{S} \setminus \sigma \notin K\}.$$

Then $\widehat{K}$ is also a simplicial complex on $\mathcal{S}$. It is called the *dual* of K.

The dual simplicial complex $\widehat{K}$ provides the following "purely simplicial interpretation" for the *Alexander duality* (see e.g. [**104**, p. 54]) between $|K|$ and $S^{m-1} \setminus |K|$ for any simplicial complex K embedded in the $(m-1)$-sphere. Let us consider the barycentric subdivision $(\partial\Delta^{m-1})'$ of the boundary of a geometrical simplex on the vertex set $[m] = \{1, \ldots, m\}$. By the definition, the faces of $(\partial\Delta^{m-1})'$ correspond to the pairs $\sigma \subset \tau$ of subsets of $[m]$ satisfying $|\sigma| \geqslant 1$, $|\tau| \leqslant m-1$. Denote the corresponding faces by $\Delta_{\sigma\subset\tau}$. (For example, $\Delta_{\{i\}\subset\{i\}}$ is the vertex $v = \{i\}$ of Δ^{m-1} regarded as a vertex of $(\partial\Delta^{m-1})'$.) Denote $\widehat{i} = [m] \setminus \{i\}$ and, more generally, $\widehat{\sigma} = [m] \setminus \sigma$ for any subset $\sigma \subset [m]$. For any simplicial complex K on $[m]$ define the following subcomplex in $(\partial\Delta^{m-1})'$:

$$D(K) = \bigcup_{\sigma,\tau : \tau \subset \sigma, \sigma \notin K} \Delta_{\widehat{\sigma} \subset \widehat{\tau}}.$$

PROPOSITION 2.27. *For any simplicial complex $K \neq \Delta^{m-1}$ on the set $[m]$ the polyhedron $D(K)$ provides a geometrical realization for the barycentric subdivision of the dual simplicial complex*

$$D(K) = |\widehat{K}'|.$$

Moreover, if the barycentric subdivision of K is realized canonically as a subpolyhedron in $(\partial\Delta^{m-1})'$, then

$$|\widehat{K}'| = \big(\partial\Delta^{m-1}\big)' \setminus \overset{\circ}{U}_{(\partial\Delta^{m-1})'}\big(|K'|\big).$$

PROOF. The complete proof is elementary but quite tedious. We just give an illustrating picture (Figure 2.1). Here K is the boundary of the square on vertices $1, 2, 3, 4$. Then $\widehat{K}$ consists of two disjoint segments. The picture shows both K' and $\widehat{K}'$ as subcomplexes in $(\partial\Delta^3)'$. $\square$

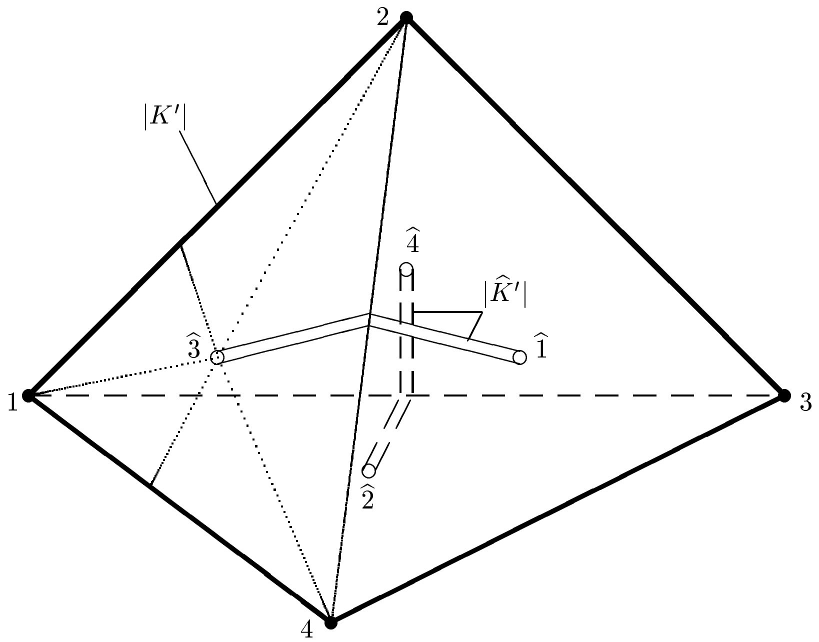


FIGURE 2.1. Dual complex and Alexander duality.

COROLLARY 2.28 (Simplicial Alexander duality). *For any simplicial complex $K \neq \Delta^{m-1}$ on the set $[m]$ it holds that*

$$\widetilde{H}_j(\widehat{K}) \cong \widetilde{H}^{m-3-j}(K), \quad -1 \leqslant j \leqslant m-2,$$

where $\widetilde{H}_k(\cdot)$ and $\widetilde{H}^k(\cdot)$ denotes the k-th reduced simplicial homology and cohomology groups (with integer coefficients) respectively. We use the agreement $\widetilde{H}_{-1}(\varnothing) = \widetilde{H}^{-1}(\varnothing) = \mathbb{Z}$ here.

PROOF. Since $(\partial\Delta^{m-1})'$ is homeomorphic to S^{m-2}, the Alexander duality theorem and Proposition 2.27 show that

$$\begin{aligned}\widetilde{H}_j(\widehat{K}) = \widetilde{H}_j\big((\partial D^{m-1})' \setminus \overset{\circ}{U}_{(\partial D^{m-1})'}(|K'|)\big)\\ \cong \widetilde{H}_j\big(S^{m-2} \setminus K\big) \cong \widetilde{H}^{m-3-j}(K), \quad -1 \leqslant j \leqslant m-2.\end{aligned}$$

$\square$

Corollary 2.28 admits the following generalization, which we will use in Chapter 8.

Proposition 2.29. *For any simplicial complex $K \neq \Delta^{m-1}$ on $[m]$ and simplex $\sigma \in \widehat{K}$ it holds that*

$$\widetilde{H}_j\big(\operatorname{link}_{\widehat{K}}\sigma\big) \cong \widetilde{H}^{m-3-j-|\sigma|}(K_{\widehat{\sigma}}),$$

where $\widehat{\sigma} = [m] \setminus \sigma$ and $K_{\widehat{\sigma}}$ is the full subcomplex in K defined in (2.1).

Corollary 2.28 is obtained by substituting $\sigma = \varnothing$ above.

Example 2.30. Let K be the boundary of a pentagon. Then $\widehat{K}$ is the Möbius band triangulated as it is shown on Figure 2.2. If we map the points $\widehat{1}, \widehat{2}, \widehat{3}, \widehat{4}$ to the vertices of a 3-simplex and $\widehat{5}$ to its barycenter, then the whole triangulated Möbius band $\widehat{K}$ becomes a subcomplex in the 3-dimensional *Schlegel diagram* (see [**145**, Lecture 5]) of a 4-dimensional simplex.

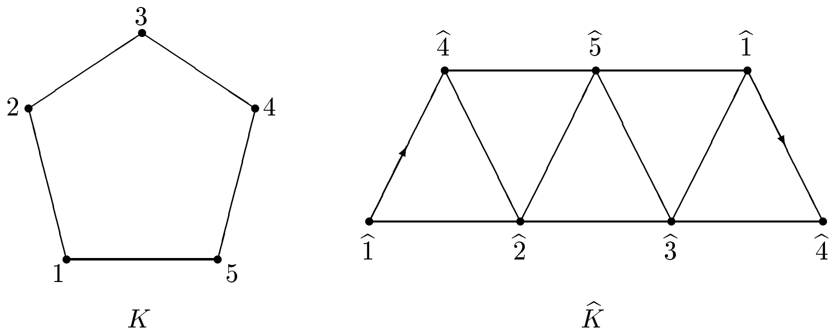


Figure 2.2. The boundary of pentagon and its dual.

2.3. Simplicial spheres

Definition 2.31. A *simplicial q-sphere* is a simplicial complex K^q homeomorphic to q-sphere. A *PL sphere* is a simplicial sphere K^q which is PL homeomorphic to the boundary of a simplex (equivalently, there is a subdivision of K^q isomorphic to a subdivision of the boundary of Δ^{q+1}). A *homology q-sphere* is a topological manifold which has the same homology as the q-sphere S^q.

The boundary of a simplicial n-polytope is an $(n-1)$-dimensional PL sphere. A PL sphere simplicially isomorphic to the boundary of a simplicial polytope is called a *polytopal sphere*. We have the following hierarchy of combinatorial objects:

$$\text{polytopal spheres} \subset PL \text{ spheres} \subset \text{simplicial spheres}. \tag{2.2}$$

In dimension 2 any simplicial sphere is polytopal (see e.g. [**69**] or [**145**, Theorem 5.8]). However, in higher dimensions both above inclusions are strict. The first inclusion in (2.2) is strict already in dimension 3. Namely, there are 39 combinatorially different triangulations of the 3-sphere with 8 vertices, out of which 2 are *non-polytopal*. The first one, now known as the *Brückner sphere* was found by Grünbaum ([**69**, §11.5], see also [**70**]) as a correction of Brückner's result of 1909 on the classification of simplicial 4-polytopes with 8 vertices. The second, known as *Barnette sphere*, is described in [**12**]. The complete classification of simplicial 3-spheres with up to 8 vertices was obtained in [**14**]. We mention also the result of Mani [**94**] that any simplicial q-sphere with up to $(q+4)$ vertices is polytopal.

As for the second inclusion in (2.2), it is known that in dimension 3 any simplicial sphere is PL. In dimension 4 the corresponding question is open (see the discussion in the next section), but starting from dimension 5 there exist *non-PL* simplicial spheres. One such thing is described in Example 2.35 below. According to the result of [**21**], for any $n \geqslant 5$ there is a non-PL triangulation of S^n with $n+13$ vertices.

Since the f-vector of a polytopal sphere coincides with the f-vector of the corresponding simplicial polytope (see Definition 2.6), the g-theorem (Theorem 1.29) holds for polytopal spheres. So it is natural to ask whether the g-theorem extends to simplicial spheres. This question was posed by McMullen [**96**] as an extension of his conjecture for simplicial polytopes. Since 1980, when McMullen's conjecture for simplicial polytopes was proved by Billera, Lee, and Stanley, the following is regarded as the main open combinatorial-geometrical problem concerning the f-vectors of simplicial complexes.

PROBLEM 2.32 (g-conjecture for simplicial spheres). *Does the g-theorem (Theorem 1.29) hold for simplicial spheres?*

The g-conjecture is open even for PL spheres. Note that only the necessity of g-theorem (i.e. that the g-vector is an M-vector) is to be verified for simplicial spheres. If correct, the g-conjecture would imply a characterisation of f-vectors of simplicial spheres.

The first part of Theorem 1.29 (the Dehn–Sommerville equations) is known to be true for simplicial spheres (see Corollary 3.41 below). Simplicial spheres also satisfy the UBC and the LBC inequalities as stated in Theorems 1.33 and 1.37. The LBC (in particular, the inequality $h_1 \leqslant h_2$) for spheres was proved by Barnette [**15**] (see also [**83**]). The UBC for spheres is due to Stanley [**123**] (see Corollary 3.19 below). This implies that the g-conjecture is true for simplicial spheres of dimension $\leqslant 4$. The inequality $h_2 \leqslant h_3$ from the GLBC (1.14) is open. Many attempts to prove the g-conjecture were made during the last two decades. Though unsuccessful, these attempts resulted in some very interesting reformulations of the g-conjecture. The results of Pachner [**109**], [**110**] reduce the g-conjecture (for PL-spheres) to some properties of *bistellar moves* (see the discussion after Theorem 2.41). We also mention the results of [**131**] showing that the g-conjecture follows from the *skeletal r-rigidity* of simplicial $(n-1)$-sphere for $r \leqslant \left[\frac{n}{2}\right]$. It was shown independently by Kalai and Stanley [**127**, Corollary 2.4] that the GLBC holds for the boundary of an n-dimensional ball that is a subcomplex of the boundary complex of a simplicial $(n+1)$-polytope. However, it is not clear now which simplicial complexes occur in this way. The lack of progress in proving the g-conjecture motivated Björner and Lutz to launch a computer-aided search for counterexamples [**21**]. Though their bistellar flip algorithm and computer program BISTELLAR produced many remarkable results on triangulations of manifolds, no counterexamples to the g-conjecture were found. For more history of g-theorem and related questions see [**128**], [**129**], [**145**, Lecture 8].

2.4. Triangulated manifolds

DEFINITION 2.33. A simplicial complex K is called a *triangulated manifold* (or *simplicial manifold*) if the polyhedron $|K|$ is a topological manifold. (All manifolds considered here are compact, connected and closed, unless otherwise stated.) A q-dimensional *PL manifold* (or *combinatorial manifold*) is a simplicial complex K^q

such that $\mathrm{link}(\sigma)$ is a PL sphere of dimension $(q-|\sigma|)$ for every non-empty simplex $\sigma \in K^q$.

Every PL manifold K^q is a (triangulated) manifold. Indeed, for each vertex $v \in K^q$ the $(q-1)$-dimensional PL-sphere $\mathrm{link}\, v$ bounds an open neighborhood U_v which is homeomorphic to an open q-ball. Since any point of $|K^q|$ is contained in U_v for some v, this defines an atlas for $|K^q|$.

Does every triangulation of a topological manifold yield a simplicial complex which is a PL manifold? The answer is "no", and the question itself ascends to a famous conjecture of the dawn of topology, known as the *Hauptvermutung der Topologie*. In the early days of topology all of the known topological invariants were defined in combinatorial terms, and it was very important to find out whether the topology of a polyhedron fully determines the combinatorics of triangulation. In the modern terminology, the *Hauptvermutung* conjecture states that any two homeomorphic polyhedrons are combinatorially equivalent (PL homeomorphic). This is valid in dimensions $\leqslant 3$ (the result is due to Rado, 1926, for 2-manifolds, Papakyriakopoulos, 1943, for 2-complexes, Moise, 1953 for 3-manifolds, and E. Brown, 1964, for 3-complexes; see [**101**] for the modern exposition). The first examples of complexes disproving the *Hauptvermutung* in dimensions $\geqslant 6$ were found by Milnor in the early 1960s. However, the *manifold Hauptvermutung*, namely the question of whether two homeomorphic triangulated manifolds are combinatorially equivalent, had remained open until the 1970s. It was finally disproved with the appearance of the following theorem.

THEOREM 2.34 (Edwards, Cannon). *The double suspension $\Sigma\Sigma S^n_H$ of any homology n-sphere S^n_H is homeomorphic to S^{n+2}.*

This theorem was proved by Edwards [**58**] for some particular homology 3-spheres and by Cannon [**39**] in the general case. The following example provides a non-PL triangulation of the 5-sphere and therefore disproves the *manifold Hauptvermutung* in dimensions $\geqslant 5$.

EXAMPLE 2.35 (non-PL simplicial 5-sphere). Let S^3_H be any simplicial homology 3-sphere which is not a topological sphere. The *Poincaré sphere* $SO(3)/A_5$ (triangulated in any way) provides an example of such a manifold. By Theorem 2.34, the double suspension $\Sigma^2 S^3_H$ is homeomorphic to S^5 (and, more generally, $\Sigma^k S^3_H$ is homeomorphic to S^{k+3} for $k \geqslant 2$). However, $\Sigma^2 S^3_H$ cannot be PL, since S^3_H appears as the link of some 1-simplex in $\Sigma^2 S^3_H$.

In the positive direction, it is known that two homeomorphic simply connected PL manifolds of dimension $\geqslant 5$ with no torsion in third homology group are combinatorially equivalent (PL homeomorphic). This is Sullivan's famous Hauptvermutung theorem. The general classification of PL structures on higher dimensional topological manifolds was obtained by Kirby and Siebenmann, see [**85**].

The following theorem gives a characterization of simplicial complexes which are triangulated manifolds of dimension $\geqslant 5$ and generalizes Theorem 2.34.

THEOREM 2.36 (Edwards [**59**]). *For $q \geqslant 5$ the polyhedron of a simplicial complex K^q is a topological q-manifold if and only if* $\mathrm{link}\,\sigma$ *has the homology of a $(q-|\sigma|)$-sphere for each non-empty simplex $|\sigma| \in K^q$ and* $\mathrm{link}\, v$ *is simply connected for each vertex $v \in K$.*

The discovery of non-PL triangulations of topological manifolds motivated further questions. Among them is whether every topological manifold admits a PL triangulation, or at least *any* triangulation, not necessarily PL. Another related question is whether the *Hauptvermutung* is valid in dimension 4. Both questions were answered (negatively) by the results of Freedman and Donaldson (early 1980s).

A smooth manifold can be triangulated by Whitney's theorem. All topological 2- and 3-dimensional manifolds can be triangulated as well (for 3-manifolds see [**101**]). Moreover, since the link of a vertex in a simplicial 3-sphere is a 2-sphere (and a 2-sphere is always PL), all 2- and 3-manifolds are PL. However, in dimension 4 there exist topological manifolds that do not admit a PL-triangulation. An example is provided by Freedman's fake $\mathbb{C}P^2$ [**63**, §8.3, §10.1], a topological manifold which is homeomorphic, but not diffeomorphic to the complex projective plane $\mathbb{C}P^2$. This shows that the *Hauptvermutung* is false for 4-dimensional manifolds. Even worse, as it is shown in [**5**], there exist topological 4-manifolds (e.g. Freedman's topological 4-manifold with the intersection form E_8) that *do not admit any triangulation*. In dimensions $\geqslant 5$ the triangulation problem is open:

Problem 2.37 (Triangulation Conjecture). *Is it true that any topological manifold of dimension $\geqslant 5$ can be triangulated?*

Another well-known problem of PL-topology concerns the uniqueness of a PL structure on a topological sphere.

Problem 2.38. *Is a PL manifold homeomorphic to the topological* 4*-sphere necessarily a PL sphere?*

Four is the only dimension where the uniqueness of a PL structure on a topological sphere is open. For dimensions $\leqslant 3$ the uniqueness was proved by Moise [**101**], and for dimensions $\geqslant 5$ it follows from the result of Kirby and Siebenmann [**85**]. In dimension 4 the category of PL manifolds is equivalent to the smooth category, hence, the above problem is equivalent to if there exists an exotic (or fake) 4-sphere.

The history of the *Hauptvermutung* conjecture is summarized in a survey article [**116**] by A. Ranicki. This source also contains a more detailed discussion of recent developments and open problems (including those mentioned above) in combinatorial and PL topology.

2.5. Bistellar moves

Bistellar moves (in other notation, bistellar flips or bistellar operations) were introduced by Pachner (see [**109**], [**110**]) as a generalization of *stellar subdivisions*. These operations allow us to decompose a PL homeomorphism into a sequence of simple "flips" and thus provide a very convenient way to compute and handle topological invariants of PL manifolds. Starting from a given PL triangulation, bistellar operations may be used to construct new triangulations with some good properties, e.g. symmetric or with a small number of vertices. On the other hand, bistellar moves can be used to produce some nasty triangulation if we start from a non-PL one. Both approaches were applied in [**21**] to find many interesting triangulations of low-dimensional manifolds. Bistellar moves also provide a combinatorial interpretation for algebraic *blow up* and *blow down* operations for projective *toric varieties* (see section 5.1) as well as for some topological *surgery operations* (see Construction 6.23). Finally, bistellar moves may be used to define a metric on the space of PL triangulations of a given PL manifold, see [**103**] for more details.

DEFINITION 2.39. Let K be a simplicial q-manifold (or any pure q-dimensional simplicial complex) and $\sigma \in K$ a $(q-i)$-simplex $(0 \leqslant i \leqslant q)$ such that $\operatorname{link}_K \sigma$ is the boundary $\partial\tau$ of an i-simplex τ that is not a face of K. Then the operation χ_σ on K defined by

$$\chi_\sigma(K) := \big(K \setminus (\sigma * \partial\tau)\big) \cup (\partial\sigma * \tau)$$

is called a *bistellar i-move.* Bistellar i-moves with $i \geqslant \left[\frac{q}{2}\right]$ are also called *reverse $(q-i)$-moves*. Note that a 0-move adds a new vertex to a triangulation (we assume that $\partial D^0 = \varnothing$), a reverse 0-move deletes a vertex, while all other bistellar moves do not change the number of vertices, see Figures 2.3 and 2.4. Two pure simplicial complexes are *bistellarly equivalent* if one is taken to another by a finite sequence of bistellar moves.

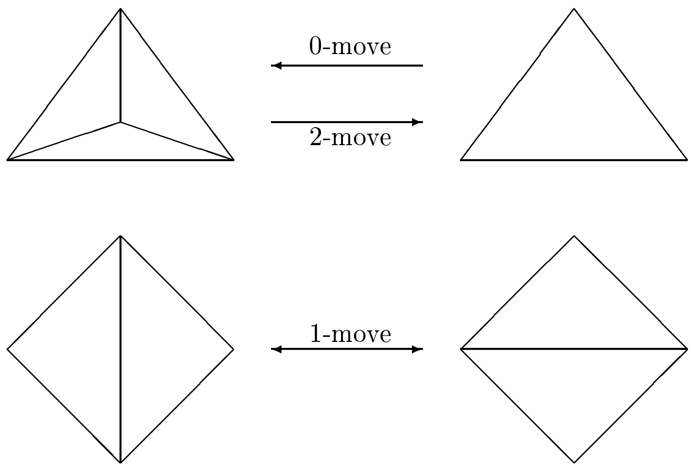


FIGURE 2.3. Bistellar moves for $q = 2$.

REMARK. The bistellar 0-move is just the stellar subdivision, or connected sum with the boundary of a simplex. In particular, *stacked spheres* (i.e., the boundaries of stacked polytopes, see Definition 1.36) are exactly those obtained from the boundary of a simplex by applying bistellar 0-moves.

It is easy to see that two bistellarly equivalent PL manifolds are PL homeomorphic. The following remarkable result shows that the converse is also true.

THEOREM 2.40 (Pachner [**109**, Theorem 1], [**110**, (5.5)]). *Two PL manifolds are bistellarly equivalent if and only if they are PL homeomorphic.*

The behavior of the face numbers of a triangulation under bistellar moves is easily controlled. Namely, the following statement holds.

THEOREM 2.41 (Pachner [**109**]). *Let L be a q-dimensional triangulated manifold obtained from K by applying a bistellar k-move, $0 \leqslant k \leqslant \left[\frac{q-1}{2}\right]$. Then*

$$g_{k+1}(L) = g_{k+1}(K) + 1;$$
$$g_i(l) = g_i(K) \quad \text{for all } i \neq k+1,$$

where $g_i(K) = h_i(K) - h_{i-1}(K)$, $0 < i \leqslant \left[\frac{n}{2}\right]$, are the components of the g-vector. Moreover, if q is even and $k = \left[\frac{q}{2}\right]$, then $g_i(L) = g_i(K)$ for all i.

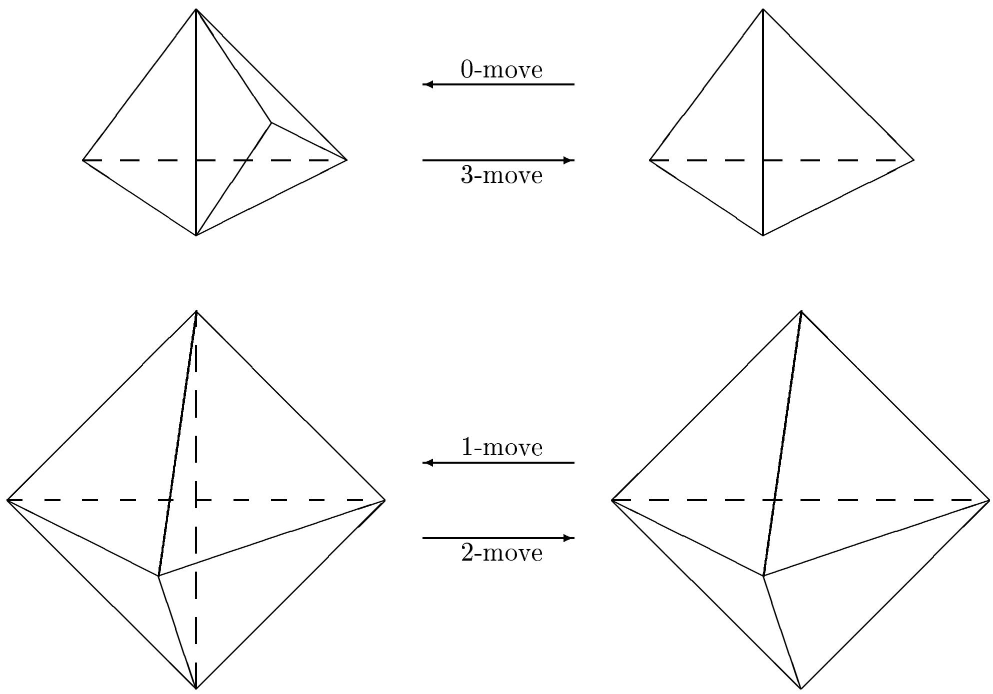


FIGURE 2.4. Bistellar moves for $q = 3$.

This theorem allows us to interpret the inequalities from the g-conjecture for PL spheres (see Theorem 1.29) in terms of the numbers of bistellar k-moves needed to transform the given PL sphere to the boundary of a simplex. For instance, the inequality $h_1 \leqslant h_2$, $n \geqslant 4$, is equivalent to the statement that the number of 1-moves in the sequence of bistellar moves taking a given $(n-1)$-dimensional PL sphere to the boundary of an n-simplex is lesser than or equal to the number of reverse 1-moves. (Note that the g-vector of $\partial\Delta^n$ has the form $(1, 0, \ldots, 0)$.)

REMARK. Pachner also proved an analogue of Theorem 2.40 for *PL manifolds with boundary*, see [**110**, (6.3)]. For this purpose he introduced another class of operations on triangulations, called *elementary shellings*.

CHAPTER 3

Commutative and homological algebra of simplicial complexes

The appearance of the Stanley–Reisner face ring of simplicial complex at the beginning of 1970s outlined a new approach to combinatorial problems concerning simplicial complexes. It relies upon the interpretation of combinatorial properties of simplicial complexes as algebraic properties of the corresponding face rings and uses commutative algebra machinery such as Cohen–Macaulay and Gorenstein algebras, local cohomology, etc. The main reference here is R. Stanley's monograph [**128**].

3.1. Stanley–Reisner face rings of simplicial complexes

Recall that $\mathbf{k}[v_1, \ldots, v_m]$ denotes the graded polynomial algebra over a commutative ring $\mathbf{k}$ with unit, $\deg v_i = 2$.

Definition 3.1. The *face ring* (or the *Stanley–Reisner ring*) of a simplicial complex K on the vertex set $[m]$ is the quotient ring

$$\mathbf{k}(K) = \mathbf{k}[v_1, \ldots, v_m]/\mathcal{I}_K,$$

where $\mathcal{I}_K$ is the homogeneous ideal generated by all square-free monomials $v_\sigma = v_{i_1} v_{i_2} \cdots v_{i_s}$ $(i_1 < \cdots < i_s)$ such that $\sigma = \{i_1, \ldots, i_s\}$ is not a simplex of K. The ideal $\mathcal{I}_K$ is called the *Stanley–Reisner ideal* of K.

Suppose P is a simple n-polytope, P^* its polar, and K_P the boundary of P^*. Then K_P is a polytopal simplicial $(n-1)$-sphere. The face ring of P from Definition 1.38 coincides with that of K_P from the above definition: $\mathbf{k}(P) = \mathbf{k}(K_P)$.

Example 3.2. 1. Let K be a 2-dimensional simplicial complex shown on Figure 3.1. Then

$$\mathcal{I}_K = (v_1v_5, v_3v_4, v_1v_2v_3, v_2v_4v_5).$$

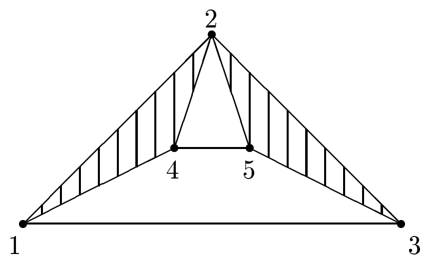


Figure 3.1

2. The Stanley–Reisner ring $\mathbf{k}(K)$ is a *quadratic algebra* (i.e. the ideal $\mathcal{I}_K$ is generated by quadratic monomials) if and only if K is a flag complex (see Definition 2.18 and compare with Proposition 2.19).

3. Let $K_1 * K_2$ be the join of K_1 and K_2 (see Construction 2.9). Then

$$\mathbf{k}(K_1 * K_2) = \mathbf{k}(K_1) \otimes \mathbf{k}(K_2).$$

In particular, for any two simple polytopes P_1 and P_2 we have

$$\mathbf{k}(P_1 \times P_2) = \mathbf{k}(P_1) \otimes \mathbf{k}(P_2)$$

(see Construction 1.12).

4. Let $K_1 \#_{\sigma_1,\sigma_2} K_2$ be the connected sum of two pure $(n-1)$-dimensional simplicial complexes on sets $\mathcal{S}_1$, $\mathcal{S}_2$ respectively, regarded as a simplicial complex on the set $\mathcal{S}_1 \cup_\sigma \mathcal{S}_2$ (see Construction 2.12). Then the ideal $\mathcal{I}_{K_1\#_{\sigma_1,\sigma_2}K_2}$ is generated by the ideals $\mathcal{I}_{K_1}$, $\mathcal{I}_{K_2}$ and the monomials v_σ and $v_{i_1}v_{i_2}$, where $i_1 \in \mathcal{S}_1 \setminus \sigma_1$ and $i_2 \in \mathcal{S}_2 \setminus \sigma_2$.

For any subset $\sigma = \{i_1, \ldots, i_k\} \subset [m]$ denote by v_σ the square-free monomial $v_{i_1} \cdots v_{i_k}$. Note that the ideal $\mathcal{I}_K$ is *monomial* and has basis of monomials v_σ corresponding to missing faces σ of K.

PROPOSITION 3.3. *Every square-free monomial ideal in the polynomial ring has the form $\mathcal{I}_K$ for some simplicial complex K.*

PROOF. Let $\mathcal{I}$ be a square-free monomial ideal. Set

$$K = \{\sigma \subset [m] \,:\, v_\sigma \notin \mathcal{I}\}.$$

Then one easily checks that K is a simplicial complex and $\mathcal{I} = \mathcal{I}_K$. □

PROPOSITION 3.4. *Let $\phi : K_1 \to K_2$ be a simplicial map (see Definition 2.7) between two simplicial complexes K_1 and K_2 on the vertex sets $[m_1]$ and $[m_2]$ respectively. Define the map $\phi^* : \mathbf{k}[w_1, \ldots, w_{m_2}] \to \mathbf{k}[v_1, \ldots, v_{m_1}]$ by*

$$\phi^*(w_j) := \sum_{\{i\}\in f^{-1}\{j\}} v_i.$$

Then ϕ^* descends to a homomorphism $\mathbf{k}(K_2) \to \mathbf{k}(K_1)$ (which we will also denote by ϕ^*).

PROOF. We have to check that $\phi^*(\mathcal{I}_{K_2}) \subset \mathcal{I}_{K_1}$. Suppose $\tau = \{j_1, \ldots, j_s\} \subset [m_2]$ is not a simplex of K_2. Then

$$\phi^*(w_{j_1} \cdots w_{j_s}) = \sum_{\{i_1\}\in\phi^{-1}\{j_1\},\ldots,\{i_s\}\in\phi^{-1}\{j_s\}} v_{i_1} \cdots v_{i_s}. \tag{3.1}$$

We claim that $\sigma = \{i_1, \ldots, i_s\}$ is not a simplex of K_1 for any monomial $v_{i_1} \cdots v_{i_s}$ in the right hand side of the above identity. Indeed, otherwise we would have $\phi(\sigma) = \tau \in K_2$ by the definition of simplicial map, which is impossible. Hence, the right hand side of (3.1) is in $\mathcal{I}_{K_1}$. □

EXAMPLE 3.5. The face ring of the barycentric subdivision K' of K is

$$\mathbf{k}(K') = \mathbf{k}[b_\sigma : \sigma \in K]/\mathcal{I}_{K'},$$

where b_σ is the polynomial generator corresponding to simplex $\sigma \in K$. We have the simplicial map $\nabla : K' \to K$ (see Example 2.16). Then it is easy to see that

$$\nabla^*(v_j) := \sum_{\sigma \in K\colon \min \sigma = j} b_\sigma.$$

for any generator $v_j \in \mathbf{k}(K)$.

EXAMPLE 3.6. The non-degenerate map $K' \to \Delta^{n-1}$ from Example 2.15 induces the following map of the corresponding Stanley–Reisner rings:

$$\begin{aligned} \mathbf{k}[v_1, \ldots, v_n] &\longrightarrow \mathbf{k}(K') \\ v_i &\longrightarrow \sum_{|\sigma|=i} b_\sigma. \end{aligned}$$

This defines a canonical $\mathbf{k}[v_1, \ldots, v_n]$-module structure in $\mathbf{k}(K')$.

DEFINITION 3.7. Let $M = M^0 \oplus M^1 \oplus \ldots$ be a graded $\mathbf{k}$-module. The series

$$F(M;t) = \sum_{i=0}^{\infty} (\dim_{\mathbf{k}} M^i) t^i$$

is called the *Poincaré series* of M.

REMARK. In the algebraic literature the series $F(M;t)$ is called the *Hilbert series* or *Hilbert–Poincaré series*.

The following lemma may be considered as an algebraic definition of the h-vector of a simplicial complex.

LEMMA 3.8 (Stanley [**128**, Theorem II.1.4]). *The Poincaré series of* $\mathbf{k}(K^{n-1})$ *can be calculated as*

$$F\big(\mathbf{k}(K^{n-1});t\big) = \sum_{i=-1}^{n-1} \frac{f_i t^{2(i+1)}}{(1-t^2)^{i+1}} = \frac{h_0 + h_1 t^2 + \cdots + h_n t^{2n}}{(1-t^2)^n},$$

where $(f_0, \ldots, f_{n-1})$ *is the* f*-vector and* $(h_0, \ldots, h_n)$ *is the* h*-vector of* K^{n-1}.

PROOF. Any monomial in $\mathbf{k}(K^{n-1})$ has the form $v_{i_1}^{\alpha_1} \cdots v_{i_{k+1}}^{\alpha_{k+1}}$, where $\{i_1, \ldots, i_{k+1}\}$ is a simplex of K^{n-1} and $\alpha_1, \ldots, \alpha_{k+1}$ are some positive integers. Thus, every k-simplex of K^{n-1} contributes the summand $\frac{t^{2(k+1)}}{(1-t^2)^{k+1}}$ to the Poincaré series, which proves the first identity. The second identity is an obvious corollary of (1.8). $\square$

EXAMPLE 3.9. 1. Let $K = \Delta^n$. Then $f_i = \binom{n+1}{i+1}$ for $-1 \leqslant i \leqslant n$, $h_0 = 1$ and $h_i = 0$ for $i > 0$. Since any subset of $[n+1]$ is a simplex of Δ^n, we have $\mathbf{k}(\Delta^n) = \mathbf{k}[v_1, \ldots, v_{n+1}]$ and $F(\mathbf{k}(\Delta^n);t) = (1-t^2)^{-(n+1)}$, which agrees with Lemma 3.8.

2. Let K be the boundary of an n-simplex. Then $h_i = 1$, $i = 0, 1, \ldots, n$, and $\mathbf{k}(K) = \mathbf{k}[v_1, \ldots, v_{n+1}]/(v_1 v_2 \cdots v_{n+1})$. By Lemma 3.8,

$$F\big(\mathbf{k}(K);t\big) = \frac{1 + t^2 + \cdots + t^{2n}}{(1-t^2)^n}.$$

3.2. Cohen–Macaulay rings and complexes

Here we suppose $\mathbf{k}$ is a field. Let A be a finitely-generated commutative graded algebra over $\mathbf{k}$. We also assume that A has only even-degree graded components, so it is commutative in both the usual and the graded sense.

DEFINITION 3.10. The *Krull dimension* of A (denoted $\mathrm{Kd}\,A$) is the maximal number of algebraically independent elements of A. A sequence $\theta_1,\dots,\theta_n$ of $n=\mathrm{Kd}\,A$ homogeneous elements of A is called an *hsop* (homogeneous system of parameters) if the Krull dimension of the quotient $A/(\theta_1,\dots,\theta_n)$ is zero. Equivalently, $\theta_1,\dots,\theta_n$ is an hsop if $n=\mathrm{Kd}\,A$ and A is a finitely-generated $\mathbf{k}[\theta_1,\dots,\theta_n]$-module. The elements of an hsop are algebraically independent.

LEMMA 3.11 (Noether normalization lemma). *For any finitely-generated graded algebra A there exists an hsop. If $\mathbf{k}$ is of zero characteristic and A is generated by degree-two elements, then a degree-two hsop can be chosen.*

In the case when A is generated by degree-two elements, a degree-two hsop is called an *lsop* (linear system of parameters).

REMARK. If $\mathbf{k}$ is of finite characteristic then an lsop may fail to exist for algebras generated in degree two, see Example 5.26 below.

In the rest of this chapter we assume that the field $\mathbf{k}$ is of zero characteristic.

DEFINITION 3.12. A sequence $\theta_1,\dots,\theta_k$ of homogeneous elements of A is called a *regular sequence* if θ_{i+1} is not a zero divisor in $A/(\theta_1,\dots,\theta_i)$ for $0\leqslant i<k$ (i.e. the multiplication by θ_{i+1} is a monomorphism of $A/(\theta_1,\dots,\theta_i)$ into itself). Equivalently, $\theta_1,\dots,\theta_k$ is a regular sequence if $\theta_1,\dots,\theta_k$ are algebraically independent and A is a *free* $\mathbf{k}[\theta_1,\dots,\theta_k]$-module.

REMARK. The concept of a regular sequence can be extended to non-finitely-generated graded algebras and to algebras over any integral domain. Regular sequences in graded polynomial rings $R[a_1,a_2,\dots,]$ on infinitely many generators, where $\deg a_i=-2i$ and R is a subring of the field $\mathbb{Q}$ of rationals, are used in the algebraic topology for constructing complex cobordism theories with coefficients, see [**89**].

Any two maximal regular sequences have the same length, which is called the *depth* of A and denoted $\operatorname{depth} A$. Obviously, $\operatorname{depth} A\leqslant \mathrm{Kd}\,A$.

DEFINITION 3.13. Algebra A is called *Cohen–Macaulay* if it admits a regular sequence $\theta_1,\dots,\theta_n$ of length $n=\mathrm{Kd}\,A$.

A regular sequence $\theta_1,\dots,\theta_n$ of length $n=\mathrm{Kd}\,A$ is an hsop. It follows that A is Cohen–Macaulay if and only if there exists an hsop $\theta_1,\dots,\theta_n$ such that A is a free $\mathbf{k}[\theta_1,\dots,\theta_n]$-module. If in addition A is generated by degree-two elements, then one can choose $\theta_1,\dots,\theta_n$ to be an lsop. In this case the following formula for the Poincaré series of A holds

$$F(A;t)=\frac{F\big(A/(\theta_1,\dots,\theta_n);t\big)}{(1-t^2)^n},$$

where $F(A/(\theta_1,\dots,\theta_n);t)=h_0+h_1t^2+\cdots$ is a polynomial. The finite vector $(h_0,h_1,\dots)$ is called the *h-vector* of A.

DEFINITION 3.14. A simplicial complex K^{n-1} is called *Cohen–Macaulay* (over $\mathbf{k}$) if its face ring $\mathbf{k}(K^{n-1})$ is Cohen–Macaulay.

Obviously, $\operatorname{Kd}\mathbf{k}(K^{n-1}) = n$. Lemma 3.8 shows that the h-vector of $\mathbf{k}(K)$ coincides with the h-vector of K.

EXAMPLE 3.15. Let K^1 be the boundary of a 2-simplex. Then $\mathbf{k}(K^1) = \mathbf{k}[v_1, v_2, v_3]/(v_1v_2v_3)$. The elements $v_1, v_2 \in \mathbf{k}(K)$ are algebraically independent, but do not form an hsop, since $\mathbf{k}(K)/(v_1, v_2) \cong \mathbf{k}[v_3]$ and $\operatorname{Kd}\mathbf{k}(K)/(v_1, v_2) = 1 \neq 0$. On the other hand, the elements $\theta_1 = v_1 - v_3$, $\theta_2 = v_2 - v_3$ of $\mathbf{k}(K)$ form an hsop, since $\mathbf{k}(K)/(\theta_1, \theta_2) \cong \mathbf{k}[t]/t^3$. It is easy to see that $\mathbf{k}(K)$ is a free $\mathbf{k}[\theta_1, \theta_2]$-module with one 0-dimensional generator 1, one 1-dimensional generator v_1, and one 2-dimensional generator v_1^2. Thus, $\mathbf{k}(K)$ is Cohen–Macaulay and (θ_1, θ_2) is a regular sequence.

THEOREM 3.16 (Stanley). *If K^{n-1} is a Cohen–Macaulay simplicial complex, then $\boldsymbol{h}(K^{n-1}) = (h_0, \dots, h_n)$ is an M-vector (see Definition 1.31).*

PROOF. Let $\theta_1, \dots, \theta_n$ be a regular sequence of degree-two elements of $\mathbf{k}(K)$. Then $A = \mathbf{k}(K)/(\theta_1, \dots, \theta_n)$ is a graded algebra generated by degree-two elements, and $\dim_{\mathbf{k}} A^{2i} = h_i$. Now the result follows from Theorem 1.32. □

The following fundamental theorem characterizes Cohen–Macaulay complexes combinatorially.

THEOREM 3.17 (Reisner [**117**]). *A simplicial complex K is Cohen–Macaulay over $\mathbf{k}$ if and only if for any simplex $\sigma \in K$ (including $\sigma = \varnothing$) and $i < \dim(\operatorname{link}\sigma)$, $\widetilde{H}_i(\operatorname{link}\sigma; \mathbf{k}) = 0$. (Here $\widetilde{H}_i(X; \mathbf{k})$ denotes the i-th reduced homology group of X with coefficients in $\mathbf{k}$.)*

COROLLARY 3.18. *A simplicial sphere is a Cohen–Macaulay complex.*

Theorem 3.16 shows that the h-vector of a simplicial sphere is an M-vector. This argument was used by Stanley to extend the UBC (Theorem 1.33) to simplicial spheres.

COROLLARY 3.19 (Upper Bound Theorem for spheres, Stanley [**123**]). *The h-vector $(h_0, h_1, \dots, h_n)$ of a simplicial $(n-1)$-sphere K^{n-1} with m vertices satisfies*

$$h_i(K^{n-1}) \leqslant \binom{m-n+i-1}{i}, \qquad 0 \leqslant i < \left[\tfrac{n}{2}\right].$$

Hence, the UBC holds for simplicial spheres, that is,

$$f_i(K^{n-1}) \leqslant f_i\bigl(C^n(m)\bigr) \quad \text{for } i = 2, \dots, n-1.$$

(see Corollary 1.35).

PROOF. Since $\boldsymbol{h}(K^{n-1})$ is an M-vector, there exists a graded algebra $A = A^0 \oplus A^2 \oplus \dots \oplus A^{2n}$ generated by degree-two elements such that $\dim_{\mathbf{k}} A^{2i} = h_i$ (Theorem 1.32). In particular, $\dim_{\mathbf{k}} A^2 = h_1 = m - n$. Since A is generated by A^2, the number h_i cannot exceed the total number of monomials of degree i in $(m-n)$ variables. The latter is exactly $\binom{m-n+i-1}{i}$. □

3.3. Homological algebra background

Here we review some homological algebra. Unless otherwise stated, all modules in this section are assumed to be finitely-generated graded $\mathbf{k}[v_1, \ldots, v_m]$-modules, $\deg v_i = 2$.

Definition 3.20. A *finite free resolution* of a module M is an exact sequence

$$0 \to R^{-h} \xrightarrow{d} R^{-h+1} \xrightarrow{d} \cdots \longrightarrow R^{-1} \xrightarrow{d} R^0 \xrightarrow{d} M \to 0, \tag{3.2}$$

where the R^{-i} are finitely-generated free modules and the maps d are degree-preserving. The minimal number h for which a free resolution (3.2) exists is called the *homological dimension* of M and denoted $\operatorname{hd} M$. By the Hilbert syzygy theorem a finite free resolution (3.2) exists and $\operatorname{hd} M \leqslant m$. A resolution (3.2) can be written as a free *bigraded differential module* $[R, d]$, where $R = \bigoplus R^{-i,j}$, $R^{-i,j} := (R^{-i})^j$ (the j-th graded component of the free module R^{-i}). The cohomology of $[R, d]$ is zero in non-zero dimensions and $H^0[R, d] = M$. Conversely, a free bigraded differential module $[R = \bigoplus_{i,j \geqslant 0} R^{-i,j},\ d : R^{-i,j} \to R^{-i+1,j}]$ with $H^0[R, d] = M$ and $H^{-i}[R, d] = 0$ for $i > 0$ defines a free resolution (3.2) with $R^{-i} := R^{-i,*} = \bigoplus_j R^{-i,j}$.

Remark. For the reasons specified below we numerate the terms of a free resolution by *non-positive* numbers, thereby viewing it as a *cochain* complex.

The Poincaré series of M can be calculated from any free resolution (3.2) by means of the following classical theorem.

Theorem 3.21. *Suppose that R^{-i} has rank q_i with free generators in degrees $d_{1i}, \ldots, d_{q_i i}$, $i = 1, \ldots, h$. Then*

$$F(M; t) = (1 - t^2)^{-m} \sum_{i=0}^{h} (-1)^i (t^{d_{1i}} + \cdots + t^{d_{q_i i}}). \tag{3.3}$$

Proof. By the definition of resolution, the following map of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & R^{-h} \arrow[r, "d"] \arrow[d, rightsquigarrow] & R^{-h+1} \arrow[r, "d"] \arrow[d, rightsquigarrow] & \cdots \arrow[r, "d"] & R^{-1} \arrow[r, "d"] \arrow[d, rightsquigarrow] & R^0 \arrow[r] \arrow[d, rightsquigarrow] & 0 \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

is a quasi-isomorphism, i.e. induces an isomorphism in the cohomology. Equating the Euler characteristics of both complexes in each degree we get (3.3) □

Construction 3.22. There is the following straightforward way to construct a free resolution for a module M. Take a set of generators $a_1, \ldots, a_{k_0}$ for M and define R^0 to be a free $\mathbf{k}[v_1, \ldots, v_m]$-module with k_0 generators in the corresponding degrees. There is an obvious epimorphism $R^0 \to M$. Then take a set of generators $a_1, \ldots, a_{k_1}$ in the kernel of $R^0 \to M$ and define R^{-1} to be a free $\mathbf{k}[v_1, \ldots, v_m]$-module with k_1 generators in the corresponding degrees, and so on. On the i-th step we take a set of generators in the kernel of the previously constructed map $d : R^{-i+1} \to R^{-i+2}$ and define R^{-i} to be a free module with the corresponding generators. The Hilbert syzygy theorem guarantees this process to end up at most at the m-th step.

EXAMPLE 3.23 (minimal resolution). For graded finitely generated modules M a *minimal generator set* (or a *minimal basis*) can be chosen. This is done as follows. Take the lowest degree in which M is non-zero and there choose a vector space basis. Span a module M_1 by this basis and then take the lowest degree in which $M \neq M_1$. In this degree choose a vector space basis in the complement of M_1, and span a module M_2 by this basis and M_1. Then continue this process. Since M is finitely generated, on some p-th step we get $M = M_p$ and a basis for M with a minimal number of generators.

If we take a minimal set of generators for modules at each step of Construction 3.22, then the produced resolution is called *minimal*. Each of its terms R^{-i} has the smallest possible rank (see Example 3.26 below). There is also the following more formal (but less convenient for particular computations) definition of minimal resolution (see [**2**]). Let M, M' be two modules. Set $\mathcal{J}(M) = v_1M + v_2M + \cdots + v_mM \subset M$. A map $f : M \to M'$ is called *minimal* if $\operatorname{Ker} f \subset \mathcal{J}(M)$. A resolution (3.2) is called *minimal* if all maps d are minimal. A minimal resolution is unique up to an isomorphism.

EXAMPLE 3.24 (Koszul resolution). Let $M = \mathbf{k}$. The $\mathbf{k}[v_1,\dots,v_m]$-module structure on $\mathbf{k}$ is defined via the map $\mathbf{k}[v_1,\dots,v_m] \to \mathbf{k}$ that sends each v_i to 0. Let $\Lambda[u_1,\dots,u_m]$ denote the exterior algebra on m generators. Turn the tensor product $R = \Lambda[u_1,\dots,u_m] \otimes \mathbf{k}[v_1,\dots,v_m]$ (here and below we use $\otimes$ for $\otimes_{\mathbf{k}}$) into a *differential bigraded algebra* by setting

$$\begin{gathered} \operatorname{bideg} u_i = (-1,2), \quad \operatorname{bideg} v_i = (0,2), \\ du_i = v_i, \quad dv_i = 0, \end{gathered} \tag{3.4}$$

and requiring that d be a derivation of algebras. An explicit construction of cochain homotopy [**92**, §7.2] shows that $H^{-i}[R,d] = 0$ for $i > 0$ and $H^0[R,d] = \mathbf{k}$. Since $\Lambda[u_1,\dots,u_m] \otimes \mathbf{k}[v_1,\dots,v_m]$ is a free $\mathbf{k}[v_1,\dots,v_m]$-module, it determines a free resolution of $\mathbf{k}$. This resolution is known as the *Koszul resolution*. Its expanded form is as follows:

$$\begin{aligned} 0 \to \Lambda^m[u_1,\dots,u_m] \otimes \mathbf{k}[v_1,\dots,v_m] \longrightarrow \cdots& \\ \longrightarrow \Lambda^1[u_1,\dots,u_m] \otimes \mathbf{k}[v_1,\dots,v_m] &\longrightarrow \mathbf{k}[v_1,\dots,v_m] \longrightarrow \mathbf{k} \to 0, \end{aligned}$$

where $\Lambda^i[u_1,\dots,u_m]$ is the submodule of $\Lambda[u_1,\dots,u_m]$ spanned by monomials of length i. Thus, in the notation of (3.2) we have $R^{-i} = \Lambda^i[u_1,\dots,u_m] \otimes \mathbf{k}[v_1,\dots,v_m]$.

Let N be another module; then applying the functor $\otimes_{\mathbf{k}[v_1,\dots,v_m]}N$ to (3.2) we obtain the following cochain complex of graded modules:

$$0 \longrightarrow R^{-h} \otimes_{\mathbf{k}[v_1,\dots,v_m]} N \longrightarrow \cdots \longrightarrow R^0 \otimes_{\mathbf{k}[v_1,\dots,v_m]} N \longrightarrow 0$$

and the corresponding bigraded differential module $[R \otimes N, d]$. The $(-i)$-th cohomology module of the above cochain complex is denoted $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$, i.e.

$$\begin{aligned} \operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N) &:= H^{-i}[R \otimes_{\mathbf{k}[v_1,\dots,v_m]} N, d] \\ &= \frac{\operatorname{Ker}[d : R^{-i} \otimes_{\mathbf{k}[v_1,\dots,v_m]} N \to R^{-i+1} \otimes_{\mathbf{k}[v_1,\dots,v_m]} N]}{d(R^{-i-1} \otimes_{\mathbf{k}[v_1,\dots,v_m]} N)}. \end{aligned}$$

Since both the R^{-i}'s and N are graded modules, we actually have

$$\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)=\bigoplus_j \operatorname{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}(M,N),$$

where

$$\operatorname{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)=\frac{\operatorname{Ker}\bigl[d:(R^{-i}\otimes_{\mathbf{k}[v_1,\dots,v_m]}N)^j\to(R^{-i+1}\otimes_{\mathbf{k}[v_1,\dots,v_m]}N)^j\bigr]}{d(R^{-i-1}\otimes_{\mathbf{k}[v_1,\dots,v_m]}N)^j}.$$

The above modules combine to a *bigraded* $\mathbf{k}[v_1,\dots,v_m]$-module,

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)=\bigoplus_{i,j}\operatorname{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}(M,N).$$

The following properties of $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$ are well known (see e.g. [**92**]).

PROPOSITION 3.25. (a) *The module* $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$ *does not depend, up to isomorphism, on a choice of resolution* (3.2).

(b) *Both* $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(\,\cdot\,,N)$ *and* $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,\,\cdot\,)$ *are covariant functors.*

(c) $\operatorname{Tor}^{0}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)\cong M\otimes_{\mathbf{k}[v_1,\dots,v_m]}N$.

(d) $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)\cong\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(N,M)$.

In homological algebra A-modules $\operatorname{Tor}_A(M,N)$ are defined for algebras A far more general than polynomial rings (and finitely-generatedness assumption for modules M and N may be also dropped). Although a finite A-free resolution (3.2) of M may fail to exist in general, there is always a *projective* resolution, which allows one to define $\operatorname{Tor}_A(M,N)$ in the same way as above. Note that projective modules over the polynomial algebra are free. In the non-graded case this was known as the *Serre problem*, now solved by Quillen and Suslin. However the graded version of this fact is much easier to prove. In this text the Tor-modules $\operatorname{Tor}_A(M,N)$ over algebras different from the polynomial ring appear only in sections 7.1 and 8.3.

3.4. Homological properties of face rings: Tor-algebras and Betti numbers

Here we apply general constructions from the previous section in the case when $M=\mathbf{k}(K)$ and $N=\mathbf{k}$. As usual, $K=K^{n-1}$ is assumed to be a simplicial complex on $[m]$. Since $\deg v_i=2$, we have

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr)=\bigoplus_{i,j=0}^{m}\operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr)$$

(i.e. $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k})$ is non-zero only in even second degrees). Define the *bigraded Betti numbers* of $\mathbf{k}(K)$ by

$$\beta^{-i,2j}\bigl(\mathbf{k}(K)\bigr):=\dim_{\mathbf{k}}\operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr),\qquad 0\leqslant i,j\leqslant m. \tag{3.5}$$

Suppose that (3.2) is a minimal free resolution of $M=\mathbf{k}(K)$ (Example 3.23). Then $R^0\cong\mathbf{k}[v_1,\dots,v_m]$ is a free $\mathbf{k}[v_1,\dots,v_m]$-module with one generator of degree 0. The basis of R^{-1} is a minimal generator set for $\mathcal{I}_K=\operatorname{Ker}[\mathbf{k}[v_1,\dots,v_m]\to\mathbf{k}(K)]$ and is represented by the missing faces of K. For each missing face $\{i_1,\dots,i_k\}$ of K denote by $v_{i_1,\dots,i_k}$ the corresponding basis element of R^{-1}. Then $\deg v_{i_1,\dots,i_k}=2k$

and the map $d : R^{-1} \to R^0$ takes $v_{i_1,\dots,i_k}$ to $v_{i_1}\cdots v_{i_k}$. Since the maps d in (3.2) are minimal, the differentials in the cochain complex

$$0 \longrightarrow R^{-h} \otimes_{\mathbf{k}[v_1,\dots,v_m]} \mathbf{k} \longrightarrow \cdots \longrightarrow R^0 \otimes_{\mathbf{k}[v_1,\dots,v_m]} \mathbf{k} \longrightarrow 0$$

are trivial. Hence, for the minimal resolution of $\mathbf{k}(K)$ it holds that

$$\begin{aligned} \operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K),\mathbf{k}\big) &\cong R^{-i} \otimes_{\mathbf{k}[v_1,\dots,v_m]} \mathbf{k}, \\ \beta^{-i,2j}\big(\mathbf{k}(K)\big) &= \operatorname{rank} R^{-i,2j}. \end{aligned} \tag{3.6}$$

Example 3.26. Let K^1 be the boundary of a square. Then

$$\mathbf{k}(K^1) \cong \mathbf{k}[v_1,\dots,v_4]/(v_1v_3, v_2v_4).$$

Let us construct a minimal resolution of $\mathbf{k}(K^1)$ using Construction 3.22. The module R^0 has one generator 1 (of degree 0), and the map $R^0 \to \mathbf{k}(K^1)$ is the quotient projection. Its kernel is the ideal $\mathcal{I}_{K^1}$, and the minimal basis consists of two monomials v_1v_3 and v_2v_4. Hence, R^{-1} has two free generators of degree 4, denoted v_{13} and v_{24}, and the map $d : R^{-1} \to R^0$ sends v_{13} to v_1v_3 and v_{24} to v_2v_4. The minimal basis for the kernel of $R^{-1} \to R^0$ consists of one element $v_2v_4v_{13} - v_1v_3v_{24}$. Hence, R^{-2} has one generator of degree 8, say a, and the map $d : R^{-2} \to R^{-1}$ is injective and sends a to $v_2v_4v_{13} - v_1v_3v_{24}$. Thus, we have the minimal resolution

$$0 \longrightarrow R^{-2} \longrightarrow R^{-1} \longrightarrow R^0 \longrightarrow M \longrightarrow 0,$$

where $\operatorname{rank} R^0 = \beta^{0,0}(\mathbf{k}(K^1)) = 1$, $\operatorname{rank} R^{-1} = \beta^{-1,4}(\mathbf{k}(K^1)) = 2$, $\operatorname{rank} R^{-2} = \beta^{-2,8}(\mathbf{k}(K^1)) = 1$.

The Betti numbers $\beta^{-i,2j}(\mathbf{k}(K))$ are important combinatorial invariants of simplicial complex K, see [**128**]. The following theorem (which was proved by combinatorial methods) reduces the calculation of $\beta^{-i,2j}(\mathbf{k}(K))$ to calculating the homology groups of subcomplexes of K.

Theorem 3.27 (Hochster [**76**] or [**128**, Theorem 4.8]). *We have*

$$\beta^{-i,2j}\big(\mathbf{k}(K)\big) = \sum_{\sigma\subset[m]:\,|\sigma|=j} \dim_{\mathbf{k}} \widetilde{H}_{j-i-1}(K_\sigma),$$

where K_σ is the full subcomplex of K corresponding to σ, see (2.1). *We assume $\widetilde{H}_{-1}(\varnothing) = \mathbf{k}$ above.*

Example 3.28. Again, let K^1 be the boundary of a square, so $m = 4$. This time we calculate the Betti numbers $\beta^{-i,2j}(\mathbf{k}(K))$ using Hochster's theorem. Among two-element subsets of $[m]$ there are four simplices and two non-simplices, namely, $\{1,3\}$ and $\{2,4\}$. Simplices contribute trivially to the sum for $\beta^{-1,4}(\mathbf{k}(K))$, while each of the two non-simplices contributes 1, hence, $\beta^{-1,4}(\mathbf{k}(K)) = 2$. Further, each of the four full subcomplexes corresponding to three-element subsets of $[m]$ is contractible, hence, its reduced homology vanishes and $\beta^{-i,6}(\mathbf{k}(K)) = 0$ for any i. Finally, the full subcomplex K_σ with $|\sigma| = 4$ is K itself, hence $\beta^{-i,8}(\mathbf{k}(K)) = \dim_{\mathbf{k}} \widetilde{H}_{4-i-1}(K_\sigma)$. The latter equals 1 for $i = 2$ and zero otherwise.

In chapter 7 we show that $\beta^{-i,2j}(\mathbf{k}(K))$ equals the corresponding bigraded Betti number of the moment-angle complex $\mathcal{Z}_K$ associated to simplicial complex K. This provides an alternative (topological) way for calculating the numbers $\beta^{-i,2j}(\mathbf{k}(K))$.

Now we turn to the Koszul resolution (Example 3.24).

LEMMA 3.29. *For any module M it holds that*

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}) \cong H\big[\Lambda[u_1,\dots,u_m]\otimes M, d\big],$$

where $H[\Lambda[u_1,\dots,u_m]\otimes M, d]$ is the cohomology of the bigraded differential module $\Lambda[u_1,\dots,u_m]\otimes M$ and d is defined as in (3.4).

PROOF. Using the Koszul resolution $[\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[v_1,\dots,v_m],d]$ in the definition of $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k},M)$, we calculate

$$\begin{aligned}&\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k},M)\\ &= H\big[\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[v_1,\dots,v_m]\otimes_{\mathbf{k}[v_1,\dots,v_m]} M\big] \cong H\big[\Lambda[u_1,\dots,u_m]\otimes M\big].\end{aligned}$$

□

COROLLARY 3.30. *Suppose that a $\mathbf{k}[v_1,\dots,v_m]$-module M is an algebra, then $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k})$ is canonically a finite-dimensional bigraded $\mathbf{k}$-algebra.*

PROOF. It is easy to see that in this case the tensor product $\Lambda[u_1,\dots,u_m]\otimes M$ is a differential algebra, and $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k})$ is its cohomology by Lemma 3.29. □

DEFINITION 3.31. The bigraded algebra $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k})$ is called the Tor-*algebra* of algebra M. If $M=\mathbf{k}(K)$ then it is called the Tor-*algebra of simplicial complex K*.

REMARK. For general $N\neq\mathbf{k}$ the module $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$ has no canonical multiplicative structure even if both M and N are algebras.

LEMMA 3.32. *A simplicial map $\phi : K_1\to K_2$ between two simplicial complexes on the vertex sets $[m_1]$ and $[m_2]$ respectively induces a homomorphism*

$$\phi_t^* : \operatorname{Tor}_{\mathbf{k}[w_1,\dots,w_{m_2}]}\big(\mathbf{k}(K_2),\mathbf{k}\big)\to \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_{m_1}]}\big(\mathbf{k}(K_1),\mathbf{k}\big) \tag{3.7}$$

of the corresponding Tor-*algebras.*

PROOF. This follows directly from Propositions 3.4 and 3.25 (b). □

CONSTRUCTION 3.33 (multigraded structure in the Tor-algebra). We may invest the polynomial ring $\mathbf{k}[v_1,\dots,v_m]$ with a multigrading (more precisely, $\mathbb{N}^m$-grading) by setting $\operatorname{mdeg} v_i=(0,\dots,0,2,0,\dots,0)$ where 2 stands at the i-th place. Then the multidegree of monomial $v_1^{i_1}\cdots v_m^{i_m}$ is $(2i_1,\dots,2i_m)$. Suppose that algebra M is a quotient of the polynomial ring by a monomial ideal. Then the multigraded structure descends to M and to the terms of resolution (3.2). We may assume that the differentials in the resolution preserve the multidegrees. Then the module $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$ acquires a canonical $\mathbb{N}\oplus\mathbb{N}^m$-grading, i.e.

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}) = \bigoplus_{i\geqslant 0,\, \boldsymbol{j}\in\mathbb{N}^m}\operatorname{Tor}^{-i,\boldsymbol{j}}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}).$$

In particular, the Tor-algebra of K is canonically an $\mathbb{N}\oplus\mathbb{N}^m$-graded algebra.

REMARK. According to our agreement, the first degree in the Tor-algebra is *non-positive*. (Remember that we numerated the terms of $\mathbf{k}[v_1,\dots,v_m]$-free Koszul resolution of $\mathbf{k}$ by non-positive integers.) In these notations the Koszul complex $[M\otimes\Lambda[u_1,\dots,u_m],d]$ becomes a *cochain* complex, and $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k})$ is its *cohomology*, not the homology as usually regarded. One of the reasons for such an agreement is that $\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k})$ is a contravariant functor from the category of simplicial complexes and simplicial maps, see Lemma 3.32. It also explains our notation $\mathrm{Tor}^{*,*}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k})$, used instead of the usual $\mathrm{Tor}^{\mathbf{k}[v_1,\dots,v_m]}_{*,*}(M,\mathbf{k})$. These notations are convenient for working with Eilenberg–Moore spectral sequences, see section 7.1.

The upper bound $\operatorname{hd} M \leqslant m$ from the Hilbert syzygy theorem can be replaced by the following sharper result.

THEOREM 3.34 (Auslander and Buchsbaum). $\operatorname{hd} M = m - \operatorname{depth} M$.

In particular, if $M = \mathbf{k}(K^{n-1})$ and K^{n-1} is Cohen–Macaulay (see Definition 3.14), then $\operatorname{hd}\mathbf{k}(K) = m - n$ and $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k}) = 0$ for $i > m - n$.

From now on we assume that M is generated by degree-two elements and the $\mathbf{k}[v_1,\dots,v_m]$-module structure in M is defined through an epimorphism $p : \mathbf{k}[v_1,\dots,v_m] \to M$ (both assumptions are satisfied by definition for $M = \mathbf{k}(K)$). Suppose that $\theta_1,\dots,\theta_k$ is a regular sequence of degree-two elements of M. Let $\mathcal{J} := (\theta_1,\dots,\theta_k) \subset M$ be the ideal generated by $\theta_1,\dots,\theta_k$. Choose degree-two elements $t_i \in \mathbf{k}[v_1,\dots,v_m]$ such that $p(t_i) = \theta_i$, $i = 1,\dots,k$. The ideal in $\mathbf{k}[v_1,\dots,v_m]$ generated by $t_1,\dots,t_k$ will be also denoted by $\mathcal{J}$. Then we have $\mathbf{k}[v_1,\dots,v_m]/\mathcal{J} \cong \mathbf{k}[w_1,\dots,w_{m-k}]$. Under these assumptions we have the following reduction lemma.

LEMMA 3.35. *The following isomorphism of algebras holds for any ideal $\mathcal{J}$ generated by a regular sequence:*

$$\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}) = \mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]/\mathcal{J}}(M/\mathcal{J},\mathbf{k}).$$

In order to prove the lemma we need the following fact from homological algebra.

THEOREM 3.36 ([**40**, p. 349]). *Let Λ be an algebra, ? its subalgebra, and $\Omega = \Lambda//?$ the quotient algebra. Suppose that Λ is a free ?-module and we are given an Ω-module A and a Λ-module C. Then there exists a spectral sequence $\{E_r, d_r\}$ such that*

$$E_r \Rightarrow \mathrm{Tor}_\Lambda(A,C), \quad E_2 = \mathrm{Tor}_\Omega\bigl(A, \mathrm{Tor}_\Gamma(C,\mathbf{k})\bigr).$$

PROOF OF LEMMA 3.35. Set $\Lambda = \mathbf{k}[v_1,\dots,v_m]$, ? $= \mathbf{k}[t_1,\dots,t_k]$, $A = \mathbf{k}$, $C = M$. Then Λ is a free ?-module and $\Omega = \Lambda//? = \mathbf{k}[v_1,\dots,v_m]/\mathcal{J}$. Therefore, Theorem 3.36 gives a spectral sequence

$$E_r \Rightarrow \mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}), \quad E_2 = \mathrm{Tor}_\Omega\bigl(\mathrm{Tor}_\Gamma(M,\mathbf{k}),\mathbf{k}\bigr).$$

Since $\theta_1,\dots,\theta_k$ is a regular sequence, M is a free ?-module. Therefore,

$$\mathrm{Tor}_\Gamma(M,\mathbf{k}) = M \otimes_\Gamma \mathbf{k} = M/\mathcal{J} \quad \text{and} \quad \mathrm{Tor}^q_\Gamma(M,\mathbf{k}) = 0 \quad \text{for} \quad q \neq 0.$$

It follows that $E_2^{p,q} = 0$ for $q \neq 0$. Thus, the spectral sequence collapses at the E_2 term, and we have

$$\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}) = \mathrm{Tor}_\Omega\bigl(\mathrm{Tor}_\Gamma(M,\mathbf{k}),\mathbf{k}\bigr) = \mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]/\mathcal{J}}(M/\mathcal{J},\mathbf{k}),$$

which concludes the proof. ∎

3.5. Gorenstein complexes and Dehn–Sommerville equations

It follows from Theorem 3.34 that if M is Cohen–Macaulay of Krull dimension n, then $\operatorname{depth} M = n$, $\operatorname{hd} M = m - n$, and $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}) = 0$ for $i > m - n$.

DEFINITION 3.37. Suppose M is a Cohen–Macaulay algebra of Krull dimension n. Then M is called a *Gorenstein algebra* if $\operatorname{Tor}^{-(m-n)}_{\mathbf{k}[v_1,\dots,v_m]}(M,\mathbf{k}) \cong \mathbf{k}$.

Following Stanley [**128**], we call a simplicial complex K *Gorenstein* if $\mathbf{k}(K)$ is a Gorenstein algebra. Further, K is called *Gorenstein** if $\mathbf{k}(K)$ is Gorenstein and $K = \operatorname{core} K$ (see Definition 2.24). The following theorem characterizes Gorenstein* simplicial complexes.

THEOREM 3.38 ([**128**, §II.5]). *A simplicial complex K is Gorenstein* over $\mathbf{k}$ if and only if for any simplex $\sigma \in K$ (including $\sigma = \varnothing$) the subcomplex* $\operatorname{link}\sigma$ *has the homology of a sphere of dimension* $\dim(\operatorname{link}\sigma)$.

In particular, simplicial spheres and simplicial homology spheres (triangulated manifolds with the homology of a sphere) are Gorenstein* complexes. However, the Gorenstein* property does not guarantee a complex to be a triangulated manifold (links of vertices are not necessarily simply connected, compare with Theorem 2.36). The Poincaré series of the Tor-algebra and the face ring of a Gorenstein* complex are "self dual" in the following sense.

THEOREM 3.39 ([**128**, §II.5]). *Suppose K^{n-1} is a Gorenstein* complex. Then the following identities hold for the Poincaré series of* $\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k})$, $0 \leqslant i \leqslant m-n$:

$$F\Bigl(\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr);\ t\Bigr) = t^{2m} F\Bigl(\operatorname{Tor}^{-(m-n)+i}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr);\ \tfrac{1}{t}\Bigr).$$

Equivalently,

$$\beta^{-i,2j}\bigl(\mathbf{k}(K)\bigr) = \beta^{-(m-n)+i,2(m-j)}\bigl(\mathbf{k}(K)\bigr), \quad 0 \leqslant i \leqslant m-n,\ 0 \leqslant j \leqslant m.$$

COROLLARY 3.40. *If K^{n-1} is Gorenstein* then*

$$F\bigl(\mathbf{k}(K),t\bigr) = (-1)^n F\bigl(\mathbf{k}(K),\tfrac{1}{t}\bigr).$$

PROOF. We apply Theorem 3.21 to a minimal resolution of $\mathbf{k}(K)$. It follows from (3.6) that the numerators of the summands in the right hand side of (3.3) are exactly $F\bigl(\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k});t\bigr)$, $i = 1,\dots,m-n$. Hence,

$$F\bigl(\mathbf{k}(K);t\bigr) = (1-t^2)^{-m} \sum_{i=0}^{m-n} (-1)^i F\Bigl(\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr);t\Bigr).$$

Using Theorem 3.39, we calculate

$$\begin{aligned} F\bigl(\mathbf{k}(K);t\bigr) &= (1-t^2)^{-m} \sum_{i=0}^{m-n} (-1)^i t^{2m} F\Bigl(\operatorname{Tor}^{-(m-n)+i}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr);\tfrac{1}{t}\Bigr) \\ &= \bigl(1-(\tfrac{1}{t})^2\bigr)^{-m}(-1)^m \sum_{j=0}^{m-n} (-1)^{m-n-j} F\Bigl(\operatorname{Tor}^{-j}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr);\tfrac{1}{t}\Bigr) \\ &= (-1)^n F\bigl(\mathbf{k}(K);\tfrac{1}{t}\bigr). \end{aligned}$$

$\square$

COROLLARY 3.41. *The Dehn–Sommerville relations $h_i = h_{n-i}$, $0 \leqslant i \leqslant n$, hold for any Gorenstein* complex K^{n-1} (in particular, for any simplicial sphere).*

PROOF. This follows from Lemma 3.8 and Corollary 3.40. □

As it was pointed out by Stanley in [**127**], Gorenstein* complexes are the most appropriate candidates for generalizing the g-theorem to. (As we have seen, polytopal spheres, PL spheres, simplicial spheres and simplicial homology spheres are examples of Gorenstein* complexes.)

The Dehn–Sommerville equations can be generalized even beyond Gorenstein* complexes. In [**86**] Klee reproved the f-vector version (1.10) of the Dehn–Sommerville equations in the more general context of *Eulerian complexes*. (A pure simplicial complex K^{n-1} is called Eulerian if for any simplex $\sigma \in K$, including $\varnothing$, holds $\chi(\operatorname{link}\sigma) = \chi(S^{n-|\sigma|-1}) = 1 + (-1)^{n-|\sigma|-1}$.) Generalizations of equations (1.10) were obtained by Bayer and Billera [**17**] (for *Eulerian posets*) and Chen and Yan [**42**] (for arbitrary polyhedra).

In section 7.6 we deduce the generalized Dehn–Sommerville equations for triangulated manifolds as a consequence of the bigraded Poincaré duality for moment-angle complexes. In particular, this gives the following short form of the equations in terms of the h-vector:

$$h_{n-i} - h_i = (-1)^i\big(\chi(K^{n-1}) - \chi(S^{n-1})\big)\tbinom{n}{i}, \quad i = 0, 1, \ldots, n.$$

Here $\chi(K^{n-1}) = f_0 - f_1 + \ldots + (-1)^{n-1}f_{n-1} = 1 + (-1)^{n-1}h_n$ is the Euler characteristic of K^{n-1} and $\chi(S^{n-1}) = 1 + (-1)^{n-1}$ is that of a sphere. Note that the above equations reduce to the classical $h_{n-i} = h_i$ in the case when K is a simplicial sphere or has odd dimension.

CHAPTER 4

Cubical complexes

At some stage of development of the combinatorial topology, cubical complexes were considered as an alternative to triangulations, a new way to study topological invariants combinatorially. Later it turned out, however, that the cubical (co)homomology itself is not very advantageous in comparison with the simplicial one. Nevertheless, as we see below, cubical complexes as particular combinatorial structures are very helpful in different geometrical and topological considerations.

4.1. Definitions and cubical maps

A q-dimensional *topological cube* as a q-ball with a face structure defined by a homeomorphism with the standard q-cube I^q. A *face* of a topological q-cube is thus the homeomorphic image of a face of I^q.

DEFINITION 4.1. A (finite topological) *cubical complex* is a subset $\mathcal{C} \subset \mathbb{R}^n$ represented as a finite union U of topological cubes of any dimensions, called *faces*, in such a way that the following two conditions are satisfied:

(a) Each face of a cube in U belongs to U;
(b) The intersection of any two cubes in U is a face of each.

The *dimension* of $\mathcal{C}$ is the maximal dimension of its faces. The *f-vector* of a cubical complex $\mathcal{C}$ is $\boldsymbol{f}(\mathcal{C}) = (f_0, f_1, \dots)$, where f_i is the number of i-faces.

REMARK. The above definition of cubical complex is a weaker cubical version of Definition 2.2 of geometrical simplicial complex. If we replace "topological cubes" in Definition 4.1 by "convex polytopes combinatorially equivalent to I^q", then we get the definition of a *combinatorial-geometrical cubical complex*, or *cubical polyhedron*. One can also define an *abstract cubical complex* as a poset (more precisely, a semilattice) such that each interval $[0, t]$ is isomorphic to the face lattice of a cube. We would not discuss here relationships between topological, geometrical and abstract cubical complexes, since all examples we need for our further constructions constitute a rather restricted family.

The theory of f-vectors of cubical complexes is parallel, to a certain extent, to that of simplicial complexes, but is much less developed. It includes the notions of h-vector, Cohen–Macaulay and Gorenstein* cubical complexes, and there are cubical analogues of the UBC, LBC and g-conjecture. See [**3**] and [**10**] for more details. A brief review of this theory and references can be found in [**129**, §2]. Since the combinatorial theory of cubical complexes is still in the early stage of its development, it may be helpful to look at some possible applications. It turns out that some particular problems from the discrete geometry and combinatorics of cubical complexes arise naturally in statistical physics, namely in connection with the *3-dimensional Ising model*. Since this aspect is not widely known to combinatorialists, we make a brief digression to the corresponding problems.

The standard unit cube $I^q = [0,1]^q$, together with all its faces, is a q-dimensional cubical complex, which we will also denote I^q. Unlike simplicial complexes, which are always realizable as subcomplexes in a simplex, not every cubical complex appears as a subcomplex of some I^q. One example of a cubical complex not embeddable as a subcomplex in any I^q is shown on Figure 4.1. Moreover, this complex is not embeddable into the standard cubical lattice in $\mathbb{R}^q$ (for any q). The authors are thankful to M. I. Shtogrin for presenting this example.

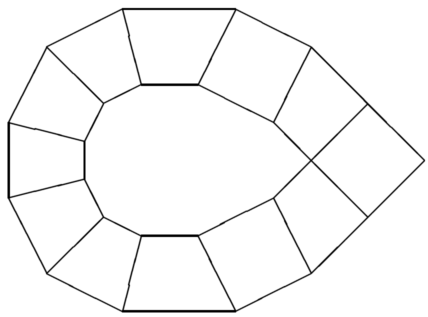


FIGURE 4.1. Cubical complex not embeddable into cubical lattice.

PROBLEM 4.2 (S. P. Novikov). *Characterize k-dimensional cubical complexes $\mathcal{C}$ (in particular, cubical manifolds) which admit*

(a) *a (cubical) embedding into the standard cubical lattice in $\mathbb{R}^q$;*

(b) *a map to the standard cubical lattice in $\mathbb{R}^q$ whose restriction to every k-dimensional cube is an isomorphism with a certain k-face of the lattice.*

In the case when $\mathcal{C}$ is homeomorphic to a 2-sphere the above problem was solved in [**57**]. Problem 4.2 is an extension of the following question, also formulated in [**57**].

PROBLEM 4.3 (S. P. Novikov). *Suppose we are given a 2-dimensional cubical* mod 2 *cycle α in the standard cubical lattice in $\mathbb{R}^3$. Describe all maps of cubical subdivisions of 2-dimensional surfaces onto α such that no two different squares are mapped to the same square of α.*

As it was told to the authors by S. P. Novikov, the above question was raised during his discussions with A. M. Polyakov on the 3-dimensional Ising model.

4.2. Cubical subdivisions of simple polytopes and simplicial complexes

Here we introduce some particular cubical complexes, which will play a pivotal rôle in our further constructions (in particular in the theory of moment-angle complexes). We make no claims for originality of constructions appearing in this section — most of them are part of mathematical folklore. At the end we give several references to the sources where some similar considerations can be found.

All cubical complexes discussed here admit a canonical cubical embedding into the standard cube. To conclude the discussion in the end of the previous section we note that the problem of embeddability into the cubical lattice is closely connected with that of embeddability into the standard cube. For instance, it is shown in [**57**] that if a cubical subdivision of a 2-dimensional surface is embeddable into the standard cubical lattice in $\mathbb{R}^q$, then it also admits a cubical embedding into I^q.

Any face of I^q can be written as

$$C_{\sigma\subset\tau} = \{(y_1,\dots,y_q)\in I^q \;:\; y_i = 0 \text{ for } i\in\sigma,\ y_i = 1 \text{ for } i\notin\tau\}, \tag{4.1}$$

where $\sigma\subset\tau$ are two (possibly empty) subsets of $[q]$. We set $C_\tau := C_{\varnothing\subset\tau}$.

CONSTRUCTION 4.4 (canonical simplicial subdivision of I^m). Let $\Delta = \Delta^{m-1}$ be the simplex on the set $[m]$, i.e. the collection of all subsets of $[m]$. Assign to each subset $\sigma = \{i_1,\dots,i_k\}\subset[m]$ the vertex $v_\sigma := C_{\sigma\subset\sigma}$ of I^m. More explicitly, $v_\sigma = (\varepsilon_1,\dots,\varepsilon_m)$, where $\varepsilon_i = 0$ if $i\in\sigma$ and $\varepsilon_i = 1$ otherwise. Regarding each σ as a vertex of the barycentric subdivision of Δ, we can extend the correspondence $\sigma\mapsto v_\sigma$ to a piecewise linear embedding of the barycentric subdivision Δ' into the (boundary complex of) standard cube I^m. Under this embedding, denoted i_c, the vertices of Δ are mapped to the vertices of I^m having only one zero coordinate, while the barycenter of Δ is mapped to $(0,\dots,0)\in I^m$ (see Figure 4.2). The image $i_c(\Delta')$ is the union of m facets of I^m meeting at the vertex $(0,\dots,0)$. For each pair $\sigma\subset\tau$ of non-empty subsets of $[m]$ all simplices of Δ' of the form $\sigma=\sigma_1\subset\sigma_2\subset\dots\subset\sigma_k=\tau$ are mapped to the same face $C_{\sigma\subset\tau}\subset I^m$. The map $i_c : \Delta'\to I^m$ extends to $\operatorname{cone}(\Delta')$ by taking the vertex of the cone to $(1,\dots,1)\in I^m$. We denote the resulting map by $\operatorname{cone}(i_c)$. Its image is the whole I^m. Hence, $\operatorname{cone}(i_c) : \operatorname{cone}(\Delta')\to I^m$ is a PL homeomorphism linear on the simplices of $\operatorname{cone}(\Delta')$. This defines a triangulation of I^m which coincides with the canonical triangulation of the product of m one-dimensional simplices, see Construction 2.11. It is also known as the "standard triangulation along the main diagonal".

In short, it can be said that the canonical triangulation of I^m arises from the identification of I^m with the cone over the barycentric subdivision of Δ^{m-1}.

CONSTRUCTION 4.5 (cubical subdivision of a simple polytope). Let $P^n\subset\mathbb{R}^n$ be a simple polytope with m facets $F_1^{n-1},\dots,F_m^{n-1}$. Choose a point in the relative interior of every face of P^n, including the vertices and the polytope itself. We get the set $\mathcal{S}$ of $1+f_0+f_1+\ldots+f_{n-1}$ points (here $\boldsymbol{f}(P^n)=(f_0,f_1,\dots,f_{n-1})$ is the f-vector of P^n). For each vertex $v\in P^n$ define the subset $\mathcal{S}_v\subset\mathcal{S}$ consisting of the points chosen inside the faces containing v. Since P^n is simple, the number of k-faces meeting at v is $\binom{n}{k}$, $0\leqslant k\leqslant n$. Hence, $|\mathcal{S}_v| = 2^n$. The set $\mathcal{S}_v$ will be the vertex set of an n-cube, which we denote C_v^n. The faces of C_v^n can be described as follows. Let G_1^k and G_2^l be two faces of P^n such that $v\in G_1^k\subset G_2^l$. Then there are exactly 2^{l-k} faces of P^n between G_1^k and G_2^l. The corresponding 2^{l-k} points from $\mathcal{S}$ form the vertex set of an $(l-k)$-face of C_v^n. We denote this face $C^{l-k}_{G_1\subset G_2}$. Every face of C_v^n is $C^i_{G_1\subset G_2}$ for some G_1, G_2 containing v. The intersection of any two cubes C_v^n, $C_{v'}^n$ is a face of each. Indeed, let $G^i\subset P^n$ be the smallest face containing both vertices v and v'. Then $C_v^n\cap C_{v'}^n = C^{n-i}_{G^i\subset P^n}$ is the face of both I_v^n and $I_{v'}^n$. Thereby we have constructed a cubical subdivision of P^n with $f_{n-1}(P^n)$ cubes of dimension n. We denote this cubical complex by $\mathcal{C}(P^n)$.

There is an embedding of $\mathcal{C}(P^n)$ to I^m constructed as follows. Every $(n-k)$-face of P^n is the intersection of k facets: $G^{n-k} = F_{i_1}^{n-1}\cap\ldots\cap F_{i_k}^{n-1}$. We map the corresponding point of $\mathcal{S}$ to the vertex $(\varepsilon_1,\dots,\varepsilon_m)\in I^m$ where $\varepsilon_i = 0$ if $i\in\{i_1,\dots,i_k\}$ and $\varepsilon_i = 1$ otherwise. This defines a mapping from the vertex set $\mathcal{S}$ of $\mathcal{C}(P^n)$ to the vertex set of I^m. Using the canonical triangulation of I^m from Construction 4.4, we extend this mapping to a PL embedding $i_P : P^n\to I^m$. For

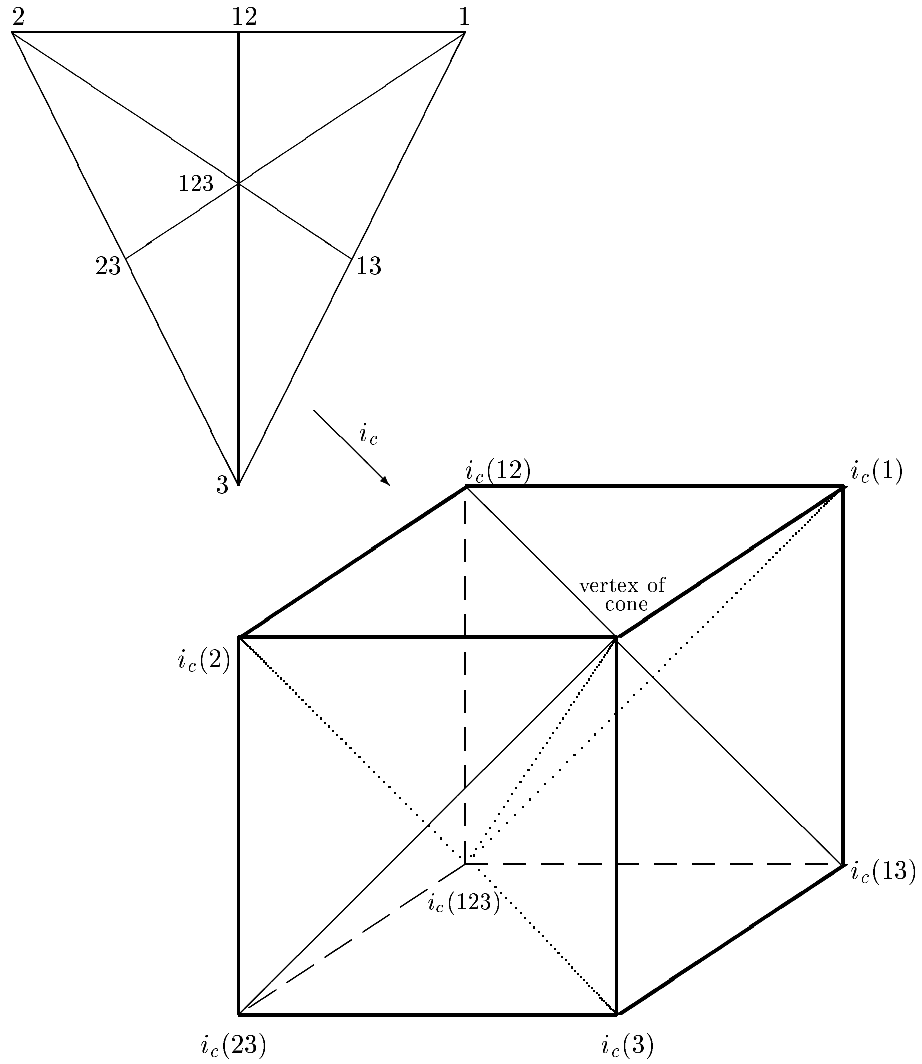


FIGURE 4.2. Cone over Δ' as the standard triangulation of cube.

each vertex $v = F_{i_1}^{n-1} \cap \cdots \cap F_{i_n}^{n-1} \in P^n$ we have

$$i_P(C_v^n) = \big\{(y_1, \dots, y_m) \in I^m \;:\; y_j = 1 \text{ for } j \notin \{i_1, \dots, i_n\}\big\}, \tag{4.2}$$

i.e. $i_P(C_v^n) = C_{\{i_1,\dots,i_n\}} \subset I^m$ in the notation of (4.1). The embedding $i_P : P^n \to I^m$ for $n = 2$, $m = 3$ is shown in Figure 4.3.

We summarize the facts from the above construction in the following statement.

PROPOSITION 4.6. *A simple polytope P^n with m facets can be split into cubes C_v^n, one for each vertex $v \in P^n$. The resulting cubical complex $\mathcal{C}(P^n)$ embeds canonically into the boundary of I^m, as described by* (4.2).

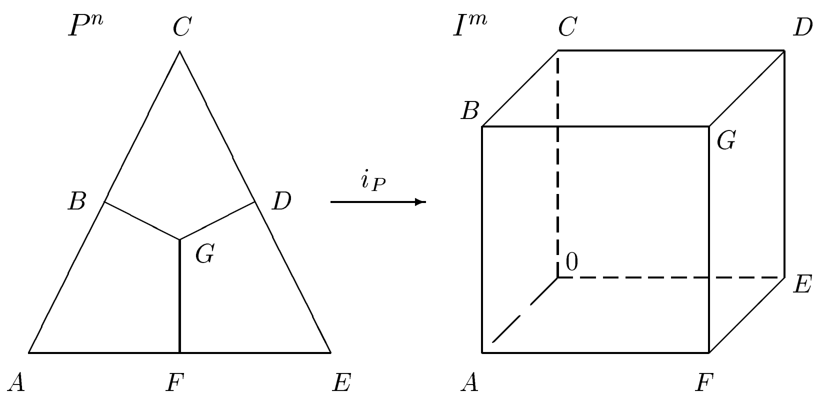


FIGURE 4.3. The embedding $i_P : P^n \to I^m$ for $n = 2$, $m = 3$.

LEMMA 4.7. *The number of k-faces of the cubical complex $\mathcal{C}(P^n)$ is given by*

$$\begin{aligned} f_k\bigl(\mathcal{C}(P^n)\bigr) &= \sum_{i=0}^{n-k} \tbinom{n-i}{k} f_{n-i-1}(P^n) \\ &= \tbinom{n}{k} f_{n-1}(P^n) + \tbinom{n-1}{k} f_{n-2}(P^n) + \dots + f_{k-1}(P^n), \quad k = 0, \dots, n. \end{aligned}$$

PROOF. This follows from the fact that the k-faces of $\mathcal{C}(P^n)$ are in one-to-one correspondence with the pairs $G_1^i \subset G_2^{i+k}$ of embedded faces of P^n. □

CONSTRUCTION 4.8. Let K^{n-1} be a simplicial complex on $[m]$. Then K is naturally a subcomplex of Δ^{m-1} and K' is a subcomplex of $(\Delta^{m-1})'$. As it follows from Construction 4.4, there is a PL embedding $i_c|_{K'} : |K'| \to I^m$. The image $i_c(|K'|)$ is an $(n-1)$-dimensional cubical subcomplex of I^m, which we denote $\mathrm{cub}(K)$. We have

$$\mathrm{cub}(K) = \bigcup_{\varnothing \neq \sigma \subset \tau \in K} C_{\sigma \subset \tau} \subset I^m, \tag{4.3}$$

i.e. $\mathrm{cub}(K)$ is the union of faces $C_{\sigma\subset\tau} \subset I^m$ over all pairs $\sigma \subset \tau$ of non-empty simplices of K.

CONSTRUCTION 4.9. Since $\mathrm{cone}(K')$ is a subcomplex of $\mathrm{cone}((\Delta^{m-1})')$, Construction 4.4 also provides a PL embedding

$$\mathrm{cone}(i_c)|_{\mathrm{cone}(K')} : |\,\mathrm{cone}(K')| \to I^m.$$

The image of this embedding is an n-dimensional cubical subcomplex of I^m, which we denote $\mathrm{cc}(K)$. It can be easily seen that

$$\mathrm{cc}(K) = \bigcup_{\tau \in K} C_{\sigma\subset\tau} = \bigcup_{\tau\in K} C_\tau \tag{4.4}$$

(the latter identity holds since $C_{\sigma\subset\tau} \subset C_{\varnothing\subset\tau} = C_\tau$).

REMARK. If $\{i\} \in [m]$ is not a vertex of K, then $\mathrm{cc}(K)$ is contained in the facet $\{y_i = 1\}$ of I^m.

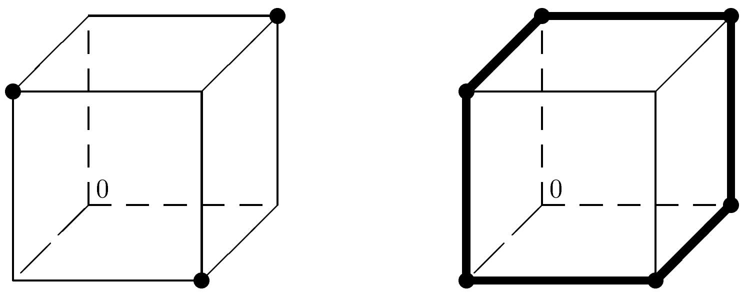
The following statement summarizes the results of two previous constructions.

PROPOSITION 4.10. *For any simplicial complex K on the set $[m]$ there is a PL embedding of the polyhedron $|K|$ into I^m linear on the simplices of K'. The image of this embedding is the cubical subcomplex* (4.3). *Moreover, there is a PL embedding of the polyhedron $|\operatorname{cone}(K)|$ into I^m linear on the simplices of* $\operatorname{cone}(K')$, *whose image is the cubical subcomplex* (4.4).

A cubical complex $\mathcal{C}'$ is called a *cubical subdivision* of cubical complex $\mathcal{C}$ if each cube of $\mathcal{C}$ is a union of finitely many cubes of $\mathcal{C}'$.

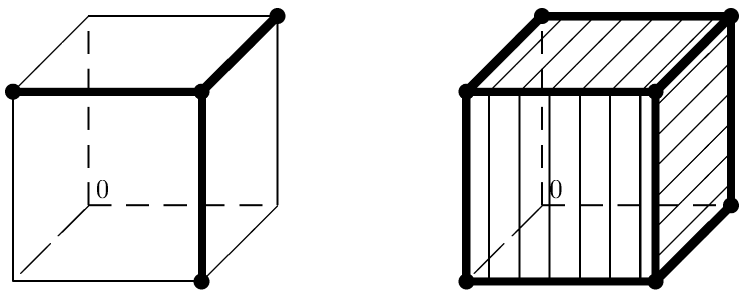
PROPOSITION 4.11. *For every cubical subcomplex $\mathcal{C}$ there exists a cubical subdivision that is embeddable into some I^q as a subcomplex.*

PROOF. Subdividing each cube of $\mathcal{C}$ as described in Construction 4.4 we obtain a simplicial complex, say $K_{\mathcal{C}}$. Then applying Construction 4.8 to $K_{\mathcal{C}}$ we get a cubical complex $\operatorname{cub}(K_{\mathcal{C}})$ that subdivides $K_{\mathcal{C}}$ and therefore $\mathcal{C}$. It is embeddable into some I^q by Proposition 4.10. □



(a) $K = 3$ points (b) $K = \partial\Delta^2$

FIGURE 4.4. The cubical complex $\operatorname{cub}(K)$.



(a) $K = 3$ points (b) $K = \partial\Delta^2$

FIGURE 4.5. The cubical complex $\operatorname{cc}(K)$.

EXAMPLE 4.12. The cubical complex $\operatorname{cub}(K)$ in the case when K is a disjoint union of 3 vertices is shown in Figure 4.4 (a). Figure 4.4 (b) shows that for the case $K = \partial\Delta^2$, the boundary complex of a 2-simplex. The corresponding cubical complexes $\operatorname{cc}(K)$ are indicated in Figure 4.5 (a) and (b).

REMARK. As a topological space, $\mathrm{cub}(K)$ is homeomorphic to $|K|$, while $\mathrm{cc}(K)$ is homeomorphic to $|\operatorname{cone}(K)|$. On the other hand, there is the cubical complex $\mathrm{cub}(\operatorname{cone}(K))$, also homeomorphic to $|\operatorname{cone}(K)|$. However, as *cubical complexes*, $\mathrm{cc}(K)$ and $\mathrm{cub}(\operatorname{cone}(K))$ differ (since $\operatorname{cone}(K') \neq (\operatorname{cone}(K))'$).

Let P be a simple n-polytope and K_P the corresponding simplicial $(n-1)$-sphere (the boundary of the polar simplicial polytope P^*). Then $\mathrm{cc}(K_P)$ coincides with the cubical complex $\mathcal{C}(P)$ from Construction 4.5. More precisely, $\mathrm{cc}(K_P) = i_P(\mathcal{C}(P))$. Thus, Construction 4.5 is a particular case of Construction 4.9 (compare Figures 4.2–4.5).

REMARK. Different versions of Construction 4.9 can be found in [**10**] and in some earlier papers listed there on p. 299. In [**48**, p. 434] a similar construction was introduced while studying certain toric spaces; we will return to this in the next chapters. A version of the cubical subcomplex $\mathrm{cub}(K) \subset I^m$ appeared in [**120**] in connection with Problem 4.2.

CHAPTER 5

Toric and quasitoric manifolds

5.1. Toric varieties

Toric varieties appeared in algebraic geometry in the beginning of 1970s in connection with compactification problems for algebraic torus actions. The geometry of toric varieties very quickly has become one of the most fascinating topics in algebraic geometry and found applications in many mathematical sciences, which otherwise seemed far from each other. We have already mentioned the proof for the "only if" part of the g-theorem for simplicial polytopes given by Stanley. Other remarkable applications include counting lattice points and volumes of lattice polytopes; relations with Newton polytopes and singularities (after Khovanskii and Kushnirenko); discriminants, resultants and hypergeometric functions (after Gelfand, Kapranov and Zelevinsky); reflexive polytopes and mirror symmetry for Calabi–Yau toric hypersurfaces (after Batyrev). Standard references in the toric geometry are Danilov's survey [**46**] and books by Oda [**105**], Fulton [**64**] and Ewald [**61**]. A more recent survey article by Cox [**45**] covers new applications, including those mentioned above. We are not going to give another review of the toric geometry here. Instead, in this section we stress upon some topological and combinatorial aspects of toric varieties. We also give Stanley's argument for the g-theorem.

5.1.1. Toric varieties and fans. Let $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ denote the multiplicative group of complex numbers. The product $(\mathbb{C}^*)^n$ of n copies of $\mathbb{C}^*$ is known as the torus in the theory of algebraic groups. In topology, the *torus* T^n is the product of n circles. We keep the topological notations, referring to $(\mathbb{C}^*)^n$ as the *algebraic torus*. The torus T^n is a subgroup of the algebraic torus $(\mathbb{C}^*)^n$ in the standard way:

$$T^n = \bigl\{\bigl(e^{2\pi i\varphi_1},\ldots,e^{2\pi i\varphi_n}\bigr)\in\mathbb{C}^n\bigr\}, \tag{5.1}$$

where $(\varphi_1,\ldots,\varphi_n)$ is running through $\mathbb{R}^n$.

Definition 5.1. A *toric variety* is a normal algebraic variety M containing the algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^*)^n$ on itself extends to an action on M.

Hence, $(\mathbb{C}^*)^n$ acts on M with a dense orbit.

One of the most beautiful properties of toric varieties is that all of their subtlest algebro-geometrical properties can be translated into the language of combinatorics and convex geometry. The following definition introduces necessary combinatorial notions.

Definition 5.2 (Fans terminology). Let $\mathbb{R}^n$ be the Euclidean space and $\mathbb{Z}^n\subset\mathbb{R}^n$ the integral lattice. Given a finite set of vectors $\boldsymbol{l}_1,\ldots,\boldsymbol{l}_s\in\mathbb{R}^n$, define the *convex polyhedral cone* σ spanned by $\boldsymbol{l}_1,\ldots,\boldsymbol{l}_s$ by

$$\sigma=\{r_1\boldsymbol{l}_1+\cdots+r_s\boldsymbol{l}_s\in\mathbb{R}^n\;:\;r_i\geqslant 0\}. \tag{5.2}$$

Any convex polyhedral cone is a convex polyhedron in the sense of Definition 1.2. Hence, the *faces* of a convex polyhedral cone are defined. A cone σ is *rational* if its generator vectors $\boldsymbol{l}_1,\ldots,\boldsymbol{l}_s$ can be taken from $\mathbb{Z}^n$ and is *strongly convex* if it contains no line through the origin. All cones considered below are strongly convex and rational. A cone is *simplicial* (respectively, *non-singular*) if it is generated by a part of a basis of $\mathbb{R}^n$ (respectively, $\mathbb{Z}^n$). A *fan* is a set Σ of cones in $\mathbb{R}^n$ such that each face of a cone in Σ is also a cone in Σ, and the intersection of two cones in Σ is a face of each. A fan Σ in $\mathbb{R}^n$ is called *complete* if the union of all cones from Σ is $\mathbb{R}^n$. A fan Σ is *simplicial* (respectively, *non-singular*) if all cones of Σ are simplicial (respectively, non-singular). Let Σ be a simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones (or *rays*). Choose generator vectors $\boldsymbol{l}_1,\ldots,\boldsymbol{l}_m$ for these rays to be integer and primitive, i.e. with relatively prime integer coordinates. The fan Σ defines a simplicial complex K_Σ on the vertex set $[m]$, which is called the *underlying complex* of Σ. By definition, $\{i_1,\ldots,i_k\}\subset[m]$ is a simplex of K_Σ if and only if $\boldsymbol{l}_{i_1},\ldots,\boldsymbol{l}_{i_k}$ span a cone of Σ. Obviously, Σ is complete if and only if K_Σ is a simplicial $(n-1)$-sphere.

As it is explained in any of the above mentioned sources, there is a one-to-one correspondence between fans in $\mathbb{R}^n$ and toric varieties of complex dimension n. We will denote the toric variety corresponding to a fan Σ by M_Σ. It follows that, in principle, all geometrical and topological properties of a toric variety can be retrieved from the combinatorics of the underlying fan.

The inclusion poset of $(\mathbb{C}^*)^n$-orbits of M_Σ is isomorphic to the poset of faces of Σ with reversed inclusion. That is, the k-dimensional cones of Σ correspond to the codimension-k orbits of the algebraic torus action on M_Σ. In particular, the n-dimensional cones correspond to the fixed points, while the origin corresponds to the unique dense orbit. The toric variety M_Σ is compact if and only if Σ is complete. If Σ is simplicial then M_Σ is an *orbifold* (i.e. is locally homeomorphic to the quotient of $\mathbb{R}^{2n}$ by a finite group action). Finally, M_Σ is non-singular (smooth) if and only if Σ is non-singular, which explains the notation. Smooth toric varieties sometimes are called *toric manifolds* in the algebraic geometry literature.

REMARK. Bistellar moves (see Definition 2.39) on the simplicial complex K_Σ can be interpreted as operations on the fan Σ. On the level of toric varieties, such an operation corresponds to a *flip* (a blow-up followed by a subsequent blow-down along different subvariety). This issue is connected with the question of factorization of a proper birational morphism between two complete smooth (or normal) algebraic varieties of dimension $\geqslant 3$ into a sequence of blow-ups and blow-downs with smooth centers, a fundamental problem in the birational algebraic geometry. Two versions of this problem are usually distinguished: the *Strong factorization conjecture*, which asks if it is possible to represent a birational morphism by a sequence of blow-ups followed by a sequence of blow-downs, and the *Weak factorization conjecture*, in which the order of blow-ups and blow-downs is insignificant. Since all toric varieties are rational, any two toric varieties of the same dimension are birationally equivalent. Weak (equivariant) factorization conjecture for smooth complete toric varieties was proved by Włodarczyk [**140**] (announced in 1991) using interpretation of equivariant flips on toric varieties as bistellar move-type operations on the corresponding fans. Thereby the weak factorization theorem for smooth toric varieties reduces to the statement that any two complete non-singular fans in $\mathbb{R}^n$ can be taken one to another by a finite sequence of bistellar move-type operations in which all intermediate fans are non-singular. This result is the essence of [**140**]. (Note that the statement does not reduce to Pachner's Theorem 2.40 because of the additional smoothness condition.) The equivariant toric strong factorization conjecture was proved by Morelli [**102**].

5.1.2. Cohomology of non-singular toric varieties. The Danilov–Jurkiewicz theorem allows us to read the integer cohomology ring of a non-singular toric variety directly from the underlying fan Σ. Write the primitive integer vectors along the rays of Σ in the standard basis of $\mathbb{Z}^n$:

$$\boldsymbol{l}_j = (l_{1j}, \dots, l_{nj})^t, \quad j = 1, \dots, m.$$

Assign to each vector $\boldsymbol{l}_j$ the indeterminate v_j of degree 2, and define linear forms

$$\theta_i := l_{i1}v_1 + \dots + l_{im}v_m \in \mathbb{Z}[v_1, \dots, v_m], \quad 1 \leqslant i \leqslant n.$$

Denote by $\mathcal{J}_\Sigma$ the ideal in $\mathbb{Z}[v_1, \dots, v_m]$ spanned by these linear forms, i.e. $\mathcal{J}_\Sigma = (\theta_1, \dots, \theta_n)$. The images of $\theta_1, \dots, \theta_n$ and $\mathcal{J}_\Sigma$ in the Stanley–Reisner ring $\mathbb{Z}(K_\Sigma) = \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}_{K_\Sigma}$ (see Definition 3.1) will be denoted by the same symbols.

THEOREM 5.3 (Danilov and Jurkiewicz). *Let Σ be a complete non-singular fan in $\mathbb{R}^n$, and M_Σ the corresponding toric variety. Then*

(a) *The Betti numbers (the ranks of homology groups) of M_Σ vanish in odd dimensions, while in even dimensions are given by*

$$b_{2i}(M_\Sigma) = h_i(K_\Sigma), \quad i = 0, 1, \dots, n,$$

where $\boldsymbol{h}(K_\Sigma) = (h_0, \dots, h_n)$ is the h-vector of K_Σ.

(b) *The cohomology ring of M_Σ is given by*

$$H^*(M_\Sigma; \mathbb{Z}) = \mathbb{Z}[v_1, \dots, v_m]/(\mathcal{I}_{K_\Sigma} + \mathcal{J}_\Sigma) = \mathbb{Z}(K_\Sigma)/\mathcal{J}_\Sigma,$$

where v_i, $1 \leqslant i \leqslant m$, denote the 2-dimensional cohomology classes dual to invariant divisors (codimension-two submanifolds) D_i corresponding to the rays of Σ. Moreover, $\theta_1, \dots, \theta_n$ is a regular sequence in $\mathbb{Z}(K_\Sigma)$.

This theorem was proved by Jurkiewicz [**82**] for projective smooth toric varieties and by Danilov [**46**, Theorem 10.8] in the general case. Note that the ideal $\mathcal{I}_{K_\Sigma}$ is determined only by the combinatorics of the fan (i.e. by the intersection poset of Σ), while $\mathcal{J}_\Sigma$ depends on the geometry of Σ. One can observe that the first part of Theorem 5.3 follows from the second part and Lemma 3.8.

REMARK. As it was shown by Danilov, the $\mathbb{Q}$-coefficient version of Theorem 5.3 is also true for simplicial fans and toric varieties.

It follows from Theorem 5.3 that the cohomology of M_Σ is generated by two-dimensional classes. This is the first thing to check if one wishes to determine whether or not a given algebraic variety or smooth manifold arises as a non-singular (or simplicial) toric variety. Another interesting algebraic-geometrical property of non-singular toric varieties, suggested by Theorem 5.3, is that the *Chow ring* [**64**, § 5.1] of M_Σ coincides with its integer cohomology ring.

5.1.3. Toric varieties from polytopes.

CONSTRUCTION 5.4 (Normal fan and toric varieties from polytopes). Suppose we are given an n-polytope (1.1) with vertices in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Such a polytope is called integral, or *lattice*. Then the vectors $\boldsymbol{l}_i$ in (1.1), $1 \leqslant i \leqslant m$, can be chosen integer and primitive, and the numbers a_i can be chosen integer. Note that $\boldsymbol{l}_i$ is normal to the facet $F_i \subset P^n$ and is pointing inside the polytope P. Define the complete fan $\Sigma(P)$ whose cones are generated by those sets of normal vectors $\boldsymbol{l}_{i_1}, \ldots, \boldsymbol{l}_{i_k}$ whose corresponding facets $F_{i_1}, \ldots, F_{i_k}$ have non-empty intersection in P. The fan $\Sigma(P)$ is called the *normal fan* of P. Alternatively, if $0 \in P$ then the normal fan consists of cones over the faces of the polar polytope P^*. Define the toric variety $M_P := M_{\Sigma(P)}$. The variety M_P is smooth if and only if P is simple and the normal vectors $\boldsymbol{l}_{i_1}, \ldots, \boldsymbol{l}_{i_n}$ of any set of n facets $F_{i_1}, \ldots, F_{i_n}$ meeting at the same vertex form a basis of $\mathbb{Z}^n$.

REMARK. Every combinatorial simple polytope is *rational*, that is, admits a convex realization with rational vertex coordinates. Indeed, there is a small perturbation of defining inequalities in (1.1) that makes all of them rational but does not change the combinatorial type (since the half-spaces defined by the inequalities are in general position). As a result, one gets a simple polytope P' of the same combinatorial type with rational vertex coordinates. To obtain a realization with integral vertex coordinates we just take the magnified polytope kP' for appropriate $k \in \mathbb{Z}$. We note that this is not the case in general: in every dimension $\geqslant 5$ there exist *non-rational* convex polytopes (non-simple and non-simplicial), see e.g. [**145**, Example 6.21] and discussion there. In dimension 3 all convex polytopes are rational, and in dimension 4 the existence of non-rational polytopes is an open problem. Returning to simple polytopes, we note that different realizations of a given combinatorial simple polytope as lattice polytopes may produce different (even topologically) toric varieties M_P. At the same time there exist combinatorial simple polytopes that do not admit *any* lattice realization with smooth M_P. We present one such example in the next section, see Example 5.26.

The underlying topological space of a toric variety M_P can be identified with the quotient space $T^n \times P^n/\sim$ for some equivalence relation $\sim$ using the following construction (see e.g. [**64**, § 4.1]).

CONSTRUCTION 5.5 (Toric variety as an identification space). We identify the torus T^n (5.1) with the quotient $\mathbb{R}^n/\mathbb{Z}^n$. For each point $q \in P^n$ define $G(q)$ as the smallest face that contains q in its relative interior. The normal subspace to $G(q)$, denoted N, is spanned by the primitive vectors $\boldsymbol{l}_i$ (see (1.1)) corresponding to those facets F_i which contain $G(q)$. (If P^n is simple then there are exactly $\operatorname{codim} G(q)$ such facets; in general there are more of them.) Since N is a rational subspace, it projects to a subtorus of T^n, which we denote $T(q)$. Note that $\dim T(q) = n - \dim G(q)$. Then, as a topological space,

$$M_P = T^n \times P^n/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T(q)$. The subtori $T(q)$ are the isotropy subgroups for the action of T^n on M_P, and P^n is identified with the orbit space. Note that if q is a vertex of P^n then $T(q) = T^n$, so the vertices correspond to the T^n-fixed points of M_P. At the other extreme, if $q \in \operatorname{int} P^n$ then $T(q) = \{e\}$, so the T^n-action is free over the interior of the polytope. More generally, if $\pi : M_P \to P^n$ is the quotient projection then

$$\pi^{-1}\big(\operatorname{int} G(q)\big) = \big(T^n/T(q)\big) \times \operatorname{int} G(q).$$

REMARK. The above construction can be generalized to all complete toric varieties (not necessarily coming from polytopes) by replacing P^n by an n-ball with cellular decomposition on the boundary. This cellular decomposition is "dual" to that defined by the complete fan.

Construction 5.4 allows us to define the simplicial fan $\Sigma(P)$ and the toric variety M_P from any lattice simple polytope P. However, the lattice polytope P contains more geometrical information than the fan $\Sigma(P)$. Indeed, besides the normal vectors $\boldsymbol{l}_i$, which determine the fan, we also have numbers $a_i \in \mathbb{Z},\ 1 \leqslant i \leqslant m$, (see (1.1)). In the notations of Theorem 5.3, it is well known in the toric geometry that the linear combination $D = a_1 D_1 + \cdots + a_m D_m$ is an *ample divisor* on M_P. It defines a projective embedding $M_P \subset \mathbb{C}P^r$ for some r (which can be taken to be the number of vertices of P). This implies that all toric varieties from polytopes are projective. Conversely, given a smooth projective toric variety $M \subset \mathbb{C}P^r$, one gets very ample divisor (line bundle) D of a hyperplane section whose zero cohomology is generated by the sections corresponding to lattice points in a certain lattice simple polytope P. For this P one has $M = M_P$. Let $\omega := a_1 v_1 + \cdots + a_m v_m \in H^2(M_P;\mathbb{Q})$ be the cohomology class of D.

THEOREM 5.6 (Hard Lefschetz theorem for toric varieties). *Let P^n be a lattice simple polytope* (1.1)*, M_P the toric variety defined by P, and $\omega = a_1 v_1 + \cdots + a_m v_m \in H^2(M_P;\mathbb{Q})$ the above defined cohomology class. Then the maps*

$$H^{n-i}(M_P;\mathbb{Q}) \xrightarrow{\ \cdot\,\omega^i\ } H^{n+i}(M_P;\mathbb{Q}), \qquad 1 \leqslant i \leqslant n,$$

are isomorphisms.

It follows from the projectivity that if M_P is smooth then it is Kähler, and ω is the class of the Kähler 2-form.

REMARK. As it is stated, Theorem 5.6 applies only to simplicial projective toric varieties. However it remains true for any projective toric variety if we replace the ordinary cohomology by the (middle perversity) *intersection cohomology*. For more details see the discussion in [**64**, § 5.2].

EXAMPLE 5.7. The complex projective space $\mathbb{C}P^n = \{(z_0 : z_1 : \cdots : z_n),\ z_i \in \mathbb{C}\}$ is a toric variety. The algebraic torus $(\mathbb{C}^*)^n$ acts on $\mathbb{C}P^n$ by

$$(t_1, \ldots, t_n) \cdot (z_0 : z_1 : \cdots : z_n) = (z_0 : t_1 z_1 : \cdots : t_n z_n).$$

Obviously, $(\mathbb{C}^*)^n \subset \mathbb{C}^n \subset \mathbb{C}P^n$ is a dense open subset. A sample fan defining $\mathbb{C}P^n$ consists of the cones spanned by all proper subsets of the set of $(n+1)$ vectors $e_1, \ldots, e_n, -e_1 - \cdots - e_n$ in $\mathbb{R}^n$. Theorem 5.3 identifies the cohomology ring $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[u]/(u^{n+1})$, $\dim u = 2$, with the quotient ring

$$\mathbb{Z}[v_1, \ldots, v_{n+1}]/(v_1 \cdots v_{n+1}, v_1 - v_{n+1}, \ldots, v_n - v_{n+1}).$$

The toric variety $\mathbb{C}P^n$ arises from a polytope: $\mathbb{C}P^n = M_P$, where P is the standard n-simplex (1.2). The corresponding class $\omega \in H^2(\mathbb{C}P^n;\mathbb{Q})$ from Theorem 5.6 is represented by v_{n+1}.

Now we are ready to give Stanley's argument for the "only if" part of the g-theorem for simple polytopes.

PROOF OF THE NECESSITY PART OF THEOREM 1.29. Realize the simple polytope as a lattice polytope $P^n \subset \mathbb{R}^n$. Let M_P be the corresponding toric variety. Part (a) is already proved (Theorem 1.20). It follows from Theorem 5.6 that the multiplication by $\omega \in H^2(M_P;\mathbb{Q})$ is a monomorphism $H^{2i-2}(M_P;\mathbb{Q}) \to H^{2i}(M_P;\mathbb{Q})$ for $i \leqslant \left[\frac{n}{2}\right]$. This together with part (a) of Theorem 5.3 gives $h_{i-1} \leqslant h_i$, $0 \leqslant i \leqslant \left[\frac{n}{2}\right]$, thus proving (b). To prove (c), define the graded commutative $\mathbb{Q}$-algebra $A := H^*(M_P;\mathbb{Q})/(\omega)$. Then $A^0 = \mathbb{Q}$, $A^{2i} = H^{2i}(M_P;\mathbb{Q})/\omega \cdot H^{2i-2}(M_P;\mathbb{Q})$ for $1 \leqslant i \leqslant \left[\frac{n}{2}\right]$, and A is generated by degree-two elements (since so is $H^*(M_P;\mathbb{Q})$). It follows from Theorem 1.32 that the numbers $\dim A^{2i} = h_i - h_{i-1},\ 0 \leqslant i \leqslant \left[\frac{n}{2}\right]$, are the components of an M-vector, thus proving (c) and the whole theorem. $\square$

REMARK. The Dehn–Sommerville equations now can be interpreted as the Poincaré duality for M_P. Even though M_P needs not to be smooth, the rational cohomology algebra of a simplicial toric variety (or toric orbifold) still satisfies the Poincaré duality.

The Hard Lefschetz theorem (Theorem 5.6) holds only for projective toric varieties. This implies that Stanley's argument cannot be directly generalized beyond the polytopal sphere case. So far this case is the only generality in which methods involving the Hard Lefschetz theorem are efficient for proving the g-theorem (see also the discussion at the end of section 7.6). However, the cohomology of toric varieties has been shown to be quite helpful in generalizing statements like the g-theorem in a different direction, namely, to the case of general (not necessarily simple or simplicial) convex polytopes. So suppose P^n is a convex lattice n-polytope. It gives rise, as described in Construction 5.4, to a projective toric variety M_P. If P^n is not simple then M_P has worse than just orbifold singularities and its ordinary cohomology behaves badly. The Betti numbers of M_P are not determined by the combinatorial type of P^n and do not satisfy the Poincaré duality. On the other hand, it turns out that the dimensions $\widehat{h}_i$ of the *intersection* cohomology of M_P are combinatorial invariants of P^n. The vector

$$\widehat{\boldsymbol{h}}(P^n) = (\widehat{h}_0, \widehat{h}_1, \ldots, \widehat{h}_n)$$

is called the *intersection h-vector* of P^n. If P^n is simple, then the intersection h-vector coincides with the ordinary one, but in general $\widehat{\boldsymbol{h}}(P^n)$ is not determined by the face vector of P^n and its combinatorial definition is quite subtle, see [**126**] for details. The intersection h-vector satisfies the "Dehn–Sommerville equations" $\widehat{h}_i = \widehat{h}_{n-i}$, and the Hard Lefschetz theorem shows that it also satisfies the GLBC inequalities:

$$\widehat{h}_0 \leqslant \widehat{h}_1 \leqslant \ldots \leqslant \widehat{h}_{\left[\frac{n}{2}\right]}.$$

In the case when P^n cannot be realized as a lattice polytope (that is, P^n is non-rational, see the remark after Construction 5.4) the combinatorial definition of intersection h-vector still works, but it is not known whether the above inequalities continue to hold. Some progress in this direction has been achieved in [**27**], [**134**].

To summarise, we may say that although Hard Lefschetz and intersection cohomology methods so far are not very helpful in the non-convex situation (like PL or simplicial spheres), they still are quite powerful in the case of general convex polytopes.

Now we look more closely at the action of the torus $T^n \subset (\mathbb{C}^*)^n$ on a non-singular compact toric variety M. This action is "locally equivalent" to the standard action of T^n on $\mathbb{C}^n$, see the next section for the precise definition. The orbit space M/T^n is homeomorphic to an n-ball, invested with the topological structure of *manifold with corners* by the fixed point sets of appropriate subtori, see [**64**, § 4.1]. Roughly speaking, a manifold with corners is a space that is locally modelled by open subsets of the positive cone $\mathbb{R}^n_+$ (1.4). From this description it is easy to deduce the strict definition [**80**], which we omit here.

CONSTRUCTION 5.8. Let P^n be a simple polytope. For any vertex $v \in P^n$ denote by U_v the open subset of P^n obtained by deleting all faces not containing v. Obviously, U_v is diffeomorphic to $\mathbb{R}^n_+$ (and even affinely isomorphic to an open set of $\mathbb{R}^n_+$ containing 0). It follows that P^n is a manifold with corners, with atlas $\{U_v\}$.

As suggested by Construction 5.5, if smooth M arises from a lattice polytope P^n (which is therefore simple) then the orbit space M/T^n is diffeomorphic, as a manifold with corners, to P^n. Furthermore, in this case there exists an explicit map $M \to \mathbb{R}^n$ (the *moment map*) with image $P^n \subset \mathbb{R}^n$ and T^n-orbits as fibres, see [**64**, §4.2]. (We will return to moment maps and some aspects of symplectic geometry in section 8.2.) The identification space description of a non-singular projective toric variety (Construction 5.5) motivated Davis and Januszkiewicz [**48**] to introduce a topological counterpart of the toric geometry, namely, the study of *quasitoric manifolds*. We proceed with their description in the next section.

5.2. Quasitoric manifolds

Quasitoric manifolds can be viewed as a "topological approximation" to algebraic non-singular projective toric varieties. This notion appeared in [**48**] under the name "toric manifolds". We use the term "quasitoric manifold", since "toric manifold" is reserved in the algebraic geometry for "non-singular toric variety". In the consequent definitions we follow [**48**], taking into account adjustments and specifications from [**38**]. As in the case of toric varieties, we first give a definition of a quasitoric manifold from the general topological point of view (as a manifold with a certain nice torus action), and then specify a combinatorial construction (similar to the construction of toric varieties from fans or polytopes).

5.2.1. Quasitoric manifolds and characteristic maps. As in the previous section, we regard the torus T^n as the standard subgroup (5.1) in $(\mathbb{C}^*)^n$, thereby specifying the orientation and the coordinate subgroups $T_i \cong S^1$ $(i = 1, \dots, n)$ in T^n. We refer to the representation of T^n by diagonal matrices in $U(n)$ as the *standard* action on $\mathbb{C}^n$. The orbit space of this action is the positive cone $\mathbb{R}^n_+$ (1.4). The canonical projection

$$T^n \times \mathbb{R}^n_+ \to \mathbb{C}^n \ : \ (t_1, \dots, t_n) \times (x_1, \dots, x_n) \to (t_1 x_1, \dots, t_n x_n)$$

identifies $\mathbb{C}^n$ with a quotient space $T^n \times \mathbb{R}^n_+/\!\sim$. This quotient will serve as the "local model" for some other identification spaces below.

Let M^{2n} be a $2n$-dimensional manifold with an action of the torus T^n (a T^n-*manifold* for short).

DEFINITION 5.9. A *standard* chart on M^{2n} is a triple (U, f, ψ), where U is a T^n-stable open subset of M^{2n}, ψ is an automorphism of T^n, and f is a ψ-equivariant homeomorphism $f : U \to W$ with some (T^n-stable) open subset $W \subset \mathbb{C}^n$. (The latter means that $f(t \cdot y) = \psi(t) f(y)$ for all $t \in T^n$, $y \in U$.) Say that a T^n-action on M^{2n} is *locally standard* if M^{2n} has a standard atlas, that is, every point of M^{2n} lies in a standard chart.

The orbit space for a locally standard action of T^n on M^{2n} is an n-dimensional manifold with corners. Quasitoric manifolds correspond to the case when this orbit space is diffeomorphic, as a manifold with corners, to a simple polytope P^n. Note that two simple polytopes are diffeomorphic as manifolds with corners if and only if they are combinatorially equivalent.

DEFINITION 5.10. Given a combinatorial simple polytope P^n, a T^n-manifold M^{2n} is called a *quasitoric manifold over* P^n if the following two conditions are satisfied:

(a) the T^n-action is locally standard;
(b) there is a projection map $\pi : M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of a codimension-k face of P^n, $k = 0, \dots, n$.

It follows that the T^n-action on a quasitoric manifold M^{2n} is free over the interior of the quotient polytope P^n, while the vertices of P^n correspond to the T^n-fixed points of M^{2n}. Direct comparison with Construction 5.5 suggests that every smooth (projective) toric variety M_P coming from a simple lattice polytope P^n is a quasitoric manifold over the corresponding combinatorial polytope. We will return to this below in Example 5.19.

Suppose P^n has m facets $F_1, \dots, F_m$. By the definition, for every facet F_i, the pre-image $\pi^{-1}(\operatorname{int} F_i)$ consists of codimension-one orbits with the same 1-dimensional isotropy subgroup, which we denote $T(F_i)$. It can be easily seen that $\pi^{-1}(F_i)$ is an $2(n-1)$-dimensional quasitoric (sub)manifold over F_i, with respect to the action of $T^n/T(F_i)$. We denote it $M_i^{2(n-1)}$ and refer to it as the *facial submanifold* corresponding to F_i. Its isotropy subgroup $T(F_i)$ can be written as

$$T(F_i) = \{ (e^{2\pi i \lambda_{1i}\varphi}, \dots, e^{2\pi i \lambda_{ni}\varphi}) \in T^n \}, \tag{5.3}$$

where $\varphi \in \mathbb{R}$ and $\boldsymbol{\lambda}_i = (\lambda_{1i}, \dots, \lambda_{ni})^t \in \mathbb{Z}^n$ is a primitive vector. This $\boldsymbol{\lambda}_i$ is determined by $T(F_i)$ only up to a sign. A choice of sign specifies an orientation for $T(F_i)$. For now we do not care about this sign and choose it arbitrarily. A more detailed treatment of signs and orientations is the subject of the next section. We refer to $\boldsymbol{\lambda}_i$ as the *facet vector* corresponding to F_i. The correspondence

$$\ell : F_i \mapsto T(F_i) \tag{5.4}$$

is called the *characteristic map* of M^{2n}.

Suppose we have a codimension-k face G^{n-k}, written as an intersection of k facets: $G^{n-k} = F_{i_1} \cap \dots \cap F_{i_k}$. Then the submanifolds $M_{i_1}, \dots, M_{i_k}$ intersect transversally in a submanifold $M(G)^{2(n-k)}$, which we refer to as the *facial submanifold* corresponding to G. The map $T(F_{i_1}) \times \dots \times T(F_{i_k}) \to T^n$ is injective since $T(F_{i_1}) \times \dots \times T(F_{i_k})$ is identified with the k-dimensional isotropy subgroup of $M(G)^{2(n-k)}$. It follows that the vectors $\boldsymbol{\lambda}_{i_1}, \dots, \boldsymbol{\lambda}_{i_k}$ form a part of an integral basis of $\mathbb{Z}^n$.

Let Λ be integer $(n \times m)$-matrix whose i-th column is formed by the coordinates of the facet vector $\boldsymbol{\lambda}_i$, $i = 1, \dots, m$. Every vertex $v \in P^n$ is an intersection of n facets: $v = F_{i_1} \cap \dots \cap F_{i_n}$. Let $\Lambda_{(v)} := \Lambda_{(i_1, \dots, i_n)}$ be the maximal minor of Λ formed by the columns $i_1, \dots, i_n$. Then

$$\det \Lambda_{(v)} = \pm 1. \tag{5.5}$$

The correspondence

$$G^{n-k} \mapsto \text{ isotropy subgroup of } M(G)^{2(n-k)}$$

extends the characteristic map (5.4) to a map from the face poset of P^n to the poset of subtori of T^n.

DEFINITION 5.11. Let P^n be a combinatorial simple polytope and ℓ is a map from facets of P^n to one-dimensional subgroups of T^n. Then (P^n, ℓ) is called a *characteristic pair* if $\ell(F_{i_1}) \times \cdots \times \ell(F_{i_k}) \to T^n$ is injective whenever $F_{i_1} \cap \cdots \cap F_{i_k} \neq \varnothing$.

The map ℓ directly extends to a map from the face poset of P^n to the poset of subtori of T^n, so we have subgroup $\ell(G) \subset T^n$ for every face G of P^n.

As in the case of standard action of T^n on $\mathbb{C}^n$, there is a projection $T^n \times P^n \to M^{2n}$ whose fibre over $x \in M^{2n}$ is the isotropy subgroup of x. This argument can be used for reconstructing the quasitoric manifold from any given characteristic pair (P^n, ℓ).

CONSTRUCTION 5.12 (Quasitoric manifold from characteristic pair). Given a point $q \in P^n$, we denoted by $G(q)$ the minimal face containing q in its relative interior. Now set

$$M^{2n}(\ell) := (T^n \times P^n)/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in \ell(G(q))$ (compare with Construction 5.5 for toric varieties). The free action of T^n on $T^n \times P^n$ obviously descends to an action on $(T^n \times P^n)/\sim$, with quotient P^n. The latter action is free over the interior of P^n and has a fixed point for each vertex of P^n. Just as P^n is covered by the open sets U_v, based on the vertices and diffeomorphic to $\mathbb{R}^n_+$ (see Construction 5.8), so the space $(T^n \times P^n)/\sim$ is covered by open sets $(T^n \times U_v)/\sim$ homeomorphic to $(T^n \times \mathbb{R}^n_+)/\sim$, and therefore to $\mathbb{C}^n$. This implies that the T^n-action on $(T^n \times P^n)/\sim$ is locally standard, and therefore $(T^n \times P^n)/\sim$ is a quasitoric manifold.

DEFINITION 5.13. Given an automorphism $\psi : T^n \to T^n$, say that two quasitoric manifolds M_1^{2n}, M_2^{2n} over the same P^n are *ψ-equivariantly diffeomorphic* if there is a diffeomorphism $f : M_1^{2n} \to M_2^{2n}$ such that $f(t \cdot x) = \psi(t) f(x)$ for all $t \in T^n$, $x \in M_1^{2n}$. The automorphism ψ induces an automorphism ψ_* of the poset of subtori of T^n. Any such automorphism descends to a *ψ-translation* of characteristic pairs, in which the two characteristic maps differ by ψ_*.

The following proposition is proved as Proposition 1.8 in [**48**] (see also [**38**, Proposition 2.6]).

PROPOSITION 5.14. *Construction* 5.12 *defines a bijection between ψ-equivariant diffeomorphism classes of quasitoric manifolds and ψ-translations of pairs (P^n, ℓ).*

When ψ is the identity, we deduce that two quasitoric manifolds are equivariantly diffeomorphic if and only if their characteristic maps are the same.

5.2.2. Cohomology of quasitoric manifolds. The cohomology ring structure of a quasitoric manifold is similar to that of a non-singular toric variety. To see this analogy we first describe a cell decomposition of M^{2n} with even dimensional cells (a perfect cellular structure) and calculate the Betti numbers accordingly, following [**48**].

CONSTRUCTION 5.15. We recall the "Morse-theoretical arguments" from the proof of Dehn–Sommerville relations (Theorem 1.20). There we turned the 1-skeleton of P^n into a directed graph and defined the index $\operatorname{ind}(v)$ of a vertex $v \in P^n$ as the number of incident edges that point towards v. These inward edges span a face G_v of dimension $\operatorname{ind}(v)$. Denote by $\widehat{G}_v$ the subset of G_v obtained by deleting all faces not containing v. Obviously, $\widehat{G}_v$ is diffeomorphic to $\mathbb{R}_+^{\operatorname{ind}(v)}$ and is contained in the open set $U_v \subset P^n$ from Construction 5.8. Then $e_v := \pi^{-1}\widehat{G}_v$ is identified with $\mathbb{C}^{\operatorname{ind}(v)}$, and the union of the e_v over all vertices of P^n define a cellular decomposition of M^{2n}. Note that all cells are even-dimensional and the closure of the cell e_v is the facial submanifold $M(G_v)^{2\operatorname{ind}(v)} \subset M^{2n}$. This argument was earlier used by Khovanskii [**84**] for constructing perfect cellular decompositions of toric varieties.

PROPOSITION 5.16. *The Betti numbers of M^{2n} vanish in odd dimensions, while in even dimensions are given by*

$$b_{2i}(M^{2n}) = h_i(P^n), \quad i = 0, 1, \ldots, n,$$

where $\boldsymbol{h}(P^n) = (h_0, \ldots, h_n)$ is the h-vector of P^n.

PROOF. The $2i$-th Betti number equals the number of $2i$-dimensional cells in the cellular decomposition constructed above. This number equals the number of vertices of index i, which is $h_i(P^n)$ by the argument from the proof of Theorem 1.20. $\square$

Given a quasitoric manifold M^{2n} with characteristic map (5.4) and facet vectors $\boldsymbol{\lambda}_i = (\lambda_{1i}, \ldots, \lambda_{ni})^t \in \mathbb{Z}^n$, $i = 1, \ldots, m$, define linear forms

$$\theta_i := \lambda_{i1}v_1 + \cdots + \lambda_{im}v_m \in \mathbb{Z}[v_1, \ldots, v_m], \quad 1 \leqslant i \leqslant n. \tag{5.6}$$

The images of these linear forms in the Stanley–Reisner ring $\mathbb{Z}(P^n)$ will be denoted by the same letters.

LEMMA 5.17 (Davis and Januszkiewicz). *$\theta_1, \ldots, \theta_n$ is a (degree-two) regular sequence in $\mathbb{Z}(P^n)$.*

Let $\mathcal{J}_\ell$ denote the ideal in $\mathbb{Z}(P^n)$ generated by $\theta_1, \ldots, \theta_n$.

THEOREM 5.18 (Davis and Januszkiewicz). *The cohomology ring of M^{2n} is given by*

$$H^*(M^{2n}; \mathbb{Z}) = \mathbb{Z}[v_1, \ldots, v_m]/(\mathcal{I}_P + \mathcal{J}_\ell) = \mathbb{Z}(P^n)/\mathcal{J}_\ell,$$

where v_i is the 2-dimensional cohomology class dual to the facial submanifold $M_i^{2(n-1)}$ (with arbitrary orientation chosen), $i = 1, \ldots, m$.

We give proofs for the above two statements in section 6.5.

REMARK. Change of sign of vector $\boldsymbol{\lambda}_i$ corresponds to passing from v_i to $-v_i$ in the description of the cohomology ring given by Theorem 5.18. We will use this observation in the next section.

5.2.3. Non-singular toric varieties and quasitoric manifolds. In this subsection we give a more detailed comparison of the two classes of manifolds. In general, none of these classes belongs to the other, and the intersection of the two classes contains smooth projective toric varieties as a proper subclass (see Figure 5.1). Below in this subsection we provide the corresponding examples.

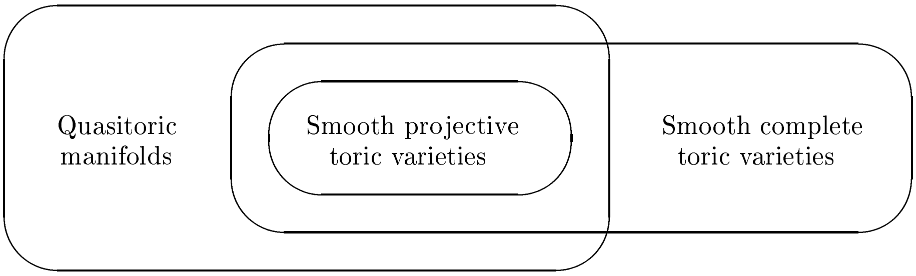


FIGURE 5.1.

EXAMPLE 5.19. As it is suggested by comparing Constructions 5.5 and 5.12, a non-singular projective toric variety M_P arising from a lattice simple polytope P^n is a quasitoric manifold over the combinatorial type P^n. The corresponding characteristic map $\ell : F_i \mapsto T(F_i)$ is defined by putting $\boldsymbol{\lambda}_i = \boldsymbol{l}_i$ in (5.3). That is, the facet vectors are the normal vectors $\boldsymbol{l}_i$ to facets of P^n, $i = 1, \ldots, m$ (see (1.1)). The corresponding characteristic $n \times m$-matrix Λ is the matrix L from Construction 1.8. In particular, if P^n is the standard simplex Δ^n (1.2) then M_P is $\mathbb{C}P^n$ (Example 5.7) and $\Lambda = (\mathrm{E}\,|\,-\mathbf{1})$, where E is the unit $n \times n$-matrix and $\mathbf{1}$ is the column of units. See also Example 5.60 below.

In general, a smooth non-projective toric variety may fail to be a quasitoric manifold: although the orbit space (for the T^n-action) is a manifold with corners (see section 5.1), it may not be diffeomorphic (or combinatorially equivalent) to a simple polytope. The authors are thankful to N. Strickland for drawing our attention to this fact. However, we do not know of any such example.

PROBLEM 5.20. *Give an example of a non-singular toric variety which is not a quasitoric manifold.*

In [**64**, p. 71] one can find an example of a complete non-singular fan Σ in $\mathbb{R}^3$ which cannot be obtained by taking the cones with vertex 0 over the faces of a convex simplicial polytope. Nevertheless, since the corresponding simplicial complex K_Σ is a simplicial 2-sphere, it is combinatorially equivalent to a polytopal 2-sphere. This means that the corresponding non-singular toric variety M_Σ, although being non-projective, is still a quasitoric manifold. It is convenient to introduce the following notation here.

DEFINITION 5.21. We say that a simplicial fan Σ in $\mathbb{R}^n$ is *strongly polytopal* (or simply *polytopal*) if it can be obtained by taking the cones with vertex 0 over the faces of a convex simplicial polytope. Equivalently, a fan is strongly polytopal if it is a normal fan of simple lattice polytope (see Construction 5.4). Say that a simplicial fan Σ is *weakly polytopal* if the underlying simplicial complex K_Σ is a polytopal sphere (that is, combinatorially equivalent to the boundary complex of a simplicial polytope).

Suppose Σ is a non-singular fan and M_Σ the corresponding toric variety. Then M_Σ is projective if and only if Σ is strongly polytopal, and M_Σ is a quasitoric manifold if and only if Σ is weakly polytopal. Thus, the answer to Problem 5.20 can be given by providing a non-singular fan which is not weakly polytopal. As it was told to the authors by Y. Civan (in private communications), this may be done by giving a (singular) fan Σ whose underlying simplicial complex K_Σ is the Barnette sphere (see section 2.3) and then desingularizing it using the standard procedure (see [**64**, §2.6]). Combinatorial properties of the Barnette sphere obstruct the resulting (non-singular) fan to be weakly polytopal.

On the other hand, it is easy to construct a quasitoric manifold which is not a toric variety. The simplest example is the manifold $\mathbb{C}P^2 \# \mathbb{C}P^2$, the connected sum of two copies of $\mathbb{C}P^2$. It is a quasitoric manifold over the square I^2 (this follows from the construction of equivariant connected sum, see [**48**, 1.11] or section 5.3 and Corollary 5.66 below). However, $\mathbb{C}P^2 \# \mathbb{C}P^2$ do not admit even an almost complex structure (i.e., its tangent bundle cannot be made complex). The following problem arises.

PROBLEM 5.22. *Let (P^n, ℓ) be a characteristic pair (see Definition 5.11), and $M^{2n}(\ell)$ the derived quasitoric manifold (see Construction 5.12). Find conditions on P^n and ℓ so that $M^{2n}(\ell)$ admits a T^n-invariant complex (or almost complex) structure.*

The almost complex case of the above problem was formulated in [**48**, Problem 7.6]. Since every non-singular toric variety is a complex manifold, characteristic pairs coming from lattice simple polytopes (as described in Example 5.19) provide a sufficient condition for Problem 5.22. However, this is not a necessary condition even for the existence of an invariant complex structure. Indeed, there exist smooth non-projective toric varieties coming from weakly polytopal fans (see the already mentioned example in [**64**, p. 71]). At the same time, we do not know any example of *non-toric* complex quasitoric manifold.

PROBLEM 5.23. *Find an example of a non-toric quasitoric manifold that admits a T^n-invariant complex structure.*

Although a general quasitoric manifold may fail to be complex or almost complex, it always admits a T^n-invariant complex structure in the *stable* tangent bundle. The corresponding constructions are the subject of the next section. We will return to Problem 5.22 in subsection 5.4.2.

Another class of problems arises in connection with the classification of quasitoric manifolds over a given combinatorial simple polytope. The general setting of this problem is discussed in section 5.5. Example 5.26 below shows that there are combinatorial simple polytopes that do not admit a characteristic map (and therefore cannot arise as orbit spaces for quasitoric manifolds).

PROBLEM 5.24. *Give a combinatorial description of the class of polytopes P^n that admit a characteristic map* (5.4).

A generalization of this problem is considered in chapter 7 (Problem 7.27).

A characteristic map is determined by an integer $n \times m$-matrix Λ which satisfies (5.5) for every vertex $v \in P^n$. The equation $(\det \Lambda_{(v)})^2 = 1$ defines a hypersurface in the space $\mathcal{M}(n, m; \mathbb{Z})$ of integer $n \times m$-matrices.

PROPOSITION 5.25. *The set of characteristic matrices coincides with the intersection*

$$\bigcap_{v \in P^n} \{(\det \Lambda_{(v)})^2 = 1\} \tag{5.7}$$

of hypersurfaces in the space $\mathcal{M}(n, m; \mathbb{Z})$, where v is running through the vertices of the polytope P^n.

Thus, Problem 5.24 is to determine for which polytopes the intersection in (5.7) is non-empty.

EXAMPLE 5.26 ([**48**, Example 1.22]). Let P^n be a 2-neighborly simple polytope with $m \geqslant 2^n$ facets (e.g., the polar of cyclic polytope $C^n(m)$ with $n \geqslant 4$ and $m \geqslant 2^n$, see Example 1.17). Then this P^n does not admit a characteristic map and therefore cannot appear as the quotient space of a quasitoric manifold. Indeed, by Proposition 5.25, it is sufficient to show that intersection (5.7) is empty. Since $m \geqslant 2^n$, any matrix $\Lambda \in \mathcal{M}(n, m; \mathbb{Z})$ (without zero columns) contains two columns, say i-th and j-th, which coincide modulo 2. Since P^n is 2-neighborly, the corresponding facets F_i and F_j have non-empty intersection in P^n. Hence, the columns i and j of Λ enter the minor $\Lambda_{(v)}$ for some vertex $v \in P^n$. This implies that the determinant of this minor is even and intersection (5.7) is empty.

In particular, the above example implies that there are no non-singular toric varieties over the combinatorial polar cyclic polytope $(C^n(m))^*$ with $\geqslant 2^n$ facets. This means that the combinatorial type $C^n(m)$ with $m \geqslant 2^n$ cannot be realized as a lattice simplicial polytope in such a way that the fan over its faces is non-singular. In the toric geometry, the question of whether for any given complete simplicial fan Σ there exists a combinatorially equivalent fan Σ' that gives rise to a smooth toric variety was known as Ewald's conjecture of 1986. The first counterexample was found in [**67**]. It was shown there that no fan over the faces of a lattice realization of $C^n(m)$ with $m \geqslant n+3$ is non-singular. The comparison of this result with Example 5.26 suggests that some cyclic polytopes $C^n(m)$ with small number of vertices (between $n+3$ and 2^n) may appear as the quotients of quasitoric manifolds, but not as quotients of non-singular toric varieties.

Another interesting corollary of Example 5.26 is that the face ring $\mathbb{Z}(C^n(m))$ of (the boundary complex of) $C^n(m)$ and the face ring $\mathbb{Z}/p(C^n(m))$ for any prime p does not admit a regular sequence of degree two (or a lsop). Of course, since $\mathbb{Z}/p(C^n(m))$ is Cohen–Macaulay, it admits a non-linear regular sequence. Note that in the case when $\mathbf{k}$ is of zero characteristic the ring $\mathbf{k}(C^n(m))$ always admits an lsop (and degree-two regular sequence) by Lemma 3.11.

5.3. Stably complex structures, and quasitoric representatives in cobordism classes

This section is the review of results obtained by N. Ray and the first author in [**37**] and [**38**], supplied with some additional comments.

A *stably complex structure* on a (smooth) manifold M is determined by a complex structure in the vector bundle $\tau(M) \oplus \mathbb{R}^k$ for some k, where $\tau(M)$ is the *tangent bundle* of M and $\mathbb{R}^k$ denotes a trivial real k-dimensional bundle over M. A *stably complex manifold* (in other notations, *weakly almost complex manifold* or *U-manifold*) is a manifold with fixed stably complex structure, which one can view as a pair (M, ξ), where ξ is a complex bundle isomorphic, as a real bundle, to

$\tau(M) \oplus \mathbb{R}^k$ for some k. If M itself is a complex manifold, then it possesses the *canonical* stably complex structure $(M, \tau(M))$. The operations of disjoint union and product endow the set of cobordism classes $[M, \xi]$ of stably complex manifolds with the structure of a graded ring, called the *complex cobordism ring* Ω^U. By the theorem of Milnor and Novikov, the complex cobordism ring is isomorphic to the polynomial ring on an infinite number of even-dimensional generators:

$$\Omega^U \cong \mathbb{Z}[a_1, a_2, \ldots], \quad \deg a_i = 2i,$$

(see [**104**], [**130**]). The ring Ω^U is the coefficient ring for generalized (co)homology theory known as the *complex (co)bordisms*. We refer to [**130**] as the standard source for the cobordism theory.

Stably complex manifolds was the main subject of F. Hirzebruch's talk at the 1958 International Congress of Mathematicians, see [**135**]. Using *Milnor hypersurfaces* (Example 5.39) and the Milnor–Novikov theorem it was shown by Milnor [**135**] that every complex cobordism class contains a non-singular algebraic variety, not necessarily connected. The following problem is still open.

PROBLEM 5.27 (Hirzebruch). *Which complex cobordism classes in Ω^U contain connected non-singular algebraic varieties?*

A weaker version of this question, which is also open, asks which cobordism classes contain connected almost complex manifolds.

EXAMPLE 5.28. The 2-dimensional cobordism group $\Omega_2^U \cong \mathbb{Z}$ is generated by the class of $[\mathbb{C}P^1]$ (Riemannian sphere). Every cobordism class $k[\mathbb{C}P^1] \in \Omega_2^U$ contains a non-singular algebraic variety, namely, the disjoint union of k copies of $\mathbb{C}P^1$ for $k > 0$ and a Riemannian surface of genus $(1-k)$ for $k \leqslant 0$. However, *connected* algebraic varieties are contained only in the cobordism classes $k[\mathbb{C}P^1]$ with $k \leqslant 1$.

The problem of choosing appropriate generators for the ring Ω^U is very important in the cobordism theory and its applications. As it was recently shown in [**37**] and [**38**], every complex cobordism class (of dimension > 2) contains a quasitoric manifold (see Theorem 5.38 below). By the definition, quasitoric manifolds are necessarily connected, so the result may be considered as an answer to the quasitoric analogue of Hirzebruch's question. The construction of quasitoric representatives in complex cobordism classes relies upon an additional structure on a quasitoric manifold, called *omniorientation*, which provides a combinatorial description for canonical stably complex structures.

Let $\pi : M^{2n} \to P^n$ be a quasitoric manifold with characteristic map ℓ. Since the torus T^n (5.1) is oriented, a choice of orientation for P^n is equivalent to a choice of orientation for M^{2n}. (An orientation of P^n is specified by orienting the ambient space $\mathbb{R}^n$.)

DEFINITION 5.29. An *omniorientation* of a quasitoric manifold M^{2n} consists of a choice of an orientation for M^{2n} and for every facial submanifold $M_i^{2(n-1)} = \pi^{-1}(F_i)$, $i = 1, \ldots, m$.

Thus, there are 2^{m+1} omniorientations in all for given M^{2n}.

An omniorientation of M^{2n} determines an orientation for every normal bundle $\nu_i := \nu(M_i \subset M^{2n})$, $i = 1, \ldots, m$. Since every ν_i is a real 2-plane bundle, an orientation of ν_i allows one to interpret it as a complex line bundle. The isotropy subgroup $T(F_i)$ (see (5.3)) of submanifold $M_i^{2(n-1)} = \pi^{-1}(F_i)$ acts on the normal bundle ν_i, $i = 1, \dots, m$. Thus, we have the following statement.

PROPOSITION 5.30. *A choice of omniorientation for M^{2n} is equivalent to a choice of orientation for P^n together with an unambiguous choice of facet vectors $\boldsymbol{\lambda}_i$, $i = 1, \dots, m$ in* (5.3).

We refer to a characteristic map ℓ as *directed* if all circles $\ell(F_i)$, $i = 1, \dots, m$, are oriented. This implies that the signs of facet vectors $\boldsymbol{\lambda}_i = (\lambda_{1i}, \dots, \lambda_{ni})^t$, $i = 1, \dots, m$, are determined unambiguously. In the previous section we organized the facet vectors into the integer $n \times m$ matrix Λ. This matrix satisfies (5.5). Due to (5.3), the matrix Λ carries exactly the same information as a directed characteristic map. Let $\mathbb{Z}^{\mathcal{F}}$ denote the m-dimensional free $\mathbb{Z}$-module spanned by the set $\mathcal{F}$ of facets of P^n. Then Λ defines an epimorphism $\lambda : \mathbb{Z}^{\mathcal{F}} \to \mathbb{Z}^n$ by $\lambda(F_i) = \boldsymbol{\lambda}_i$ and an epimorphism $T^{\mathcal{F}} \to T^m$, which we will denote by the same letter λ. In the sequel we write $\mathbb{Z}^m$ for $\mathbb{Z}^{\mathcal{F}}$ and T^m for $T^{\mathcal{F}}$, assuming that the member $\boldsymbol{e}_i$ of the standard basis of $\mathbb{Z}^m$ corresponds to the facet $F_i \in \mathbb{Z}^{\mathcal{F}}$, $i = 1, \dots, m$, and similarly for T^m.

DEFINITION 5.31. A *directed characteristic pair* (P^n, Λ) consists of a combinatorial simple polytope P^n and an integer matrix Λ (or, equivalently, an epimorphism $\lambda : \mathbb{Z}^m \to \mathbb{Z}^n$) that satisfies (5.5).

Proposition 5.30 shows that the characteristic pair of an omnioriented quasitoric manifold is directed. On the other hand, the quasitoric manifold derived from a directed characteristic pair using Construction 5.12 is omnioriented.

CONSTRUCTION 5.32. The orientation of the normal bundle ν_i over M_i defines an integral Thom class in the cohomology group $H^2(\mathcal{T}(\nu_i))$, represented by a complex line bundle over the Thom complex $\mathcal{T}(\nu_i)$. We pull this back along the Pontryagin–Thom collapse $M^{2n} \to \mathcal{T}(\nu_i)$, and denote the resulting bundle ρ_i. The restriction of ρ_i to $M_i \subset M^{2n}$ is ν_i. In the algebraic geometry this construction corresponds to assigning the line bundle to a divisor. In particular, in the case when M^{2n} is a smooth toric variety, the line bundle ρ_i corresponds to the divisor D_i, see Theorem 5.3.

THEOREM 5.33 ([**48**] and [**38**, Theorem 3.8]). *Every omniorientation of a quasitoric manifold M^{2n} determines a stably complex structure on it by means of the following isomorphism of real $2m$-bundles:*

$$\tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \dots \oplus \rho_m.$$

The above isomorphism of real vector bundles is essentially due to Davis and Januszkiewicz (see [**48**, Theorem 6.6]). The interpretation of stably complex structures in terms of omniorientations was given in [**38**].

COROLLARY 5.34. *In the notation of Theorem* 5.18, *suppose $v_i \in H^2(M^{2n})$ is the cohomology class dual to the oriented facial submanifold M_i of an omnioriented quasitoric manifold M^{2n}, $i = 1, \dots, m$. Then the total Chern class of stably complex structure on M^{2n} defined by the omniorientation is given by*

$$c(M^{2n}) = (1 + v_1) \dots (1 + v_m) \in H^*(M^{2n}).$$

It follows from Theorem 5.33 that a directed characteristic pair (P^n, Λ) determines a complex cobordism class $[M^{2n}, \rho_1 \oplus \dots \oplus \rho_m] \in \Omega^U$. The following direct extension of Theorem 5.18 provides a description of the complex cobordism ring of an omnioriented quasitoric manifold.

PROPOSITION 5.35 ([**38**, Proposition 5.3]). *Let v_i denote the first cobordism Chern class $c_1(\rho_i) \in \Omega_U^2(M^{2n})$ of the bundle ρ_i, $1 \leqslant i \leqslant m$. Then the complex cobordism ring of M^{2n} is given by*

$$\Omega_U^*(M^{2n}) = \Omega_U[v_1, \dots, v_m]/(\mathcal{I}_P + \mathcal{J}_\Lambda),$$

where the ideals $\mathcal{I}_P$ and $\mathcal{J}_\Lambda$ are defined in the same way as in Theorem 5.18.

Note that the Chern class $c_1(\rho_i)$ is Poincaré dual to the inclusion $M_i^{2(n-1)} \subset M^{2n}$ by the construction of ρ_i. This highlights the remarkable fact that the complex bordism groups $\Omega_*^U(M^{2n})$ are spanned by *embedded* submanifolds. By definition, the fundamental cobordism class $\langle M^{2n}\rangle \in \Omega_U^{2n}(M^{2n})$ is dual to the bordism class of a point. Thus, $\langle M^{2n}\rangle = v_{i_1}\cdots v_{i_n}$ for any set $\{i_1, \dots, i_n\}$ such that $F_{i_1} \cap \dots \cap F_{i_n}$ is a vertex of P^n.

The following two examples are used to construct quasitoric representatives in complex cobordisms.

EXAMPLE 5.36 (bounded flag manifold [**36**]). A *bounded flag* in $\mathbb{C}^{n+1}$ is a complete flag $U = \{U_1 \subset U_2 \subset \dots \subset U_{n+1} = \mathbb{C}^{n+1}\}$ for which U_k, $2 \leqslant k \leqslant n$, contains the coordinate subspace $\mathbb{C}^{k-1}$ spanned by the first $k-1$ standard basis vectors. As it is shown in [**38**, Example 2.8], the $2n$-dimensional manifold B_n of all bounded flags in $\mathbb{C}^{n+1}$ is a quasitoric manifold over the combinatorial cube I^n with respect to the action induced by $t \cdot z = (t_1z_1, \dots, t_nz_n, z_{n+1})$ on $\mathbb{C}^{n+1}$, where $t \in T^n$.

EXAMPLE 5.37. A family of manifolds $B_{i,j}$ $(0 \leqslant i \leqslant j)$ is introduced in [**37**]. The manifold $B_{i,j}$ consists of pairs (U, W), where U is a bounded flag in $\mathbb{C}^{i+1}$ (see Example 5.36) and W is a line in $U_1^- \oplus \mathbb{C}^{j-i}$. So $B_{i,j}$ is a smooth $\mathbb{C}P^{j-1}$-bundle over B_i. It is shown in [**38**, Example 2.9] that $B_{i,j}$ is a quasitoric manifold over the product $I^i \times \Delta^{j-1}$.

The canonical stably complex structures and omniorientations on the manifolds B_n and $B_{i,j}$ are described in [**38**, examples 4.3, 4.5].

REMARK. The product of two quasitoric manifolds $M_1^{2n_1}$ and $M_2^{2n_2}$ over polytopes $P_1^{n_1}$ and $P_2^{n_2}$ is a quasitoric manifold over $P_1^{n_1} \times P_2^{n_2}$. This construction extends to omnioriented quasitoric manifolds and is compatible with stably complex structures (details can be found in [**38**, Proposition 4.7]).

It is shown in [**37**] that the cobordism classes of $B_{i,j}$ multiplicatively generate the ring Ω^U. Hence, every $2n$-dimensional complex cobordism class may be represented by a disjoint union of products

$$B_{i_1,j_1} \times B_{i_2,j_2} \times \dots \times B_{i_k,j_k}, \tag{5.8}$$

where $\sum_{q=1}^k (i_q + j_q) - 2k = n$. Each such component is a quasitoric manifold, under the product quasitoric structure. This result is the substance of [**37**]. The stably complex structures of products (5.8) are induced by omniorientations, and are therefore also preserved by the torus action.

To give genuinely quasitoric representatives (which are, by definition, connected) for each cobordism class of dimension > 2, it remains only to replace the disjoint union of products (5.8) with their connected sum. This is done in [**38**, §6] using Construction 1.13 and its extension to omnioriented quasitoric manifolds.

THEOREM 5.38 ([**38**, Theorem 6.11]). *In dimensions > 2, every complex cobordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

The connected sum operation usually destroys the algebraicity of manifolds, so the complex cobordism representatives provided by the above theorem in general are not algebraic (compare Example 5.28).

We note that in their work [**44**, §42] Conner and Floyd constructed a class of manifolds with canonical circle actions which can be chosen as representatives for multiplicative generators in oriented cobordisms. One can show that these Conner–Floyd manifolds can be obtained as particular cases of manifolds $B_{i,j}$ from Example 5.37. However, Conner and Floyd did not consider actions of half-dimensional tori on their manifolds.

EXAMPLE 5.39. The standard set of multiplicative generators for Ω^U consists of projective spaces $\mathbb{C}P^i$, $i \geqslant 0$, and *Milnor hypersurfaces* $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$, $1 \leqslant i \leqslant j$. The hypersurface $H_{i,j}$ is defined by

$$H_{i,j} = \Big\{(z_0 : \cdots : z_i) \times (w_0 : \cdots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : \sum_{q=0}^{i} z_q w_q = 0\Big\}.$$

However, the hypersurfaces $H_{i,j}$ are *not* quasitoric manifolds for $i > 1$, see [**37**]. This can be shown in the following way.

CONSTRUCTION 5.40. Let $\mathbb{C}^{i+1} \subset \mathbb{C}^{j+1}$ be the subspace spanned by the first $i+1$ vectors of the standard basis of $\mathbb{C}^{j+1}$. Identify $\mathbb{C}P^i$ with the set of complex lines $l \subset \mathbb{C}^{i+1}$. To each line l assign the set of hyperplanes $\alpha \subset \mathbb{C}^{j+1}$ that contain l. The latter set is identified with $\mathbb{C}P^{j-1}$, so there is a bundle $E \to \mathbb{C}P^i$ with fibre $\mathbb{C}P^{j-1}$. Here E is the set of pairs (l, α), $l \subset \alpha$, and the projection takes (l, α) to l.

LEMMA 5.41. *$H_{i,j}$ is identified with the total space of bundle $E \to \mathbb{C}P^i$.*

PROOF. A line $l \subset \mathbb{C}^{i+1}$ is given by a vector $(z_0 : z_1 : \cdots : z_i)$. A hyperplane $\alpha \subset \mathbb{C}^{j+1}$ is given by a linear form. If we denote the coefficients of this linear form by $w_0, w_1, \ldots, w_j$, then the condition $l \subset \alpha$ is exactly that from the definition of $H_{i,j}$. □

THEOREM 5.42. *The cohomology of $H_{i,j}$ is given by*

$$H^*(H_{i,j}) \cong \mathbb{Z}[u,v]\Big/\Big(u^{i+1} = 0,\ v^{j-i}\sum_{k=0}^{i} u^k v^{i-k} = 0\Big),$$

where $\deg u = \deg v = 2$.

PROOF. We will use the notation from Construction 5.40. Let ζ denote the bundle over $\mathbb{C}P^i$ whose fibre over $l \in \mathbb{C}P^i$ is the j-dimensional subspace $l^- \subset \mathbb{C}^{j+1}$. Then one can identify $H_{i,j}$ with the projectivization $\mathbb{C}P(\zeta)$. Indeed, for any line $l' \subset l^-$ representing a point in the fibre of $\mathbb{C}P(\zeta)$ over $l \in \mathbb{C}P^i$ the hyperplane $\alpha = (l')^- \subset \mathbb{C}^{j+1}$ contains l, so the pair (l, α) represents a point in $H_{i,j}$ (see Lemma 5.41). The rest of the proof reproduces the general argument from the Dold theorem about the cohomology of projectivizations.

Denote by ξ the tautological line bundle over $\mathbb{C}P^i$ (its fibre over $l \in \mathbb{C}P^i$ is the line l itself). Then $\xi \oplus \zeta$ is a trivial $(j+1)$-dimensional bundle. Set $w = c_1(\bar{\xi}) \in$

$H^2(\mathbb{C}P^i)$. Let $c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \dots$ denote the total Chern class. Since $c(\xi)c(\zeta) = 1$ and $c(\xi) = 1 - w$, we get

$$c(\zeta) = 1 + w + \dots + w^i. \tag{5.9}$$

Consider the projection $p : \mathbb{C}P(\zeta) \to \mathbb{C}P^i$. Denote by η the "tautological" line bundle over $\mathbb{C}P(\zeta)$ whose fiber over a point $l' \in \mathbb{C}P(\zeta)$ is the line l' itself. Denote by η^- the $(j-1)$-bundle over $\mathbb{C}P(\zeta)$ whose fibre over a point $l' \subset l^-$ is the orthogonal complement to l' in l^-. Then it is easy to see that $p^*(\zeta) = \eta \oplus \eta^-$. Set $v = c_1(\bar{\eta}) \in H^2(\mathbb{C}P(\zeta))$ and $u = p^*(w) \in H^2(\mathbb{C}P(\zeta))$. Then $u^{i+1} = 0$. We have $c(\eta) = 1 - v$ and $c(p^*(\zeta)) = c(\eta)c(\eta^-)$, hence,

$$c(\eta^-) = p^*\big(c(\zeta)\big)(1-v)^{-1} = (1 + u + \dots + u^i)(1 + v + v^2 + \dots)$$

(see (5.9)). But η^- is $(j-1)$-dimensional, hence $c_j(\eta^-) = 0$. Calculating the homogeneous part of degree j in the above identity, we get the second identity $0 = v^{j-i}\sum_{k=0}^{i} u^k v^{i-k}$.

Since both $\mathbb{C}P^i$ and $\mathbb{C}P^{j-1}$ have only even-dimensional cells, the Leray–Serre spectral sequence of the bundle $p : \mathbb{C}P(\zeta) \to \mathbb{C}P^i$ collapses at the E_2 term. It follows that there is an epimorphism $\mathbb{Z}[u, v] \to H^*(\mathbb{C}P(\zeta))$, and additively the cohomology of $H^*(\mathbb{C}P(\zeta))$ coincides with that of $\mathbb{C}P^i \times \mathbb{C}P^{j-1}$. Hence, there are no other relations except those two mentioned in the theorem. □

PROPOSITION 5.43. *$H_{i,j}$ is not a quasitoric manifold for $i > 1$.*

PROOF. By Theorem 5.18, the cohomology of a quasitoric manifold is isomorphic to a quotient $\mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}{+}\mathcal{J}$, where the ideal $\mathcal{I}$ is generated by square-free monomials and $\mathcal{J}$ is generated by linear forms. Due to (5.5) we may assume without loss of generality that first n variables $v_1, \dots, v_n$ are expressed via the last $m - n$ by means of linear equations with integer coefficients. Hence, we have

$$\mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}{+}\mathcal{J} \cong \mathbb{Z}[w_1, \dots, w_{m-n}]/\mathcal{I}',$$

where $\mathcal{I}'$ is an ideal having a basis each of whose elements is a product of $\geqslant 2$ integer linear forms. Suppose now that $H_{i,j}$, $i > 1$, is a quasitoric manifold. Then we have an isomorphism

$$\mathbb{Z}[w_1, \dots, w_{m-n}]/\mathcal{I}' \cong \mathbb{Z}[u, v]/\mathcal{I}'',$$

where $\mathcal{I}''$ is the ideal from Theorem 5.42. It is easy to see that in this case we have $m - n = 2$ above, and w_1, w_2 can be identified with u, v. Thus, the ideal $\mathcal{I}''$ must have a basis consisting of products of $\geqslant 2$ linear forms with integer coefficients. But this is impossible for $i > 1$. □

5.4. Combinatorial formulae for Hirzebruch genera of quasitoric manifolds

The constructions from the previous section open the way to evaluation of cobordism invariants (Chern numbers, Hirzebruch genera etc.) on omnioriented quasitoric manifolds in terms of the combinatorics of the quotient. In this section we expose the results obtained in this direction by the second author in [**111**], [**112**]. Namely, using arguments similar to that from the proof of Theorem 1.20 we construct a circle action with only isolated fixed points on any quasitoric manifold M^{2n}. If M^{2n} is omnioriented then this action preserves the stably complex structure and its local representations near fixed points are described in terms of the characteristic matrix Λ. This allows us to calculate Hirzebruch's χ_y-genus as a sum of contributions corresponding to the vertices of polytope. Each of these contributions depends only on the "local combinatorics" near the vertex. In particular, we obtain formulae for the signature and the Todd genus of M^{2n}.

DEFINITION 5.44. The *Hirzebruch genus* [**74**], [**75**] associated with the series

$$Q(x) = 1 + \sum q_k x^k, \quad q_k \in \mathbb{Q},$$

is the ring homomorphism $\varphi_Q : \Omega^U \to \mathbb{Q}$ that to each cobordism class $[M^{2n}] \in \Omega^U_{2n}$ assigns the value given by the formula

$$\varphi_Q[M^{2n}] = \Big(\prod_{i=1}^{n} Q(x_i), \langle M^{2n} \rangle \Big).$$

Here M^{2n} is a smooth manifold whose stable tangent bundle $\tau(M^{2n})$ is a complex bundle with complete Chern class in cohomology

$$c(\tau) = 1 + c_1(\tau) + \dots + c_n(\tau) = \prod_{i=1}^{n}(1 + x_i),$$

and $\langle M^{2n} \rangle$ is the fundamental class in homology.

As it was shown by Hirzebruch, every ring homomorphism $\varphi : \Omega^U \to \mathbb{Q}$ arises as a genus φ_Q for some Q. There is also an oriented version of Hirzebruch genera, which deals with ring homomorphisms $\varphi : \Omega^{SO} \to \mathbb{Q}$ from the *oriented cobordism ring* Ω^{SO}.

DEFINITION 5.45. The χ_y*-genus* is the Hirzebruch genus associated with the series

$$Q(x) = \frac{x(1 + ye^{-x(1+y)})}{1 - e^{-x(1+y)}},$$

where $y \in \mathbb{R}$ is a parameter. For particular values $y = -1, 0, 1$ we obtain the n-th Chern number, the *Todd genus* and the L*-genus* of M^{2n} correspondingly.

Given a $4k$-dimensional oriented manifold X^{4k}, the *signature* $\operatorname{sign}(X^{4k})$ is defined as the signature (the number of positive squares minus the number of negative ones) of the intersection form

$$f(\alpha, \beta) := \big(\alpha \cdot \beta, \langle X^{4k} \rangle\big), \quad \alpha, \beta \in H^{2k}(X^{4k})$$

in the middle-dimensional cohomology $H^{2k}(X^{4k})$. We also extend the signature to all even-dimensional manifolds by setting $\operatorname{sign}(X^{4k+2}) = 0$. It can be shown that the signature is multiplicative and is an invariant of cobordism, so it defines a ring homomorphism $\varphi : \Omega^{SO} \to \mathbb{Z}$ (a genus). By the classical theorem of Hirzebruch [**74**], the signature coincides with the L-genus, and we will not distinguish between the two notions in the sequel.

In the case when M^{2n} is a complex manifold, the value $\chi_y(M^{2n})$ can be calculated in terms of Euler characteristics of the *Dolbeault complexes* on M^{2n}, see [**74**]. This was the original Hirzebruch's motivation for studying the χ_y-genus.

In this section we assume that we are given an omnioriented quasitoric manifold M^{2n} over some P^n with characteristic matrix Λ. This specifies a stably complex structure on M^{2n}, as described in the previous section. The orientation of M^{2n} determines the fundamental class $\langle M^{2n} \rangle \in H_{2n}(M^{2n}; \mathbb{Z})$.

5.4.1. The sign and index of a vertex, edge vectors and calculation of the χ_y-genus. Here we introduce some combinatorial invariants of the torus action and calculate the χ_y-genus.

CONSTRUCTION 5.46. Suppose v is a vertex of P^n expressed as the intersection of n facets:

$$v = F_{i_1} \cap \dots \cap F_{i_n}. \tag{5.10}$$

To each facet F_{i_k} above assign the unique edge E_k such that $E_k \cap F_{i_k} = v$ (that is, $E_k = \bigcap_{j \neq k} F_{i_j}$). Let $\boldsymbol{e}_k$ be a vector along E_k with origin v. Then $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is a basis of $\mathbb{R}^n$, which may be either positively or negatively oriented depending on the ordering of facets in (5.10). Throughout this section we assume this ordering to be such that $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is a positively oriented basis.

Once we specified an ordering of facets in (5.10), the facet vectors $\boldsymbol{\lambda}_{i_1}, \dots, \boldsymbol{\lambda}_{i_n}$ at v may in turn constitute either positively or negatively oriented basis depending on the sign of the determinant of $\Lambda_{(v)} = (\boldsymbol{\lambda}_{i_1}, \dots, \boldsymbol{\lambda}_{i_n})$ (see (5.5)).

DEFINITION 5.47. The *sign* of a vertex $v = F_{i_1} \cap \dots \cap F_{i_n}$ is

$$\sigma(v) := \det \Lambda_{(v)}.$$

One can understand the sign of a vertex geometrically as follows. For each vertex $v \in P^n$ the omniorientation of M^{2n} determines two orientations of the tangent space T_vM^{2n} at v. The first is induced by the orientation of M^{2n}. On the other hand, T_vM^{2n} decomposes into the sum of n two-dimensional vector spaces normal to the facial submanifolds $M_{i_1}, \dots, M_{i_n}$ containing v. By the definition of omniorientation, each of these two-dimensional vector spaces is oriented, so they together define another orientation of T_vM^{2n}. Then $\sigma(v) = 1$ if the two orientations coincide and $\sigma(v) = -1$ otherwise.

The collection of signs of vertices of P^n is an important invariant of an omnioriented quasitoric manifold. Note that reversing the orientation of M^{2n} changes all signs $\sigma(v)$ to the opposite. At the same time changing the direction of one facet vector reverses the signs for those vertices contained in the corresponding facet.

Let E be an edge of P^n. The isotropy subgroup of 2-dimensional submanifold $\pi^{-1}(E) \subset M^{2n}$ is an $(n-1)$-dimensional subtorus, which we denote by $T(E)$. It can be written as

$$T(E) = \left\{ \left(e^{2\pi i\varphi_1}, \dots, e^{2\pi i\varphi_n}\right) \in T^n \ : \ \mu_1\varphi_1 + \ldots + \mu_n\varphi_n = 0 \right\} \tag{5.11}$$

for some integers $\mu_1, \dots, \mu_n$. We refer to $\boldsymbol{\mu} := (\mu_1, \dots, \mu_n)^t$ as the *edge vector* corresponding to E. This $\boldsymbol{\mu}$ is a primitive vector in the dual lattice $(\mathbb{Z}^n)^*$ and is determined by E only up to a sign. There is no canonical way to choose these signs simultaneously for all edges. However, the following lemma shows that the omniorientation of M^{2n} provides a canonical way to choose signs of edge vectors "locally" at each vertex.

LEMMA 5.48. *For each vertex $v \in P^n$, the signs of edge vectors $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_n$ meeting at v can be chosen in such a way that the $n \times n$-matrix $\mathrm{M}_{(v)} := (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_n)$ satisfies the identity*

$$\mathrm{M}_{(v)}^t \cdot \Lambda_{(v)} = \mathrm{E},$$

where E is the unit matrix. In other words, $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_n$ and $\boldsymbol{\lambda}_{i_1}, \dots, \boldsymbol{\lambda}_{i_n}$ are conjugate bases.

PROOF. At the beginning we choose signs of the edge vectors at v arbitrary, and express v as in (5.10). Then $\boldsymbol{\mu}_k$ is the edge vector corresponding to the edge E_k opposite to F_{i_k}, $k = 1, \ldots, n$. It follows that $E_k \subset F_{i_l}$ for $l \neq k$, so $T(F_{i_l}) \subset T(E_k)$. Hence,

$$\langle \boldsymbol{\mu}_k, \boldsymbol{\lambda}_{i_l} \rangle = 0, \quad l \neq k, \tag{5.12}$$

(see (5.3) and (5.11)). Since $\boldsymbol{\mu}_k$ is a primitive vector and $\boldsymbol{\lambda}_{i_1}, \ldots, \boldsymbol{\lambda}_{i_n}$ is a basis of $\mathbb{Z}^n$, it follows from (5.12) that $\langle \boldsymbol{\mu}_k, \boldsymbol{\lambda}_{i_k} \rangle = \pm 1$. Changing the sign of $\boldsymbol{\mu}_{i_k}$ if necessary, we obtain

$$\langle \boldsymbol{\mu}_k, \boldsymbol{\lambda}_{i_k} \rangle = 1,$$

which together with (5.12) gives $\mathrm{M}^t_{(v)} \cdot \Lambda_{(v)} = \mathrm{E}$, as needed. □

In the sequel, while making some local calculations near a vertex v, we assume that the signs of edge vectors are chosen as in the above lemma. It follows that the edge vectors $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n$ meeting at v constitute an integer basis of $\mathbb{Z}^n$ and

$$\det \mathrm{M}_{(v)} = \sigma(v). \tag{5.13}$$

EXAMPLE 5.49. Suppose $M^{2n} = M_P$ is a smooth toric variety arising from a lattice simple polytope P defined by (1.1). Then $\boldsymbol{\lambda}_i = \boldsymbol{l}_i$, $i = 1, \ldots, m$ (see Example 5.19), whereas the edge vectors at $v \in P^n$ are the primitive integer vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ along the edges with origin at v. It follows from Construction 5.46 that $\sigma(v) = 1$ for any v (compare with Proposition 5.53 below). Lemma 5.48 in this case expresses the fact that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ and $\boldsymbol{l}_{i_1}, \ldots, \boldsymbol{l}_{i_n}$ are conjugate bases of $\mathbb{Z}^n$.

REMARK. Globally Lemma 5.48 provides two directions (signs) for an edge vector, one for each of its ends. These two signs are always different if M^{2n} is a complex manifold (e.g. a smooth toric variety), but in general they may be the same as well.

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)^t \in \mathbb{Z}^n$ be a primitive vector such that

$$\langle \boldsymbol{\mu}, \boldsymbol{\nu} \rangle \neq 0 \quad \text{for any edge vector } \boldsymbol{\mu}. \tag{5.14}$$

The vector $\boldsymbol{\nu}$ defines a one-dimensional oriented subtorus:

$$T_{\boldsymbol{\nu}} := \left\{ \left(e^{2\pi i \nu_1 \varphi}, \ldots, e^{2\pi i \nu_n \varphi} \right) \in T^n \ , \ \varphi \in \mathbb{R} \right\}.$$

LEMMA 5.50 ([**112**, Theorem 2.1]). *For any $\boldsymbol{\nu}$ satisfying* (5.14) *the circle $T_{\boldsymbol{\nu}}$ acts on M^{2n} with only isolated fixed points, corresponding to the vertices of P^n. For each vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$ the action of $T_{\boldsymbol{\nu}}$ induces a representation of S^1 in the tangent space $T_v M^{2n}$ with weights $\langle \boldsymbol{\mu}_1, \boldsymbol{\nu} \rangle, \ldots, \langle \boldsymbol{\mu}_n, \boldsymbol{\nu} \rangle$.*

REMARK. If $M^{2n} = M_P$ is a smooth toric variety, then the genericity condition (5.14) is equivalent to that from the proof of Theorem 1.20.

DEFINITION 5.51. Suppose we are given a primitive vector $\boldsymbol{\nu}$ satisfying (5.14). Define the *index* of a vertex $v \in P^n$ as the number of negative weights of the S^1-representation in $T_v M^{2n}$ from Lemma 5.50. That is, if $v = F_{i_1} \cap \cdots \cap F_{i_n}$ then

$$\operatorname{ind}_{\boldsymbol{\nu}}(v) = \{ \# k : \langle \boldsymbol{\mu}_k, \boldsymbol{\nu} \rangle < 0 \}.$$

REMARK. The index of a vertex v can be also defined in terms of the facet vectors at v. Indeed, Lemma 5.48 shows that if $v = F_{i_1} \cap \cdots \cap F_{i_n}$ then

$$\boldsymbol{\nu} = \langle \boldsymbol{\mu}_1, \boldsymbol{\nu} \rangle \boldsymbol{\lambda}_{i_1} + \cdots + \langle \boldsymbol{\mu}_n, \boldsymbol{\nu} \rangle \boldsymbol{\lambda}_{i_n}.$$

Hence, $\mathrm{ind}_{\boldsymbol{\nu}}(v)$ equals the number of negative coefficients in the representation of $\boldsymbol{\nu}$ as a linear combination of basis vectors $\boldsymbol{\lambda}_{i_1}, \dots, \boldsymbol{\lambda}_{i_n}$.

THEOREM 5.52 ([**111**, Theorem 6], [**112**, Theorem 3.1]). *For any vector $\boldsymbol{\nu}$ satisfying* (5.14)*, the χ_y-genus of M^{2n} can be calculated as*

$$\chi_y(M^{2n}) = \sum_{v \in P^n} (-y)^{\mathrm{ind}_\nu(v)} \sigma(v).$$

The proof of this theorem uses the Atiyah–Hirzebruch formula [**8**] and the circle action from Lemma 5.50.

5.4.2. Top Chern number and Euler characteristic. The value of the χ_y-genus $\chi_y(M^{2n})$ at $y = -1$ equals the n-th Chern number $c_n(\xi)\langle M^{2n}\rangle$ for any $2n$-dimensional stably complex manifold $[M^{2n}, \xi]$. If the stably complex structure on M^{2n} comes from a complex structure in the tangent bundle (i.e. if M^{2n} is almost complex), then the n-th Chern number equals the Euler characteristic of M^{2n}. However, for general stably complex manifolds, the two numbers may differ, see Example 5.61 below.

Given an omnioriented quasitoric manifold M^{2n}, Theorem 5.52 gives the following formula for its top Chern number:

$$c_n[M^{2n}] = \sum_{v \in P^n} \sigma(v). \tag{5.15}$$

If M^{2n} is a smooth projective toric variety, then $\sigma(v) = 1$ for every vertex $v \in P^n$ (see Example 5.49) and $c_n[M^{2n}]$ equals the Euler characteristic $e(M^{2n})$. Hence, for toric varieties the Euler characteristic equals the number of vertices of P^n, which of course is well known. This is also true for arbitrary quasitoric M^{2n}:

$$e(M^{2n}) = f_{n-1}(P^n). \tag{5.16}$$

To prove this, one can just use Lemma 5.50 and observe that the Euler characteristic of an S^1-manifold equals the sum of Euler characteristics of fixed submanifolds.

Comparing (5.15) and (5.16), we can deduce some results on the existence of a T^n-invariant almost complex structure on a quasitoric manifold M^{2n} (see Problem 5.22).

An almost complex structure on M^{2n} determines a canonical orientation of the manifold. A T^n-invariant almost complex structure also determines orientations for the facial submanifolds $M_i^{2(n-1)} \subset M^{2n}$, $i = 1, \dots, m$ (since they are fixed point sets for the appropriate subtori) and thus gives rise to an omniorientation of M^{2n}.

PROPOSITION 5.53. *Suppose that an omniorientation of a quasitoric manifold M^{2n} is determined by a T^n-invariant almost complex structure. Then $\sigma(v) = 1$ for any vertex $v \in P^n$ and, therefore,*

$$c_n[M^{2n}] = e(M^{2n}).$$

PROOF. Indeed, the tangent space T_vM^{2n} has canonical complex structure, and the orientations of normal subspaces to facial submanifolds meeting at v are the canonical orientations of complex subspaces. Hence, the two orientations of T_vM^{2n} coincide, and $\sigma(v) = 1$. □

As a corollary, we obtain the following necessary condition for the existence of a T^n-invariant almost complex structure on M^{2n}.

COROLLARY 5.54. *Let M^{2n} be a quasitoric manifold over P^n and Λ the corresponding characteristic matrix (with undetermined signs of column vectors). Suppose M^{2n} admits a T^n-invariant almost complex structure. Then the signs of column vectors of Λ can be chosen in such a way that the minors $\Lambda_{(v)}$ (see* (5.5)*) are positive for all vertices $v = F_{i_1} \cap \cdots \cap F_{i_n}$ of P^n.*

On the other hand, due to a theorem of Thomas [**132**, Theorem 1.7], a real orientable $2n$-bundle ξ has a complex structure if and only if it has a stable complex structure ω such that $c_n(\omega) = e(\xi)$ (the latter denotes the Euler class). It follows from (5.15) and (5.16) that the condition from the above corollary is also sufficient for a quasitoric manifold M^{2n} to admit an almost complex structure (not necessarily T^n-invariant). Note that although the *stably* complex structure determined by an omniorientation of a quasitoric manifold is T^n-invariant (see Theorem 5.38), the *almost* complex structure whose existence is claimed by the result of Thomas (provided that the condition $c_n[M^{2n}] = e(M^{2n})$ is satisfied) may fail to be invariant.

5.4.3. Signature. The value of the χ_y-genus at $y = 1$ is the *signature* (or the L-genus). Theorem 5.52 gives the following formula.

COROLLARY 5.55. *The signature of an omnioriented quasitoric manifold M^{2n} can be calculated as*

$$\operatorname{sign}(M^{2n}) = \sum_{v \in P^n} (-1)^{\operatorname{ind}_\nu(v)} \sigma(v).$$

Being an invariant of an *oriented* cobordism class, the signature does not depend on a particular choice of stably complex structure (or omniorientation) on the oriented manifold M^{2n}. The following modification of Corollary 5.55 provides a formula for $\operatorname{sign}(M^{2n})$ that does not depend on an omniorientation.

COROLLARY 5.56 ([**112**, Corollary 3.3]). *The signature of an oriented quasitoric manifold M^{2n} can be calculated as*

$$\operatorname{sign}(M^{2n}) = \sum_{v \in P^n} \det(\widetilde{\boldsymbol{\mu}}_1, \dots, \widetilde{\boldsymbol{\mu}}_n),$$

where $\widetilde{\boldsymbol{\mu}}_k$, $k = 1, \dots, n$, are the edge vectors at v oriented in such a way that $\langle \widetilde{\boldsymbol{\mu}}_k, \boldsymbol{\nu} \rangle > 0$.

If $M^{2n} = M_P$ is a smooth toric variety, then $\sigma(v) = 1$ for any $v \in P^n$, and Corollary 5.55 gives

$$\operatorname{sign}(M_P) = \sum_{v \in P^n} (-1)^{\operatorname{ind}_\nu(v)}.$$

Since in this case $\operatorname{ind}_\nu(v)$ equals the index from the proof of Theorem 1.20, we obtain

$$\operatorname{sign}(M_P) = \sum_{k=1}^{n} (-1)^k h_k(P). \tag{5.17}$$

Note that if n is odd then the right hand side of the above formula vanishes due to the Dehn–Sommerville equations. The formula (5.17) appears in a more general context in recent work of Leung and Reiner [**90**]. The quantity in the right hand side of (5.17) arises in the following combinatorial conjecture.

PROBLEM 5.57 (Charney–Davis conjecture). *Let K be a $(2q-1)$-dimensional Gorenstein* flag complex with h-vector $(h_0, h_1, \ldots, h_{2q})$. Is it true that*

$$(-1)^q(h_0 - h_1 + \cdots + h_{2q}) \geqslant 0?$$

This conjecture was posed in [**41**, Conjecture D] for flag simplicial homology spheres. Stanley [**129**, Problem 4] extended it to Gorenstein* complexes. The Charney–Davis conjecture is closely connected with the following differential-geometrical conjecture.

PROBLEM 5.58 (Hopf conjecture). *Let M^{2q} be a Riemannian manifold of non-positive sectional curvature. Is it true that the Euler characteristic $\chi(M^{2n})$ satisfies the inequality*

$$(-1)^q\chi(M^{2q}) \geqslant 0?$$

More details about the connection between the Charney–Davis and Hopf conjectures can be found in [**41**] and in a more recent paper [**50**]. The relationships between the above two problems and the signature of a toric variety are discussed in [**90**].

5.4.4. Todd genus. The next important particular case of χ_y-genus is the *Todd genus*, corresponding to $y = 0$. In this case the summands in the formula from Theorem 5.52 are not well defined for the vertices of index 0, so it requires some additional analysis.

THEOREM 5.59 ([**111**, Theorem 7], [**112**, Theorem 3.4]). *The Todd genus of an omnioriented quasitoric manifold can be calculated as*

$$\mathrm{td}(M^{2n}) = \sum_{v\in P^n:\,\mathrm{ind}_\nu(v)=0} \sigma(v)$$

(the sum is taken over all vertices of index 0*).*

In the case of smooth toric variety there is only one vertex of index 0. This is the "bottom" vertex of P^n, which has all incident edges pointing out (in the notations used in the proof of Theorem 1.20). Since $\sigma(v) = 1$ for every $v \in P^n$, Theorem 5.59 gives $\mathrm{td}(M_P) = 1$, which is well known (see e.g. [**64**, §5.3]). Note that for algebraic varieties the Todd genus equals the *arithmetic genus* [**74**].

If M^{2n} is an almost complex manifold then $\mathrm{td}(M^{2n}) \geqslant 0$ by Proposition 5.53 and Theorem 5.59.

5.4.5. Examples.

EXAMPLE 5.60. Let us look at the projective space $\mathbb{C}P^2$ regarded as a toric variety. Its stably complex structure is determined by the standard complex structure in $\mathbb{C}P^2$, that is, via the isomorphism of bundles $\tau(\mathbb{C}P^2)\oplus\mathbb{C} \simeq \bar\eta\oplus\bar\eta\oplus\bar\eta$. Here $\mathbb{C}$ is the trivial complex line bundle and η is the Hopf line bundle over $\mathbb{C}P^2$. The orientation is defined by the complex structure. The toric variety $\mathbb{C}P^2$ arises from the 2-dimensional lattice simplex with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The facet vectors here are primitive, normal to facets, and pointing inside the polytope. The edge vectors are primitive, parallel to edges, and pointing out of the corresponding vertex. This is shown in Figure 5.2. Let us calculate the Todd genus and the signature using Corollary 5.55 and Theorem 5.59. We have $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = 1$.

Take $\nu = (1,2)$, then $\operatorname{ind}(v_1) = 0$, $\operatorname{ind}(v_2) = 1$, $\operatorname{ind}(v_3) = 2$ (remember that the index is the number of negative scalar products of edge vectors with ν). Thus,

$$\operatorname{sign}(\mathbb{C}P^2) = \operatorname{sign}(\mathbb{C}P^2, \bar\eta \oplus \bar\eta \oplus \bar\eta) = 1, \quad \operatorname{td}(\mathbb{C}P^2) = \operatorname{td}(\mathbb{C}P^2, \bar\eta \oplus \bar\eta \oplus \bar\eta) = 1.$$

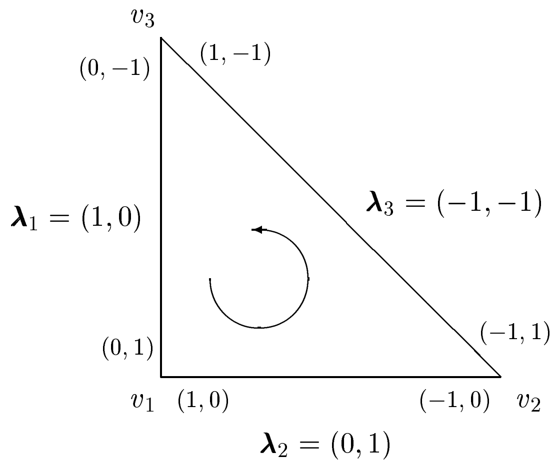


FIGURE 5.2. $\tau(\mathbb{C}P^2) \oplus \mathbb{C} \simeq \bar\eta \oplus \bar\eta \oplus \bar\eta$

EXAMPLE 5.61. Now consider $\mathbb{C}P^2$ with the omniorientation determined by the three facet vectors $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3$, shown in Figure 5.3. This omniorientation differs from the previous example by the sign of $\boldsymbol{\lambda}_3$. The corresponding stably complex structure is determined by the isomorphism $\tau(\mathbb{C}P^2) \oplus \mathbb{R}^2 \cong \bar\eta \oplus \bar\eta \oplus \eta$. Using (5.13) we calculate

$$\sigma(v_1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \sigma(v_2) = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \sigma(v_3) = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$$

Taking $\boldsymbol{\nu} = (1,2)$, we find $\operatorname{ind}_\nu(v_1) = 0$, $\operatorname{ind}_\nu(v_2) = 0$, $\operatorname{ind}_\nu(v_3) = 1$. Thus,

$$\operatorname{sign}[\mathbb{C}P^2, \bar\eta \oplus \bar\eta \oplus \eta] = 1, \quad \operatorname{td}[\mathbb{C}P^2, \bar\eta \oplus \bar\eta \oplus \eta] = 0.$$

Note that in this case formula (5.15) gives

$$c_n[\mathbb{C}P^2, \bar\eta \oplus \bar\eta \oplus \eta] = \sigma(v_1) + \sigma(v_2) + \sigma(v_3) = -1,$$

while the Euler number of $\mathbb{C}P^2$ is 3.

T^n-equivariant stably complex and almost complex manifolds were considered in works of Hattori [**71**] and Masuda [**93**] as a separate generalization (called the *unitary toric manifolds*) of toric varieties. Instead of Davis and Januszkiewicz's characteristic maps, Masuda in [**93**] used the notion of *multi-fan* to describe the combinatorial structure of the orbit space. The multi-fan is a collection of cones which may overlap unlike a usual fan. The Todd genus of a unitary toric manifold was calculated in [**93**] via the degree of the overlap of cones in the multi-fan. This result is equivalent to our Theorem 5.59 in the case of quasitoric manifolds. A formula for the χ_y-genus similar to that from Theorem 5.52 has been obtained (independently) in a more recent paper [**73**]. For more information about multi-fans see [**72**].

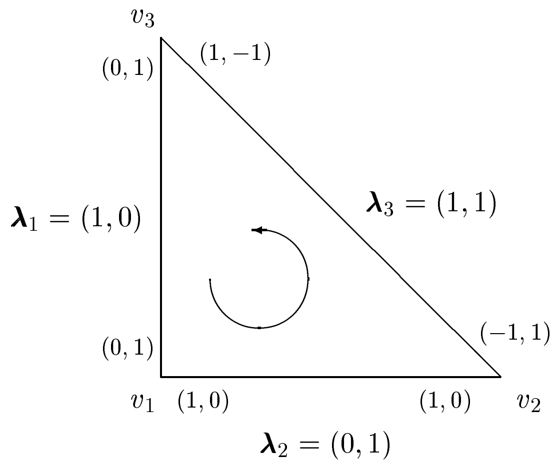


FIGURE 5.3. $\tau(\mathbb{C}P^2) \oplus \mathbb{C} \simeq \bar{\eta} \oplus \bar{\eta} \oplus \eta$

5.5. Classification problems

There are two main classification problems for quasitoric manifolds over a given simple polytope: the equivariant (i.e. up to a ψ-equivariant diffeomorphism) and the topological (i.e. up to a diffeomorphism). Due to Proposition 5.14, the equivariant classification reduces to describing all characteristic maps for given simple polytope P^n. The topological classification problem usually requires additional analysis. In general, both problems seem to be intractable. However, in some particular cases nice classification results may be achieved. Here we give a brief review of what is known on the subject.

Let M^{2n} be a quasitoric manifold over P^n with characteristic map ℓ. We assume here that the facets are ordered in such a way that the first n of them share a common vertex.

LEMMA 5.62. *Up to ψ-equivalence (see Definition* 5.13*) , we may assume that $\ell(F_i)$ is the i-th coordinate subtorus $T_i \subset T^n$, $i = 1, \ldots, n$.*

PROOF. Since the one-dimensional subtori $\ell(F_i)$, $i = 1, \ldots, n$, generate T^n, we may define ψ as any automorphism of T^n that maps $\ell(F_i)$ to T_i. $\square$

It follows that M^{2n} admits an omniorientation whose corresponding characteristic $n \times m$-matrix Λ has the form $(\mathrm{E}\,|\,*)$, where E is the unit matrix and $*$ denotes some integer $n \times (m-n)$-matrix.

In the simplest case $P^n = \Delta^n$ the equivariant (and topological) classification of quasitoric manifolds reduces to the following easy result.

PROPOSITION 5.63. *Any quasitoric manifold over Δ^n is ψ-equivariantly diffeomorphic to $\mathbb{C}P^n$ (regarded as a toric variety, see Examples* 5.7 *and* 5.19*).*

PROOF. The characteristic map for $\mathbb{C}P^n$ has the form

$$\ell_{\mathbb{C}P^n}(F_i) = T_i, \quad i = 1, \ldots, n, \qquad \ell_{\mathbb{C}P^n}(F_{n+1}) = S_d,$$

where $S_d^1 := \{(e^{2\pi i\varphi}, \ldots, e^{2\pi i\varphi}) \in T^n\}$, $\varphi \in \mathbb{R}$, is the diagonal subgroup in T^n. Let M^{2n} be a quasitoric manifold over Δ^n with characteristic map ℓ_M. We may assume that $\ell_M(F_i) = T_i$, $i = 1, \ldots, n$, by Lemma 5.62. Then it easily follows from (5.5) that

$$\ell_M(F_{n+1}) = \left\{ \left(e^{2\pi i \varepsilon_1 \varphi}, \ldots, e^{2\pi i \varepsilon_n \varphi} \right) \in T^n \right\}, \quad \varphi \in \mathbb{R},$$

where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$. Now define the automorphism $\psi : T^n \to T^n$ by

$$\psi\left(e^{2\pi i \varphi_1}, \ldots, e^{2\pi i \varphi_n} \right) = \left(e^{2\pi i \varepsilon_1 \varphi_1}, \ldots, e^{2\pi i \varepsilon_n \varphi_n} \right).$$

It can be readily seen that $\psi \cdot \ell_M = \ell_{\mathbb{C}P^n}$, which together with Proposition 5.14 completes the proof. □

Both problems of equivariant and topological classification also admit a complete solution for $n = 2$ (i.e., for quasitoric manifolds over polygons).

EXAMPLE 5.64. Given an integer k, the *Hirzebruch surface* H_k is the complex manifold $\mathbb{C}P(\zeta_k \oplus \mathbb{C})$, where ζ_k is the complex line bundle over $\mathbb{C}P^1$ with first Chern class k, and $\mathbb{C}P(\cdot)$ denotes the projectivisation of a complex bundle. In particular, each Hirzebruch surface is the total space of the bundle $H_k \to \mathbb{C}P^1$ with fibre $\mathbb{C}P^1$. The surface H_k is diffeomorphic to $S^2 \times S^2$ for even k and to $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ for odd k, where $\overline{\mathbb{C}P}^2$ denotes the space $\mathbb{C}P^2$ with reversed orientation. Each Hirzebruch surface is a non-singular projective toric variety, see [**64**, p. 8]. The orbit space for H_k (regarded as a quasitoric manifold) is a combinatorial square; the corresponding characteristic maps can be described using Example 5.19 (see also [**48**, Example 1.19]).

THEOREM 5.65 ([**106**, p. 553]). *A quasitoric manifold of dimension* 4 *is equivariantly diffeomorphic to an equivariant connected sum of several copies of* $\mathbb{C}P^2$ *and Hirzebruch surfaces* H_k.

COROLLARY 5.66. *A quasitoric manifold of dimension* 4 *is diffeomorphic to a connected sum of several copies of* $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ *and* $S^2 \times S^2$.

The classification problem for quasitoric manifolds over a given simple polytope can be considered as a generalization of the corresponding problem for non-singular toric varieties. The classification result for 4-dimensional toric varieties is similar to Theorem 5.65 and can be found e.g., in [**62**]. In [**105**], to every toric variety over a simple 3-polytope P^3 there were assigned two integer *weights* on every edge of the dual simplicial complex K_P. Using the special "monodromy conditions" for weights, the complete classification of toric varieties over simple 3-polytopes with $\leqslant 8$ facets was obtained in [**105**]. A similar construction was used in [**88**] to obtain the classification of toric varieties over P^n with $m = n + 2$ facets (note that any such simple polytope is a product of two simplices).

In [**56**] the construction of weights from [**105**] was generalized to the case of quasitoric manifolds. This resulted in a criterion [**56**, Theorem 3] for the existence of a quasitoric manifold with prescribed weight set and signs of vertices (see Definition 5.47). The methods of [**56**] allow one to simplify the equations (5.5) for characteristic map on a given polytope. As an application, results on the classification of quasitoric manifolds over a product of an arbitrary number of simplices were obtained there.

CHAPTER 6

Moment-angle complexes

6.1. Moment-angle manifolds $\mathcal{Z}_P$ defined by simple polytopes

For any combinatorial simple polytope P^n with m facets, Davis and Januszkiewicz introduced in [**48**] a T^m-manifold $\mathcal{Z}_P$ with orbit space P^n. This manifold has the following universal property: for every quasitoric manifold $\pi : M^{2n} \to P^n$ there is a principal T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}$ whose composite map with π is the orbit map for $\mathcal{Z}_P$. Topology of manifolds $\mathcal{Z}_P$ and their further generalizations is very nice itself and at the same time provides an effective tool for understanding inter-relations between algebraic and combinatorial objects such as Stanley–Reisner rings, subspace arrangements, cubical complexes etc. In this section we reproduce the original definition of $\mathcal{Z}_P$ and adjust it in a way convenient for subsequent generalizations.

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be the set of facets of P^n. For each facet $F_i \in \mathcal{F}$ denote by T_{F_i} the one-dimensional coordinate subgroup of $T^{\mathcal{F}} \cong T^m$ corresponding to F_i. Then assign to every face G the coordinate subtorus

$$T_G = \prod_{F_i \supset G} T_{F_i} \subset T^{\mathcal{F}}.$$

Note that $\dim T_G = \operatorname{codim} G$. Recall that for every point $q \in P^n$ we denoted by $G(q)$ the unique face containing q in the relative interior.

Definition 6.1. For any combinatorial simple polytope P^n introduce the identification space

$$\mathcal{Z}_P = (T^{\mathcal{F}} \times P^n)/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T_{G(q)}$.

Remark. The above definition resembles constructions 5.5 and 5.12, but this time the equivalence relation depends only on the combinatorics of P^n. Similar constructions appeared in earlier works of Vinberg [**137**] and Davis [**47**] on reflection groups.

The free action of T^m on $T^{\mathcal{F}} \times P^n$ descends to an action on $\mathcal{Z}_P$, with quotient P^n. Let $\rho : \mathcal{Z}_P \to P^n$ be the orbit map. The action of T^m on $\mathcal{Z}_P$ is free over the interior of P^n, while each vertex $v \in P^n$ represents the orbit $\rho^{-1}(v)$ with maximal isotropy subgroup of dimension n.

Lemma 6.2. *The space $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$.*

We will provide several different proofs of this lemma, each of which arises from an equivalent definition of $\mathcal{Z}_P$. To give our first proof we need the following simple topological fact.

Proposition 6.3. *The torus T^k admits an embedding into $\mathbb{R}^{k+1}$.*

Proof. The statement is obvious for $k = 1$. Suppose it holds for $k = i - 1$. We may assume that T^{i-1} is embedded into an i-ball $D^i \subset \mathbb{R}^i$. Represent the $(i+1)$-sphere as $S^{i+1} = D^i \times S^1 \cup S^{i-1} \times D^2$ (two pieces are glued by the identity diffeomorphism of the boundaries). By the assumption, the torus $T^i = T^{i-1} \times S^1$ can be embedded into $D^i \times S^1$ and therefore into S^{i+1}. Since T^i is compact and S^{i+1} is the one-point compactification of $\mathbb{R}^{i+1}$ we have $T^i \subset \mathbb{R}^{i+1}$, and the statement follows by induction. □

Proof of Lemma 6.2. Construction 5.8 provides the atlas $\{U_v\}$ for P^n as a manifold with corners. The set U_v is based on the vertex v and is diffeomorphic to $\mathbb{R}^n_+$. Then $\rho^{-1}(U_v) \cong T^{m-n} \times \mathbb{R}^{2n}$. We claim that $T^{m-n} \times \mathbb{R}^{2n}$ can be realized as an open set in $\mathbb{R}^{m+n}$, thus providing a chart for $\mathcal{Z}_P$. To see this we embed T^{m-n} into $\mathbb{R}^{m-n+1}$ as a closed hypersurface H (Proposition 6.3). Since the normal bundle is trivial, a small neighborhood of $H \subset \mathbb{R}^{m-n+1}$ is homeomorphic to $T^{m-n} \times \mathbb{R}$. Taking the cartesian product with $\mathbb{R}^{2n-1}$ we obtain an open set in $\mathbb{R}^{m+n}$ homeomorphic to $T^{m-n} \times \mathbb{R}^{2n}$. □

The following statement follows easily from the definition of $\mathcal{Z}_P$.

Proposition 6.4. *If $P = P_1 \times P_2$ for some simple polytopes P_1, P_2, then $\mathcal{Z}_P = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$. If $G \subset P$ is a face, then $\mathcal{Z}_G$ is a submanifold of $\mathcal{Z}_P$.*

Suppose now that we are given a characteristic map ℓ on P^n and $M^{2n}(\ell)$ is the derived quasitoric manifold (Construction 5.12). Choosing an omniorientation in any way we obtain a directed characteristic map $\lambda : T^{\mathcal{F}} \to T^n$. Denote its kernel by $H(\ell)$ (it depends only on ℓ); then $H(\ell)$ is an $(m-n)$-dimensional subtorus of $T^{\mathcal{F}}$.

Proposition 6.5. *The subtorus $H(\ell)$ acts freely on $\mathcal{Z}_P$, thereby defining a principal T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}(\ell)$.*

Proof. It follows from (5.5) that $H(\ell)$ meets every isotropy subgroup only at the unit. This implies that the action of $H(\ell)$ on $\mathcal{Z}_P$ is free. By definitions of $\mathcal{Z}_P$ and $M^{2n}(\ell)$, the projection $\lambda \times \mathrm{id} : T^{\mathcal{F}} \times P^n \to T^n \times P^n$ descends to the projection

$$(T^{\mathcal{F}} \times P^n)/\!\sim\; \longrightarrow (T^n \times P^n)/\!\sim,$$

which displays $\mathcal{Z}_P$ as a principal T^{m-n}-bundle over $M^{2n}(\ell)$. □

To simplify notations, from now on we will write T^m, $\mathbb{C}^m$ etc. instead of $T^{\mathcal{F}}$, $\mathbb{C}^{\mathcal{F}}$ etc.

Consider the unit poly-disc $(D^2)^m$ in the complex space:

$$(D^2)^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| \leqslant 1, \quad i = 1, \ldots, m\}.$$

Then $(D^2)^m$ is stable under the standard action of T^m on $\mathbb{C}^m$, and the quotient is the unit cube $I^m \subset \mathbb{R}^m_+$.

Lemma 6.6. *The cubical embedding $i_P : P^n \to I^m$ from Construction 4.5 is covered by an equivariant embedding $i_e : \mathcal{Z}_P \to (D^2)^m$.*

Proof. Recall that the cubical complex $\mathcal{C}(P^n)$ consists of the cubes C^n_v based on the vertices $v \in P^n$. Note that C^n_v is contained in the open set $U_v \subset P^n$ (see Construction 5.8). The inclusion $C^n_v \subset U_v$ is covered by an equivariant inclusion $B_v \subset \mathbb{C}^m$, where $B_v = \rho^{-1}(C^n_v)$ is a closed subset homeomorphic to $(D^2)^n \times T^{m-n}$. Since $\mathcal{Z}_P = \bigcup_{v \in P^n} B_v$ and B_v is stable under the T^m-action, the resulting embedding $\mathcal{Z}_P \to (D^2)^m$ is equivariant. □

It follows from the proof that the manifold $\mathcal{Z}_P$ is represented as a union of $f_{n-1}(P)$ closed T^m-invariant subspaces B_v. In section 6.3 we will use this to construct a cell decomposition of $\mathcal{Z}_P$. For now, we mention that if $v = F_{i_1} \cap \cdots \cap F_{i_n}$ then

$$i_e(B_v) = (D^2)^n_{i_1,\ldots,i_n} \times T^{m-n}_{[m]\setminus\{i_1,\ldots,i_n\}} \subset (D^2)^m,$$

or, more precisely,

$$i_e(B_v) = \{(z_1, \ldots, z_m) \in (D^2)^m \;:\; |z_i| = 1 \text{ for } i \notin \{i_1, \ldots, i_n\}\}.$$

Recalling that the vertices of P^n correspond to the maximal simplices in the polytopal sphere K_P (the boundary of polar polytope P^*), we can write

$$i_e(\mathcal{Z}_P) = \bigcup_{\sigma \in K_P} (D^2)_\sigma \times T_{\widehat{\sigma}} \subset (D^2)^m, \tag{6.1}$$

where $\widehat{\sigma} = [m] \setminus \sigma$. The above formula may be regarded as an alternative definition of $\mathcal{Z}_P$. Introducing the polar coordinates in $(D^2)^m$ we see that $i_e(B_v)$ is parametrized by n radial (or moment) and m angle coordinates. We refer to $\mathcal{Z}_P$ as the *moment-angle manifold corresponding to* P^n.

EXAMPLE 6.7. Let $P^n = \Delta^n$ (the n-simplex). Then $\mathcal{Z}_P$ is homeomorphic to the $(2n+1)$-sphere S^{2n+1}. The cubical complex $\mathcal{C}(\Delta^n)$ (see Construction 4.5) consists of $(n+1)$ cubes C^n_v. Each subset $B_v = \rho^{-1}(C^n_v)$ is homeomorphic to $(D^2)^n \times S^1$. In particular, for $n = 1$ we obtain the representation of the 3-sphere S^3 as a union of two solid tori $D^2 \times S^1$ and $S^1 \times D^2$, glued by the identity diffeomorphism of their boundaries.

Another way to construct an equivariant embedding of $\mathcal{Z}_P$ into $\mathbb{C}^m$ can be derived from Construction 1.8.

CONSTRUCTION 6.8. Consider the affine embedding $A_P : P^n \hookrightarrow \mathbb{R}^m_+$ defined by (1.6). It is easy to see that $\mathcal{Z}_P$ enters the following pullback diagram:

$$\begin{array}{ccc} \mathcal{Z}_P & \longrightarrow & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \\ P^n & \xrightarrow{A_P} & \mathbb{R}^m_+ \end{array}.$$

Thus, there is an equivariant embedding $\mathcal{Z}_P \hookrightarrow \mathbb{C}^m$ covering A_P. A choice of matrix W in Construction 1.8 gives a basis in the $(m-n)$-dimensional subspace orthogonal to the n-plane containing $A_P(P^n)$ (see (1.6)). The following statement follows.

COROLLARY 6.9 (see also [**38**, §3]). *The embedding $\mathcal{Z}_P \hookrightarrow \mathbb{C}^m$ has the trivial normal bundle. In particular, $\mathcal{Z}_P$ is null-cobordant.*

REMARK. Another way to see that $\mathcal{Z}_P$ is null-cobordant is to establish a free S^1-action on it (see e.g. Proposition 7.29). Then we get the manifold $\mathcal{Z}_P \times_{S^1} D^2$ with boundary $\mathcal{Z}_P$.

6.2. General moment-angle complexes $\mathcal{Z}_K$

In this section, for any cubical subcomplex in I^m, we define a certain T^m-stable subcomplex in the m-disc $(D^2)^m$. In particular, this provides an extension of the construction of $\mathcal{Z}_P$ to the case of general simplicial complex K. The resulting space, denoted $\mathcal{Z}_K$, is not a manifold for arbitrary K, but is so when K is a simplicial sphere. The complex $\mathcal{Z}_K$, as a generalization of manifold $\mathcal{Z}_P$, first appeared in [**48**, §4.1]. The approach used there involves the notion of "simple polyhedral complex", which extends the correspondence between polytopal simplicial spheres and simple polytopes to general simplicial complexes.

In the sequel, we denote the canonical projection $(D^2)^m \to I^m$ (and any of its restriction to a closed T^m-stable subset of $(D^2)^m$) by ρ. For each face $C_{\sigma\subset\tau}$ of I^m (see (4.1)) define

$$\begin{aligned}(6.2)\quad B_{\sigma\subset\tau} &:= \rho^{-1}(C_{\sigma\subset\tau})\\ &= \{(z_1,\dots,z_m)\in(D^2)^m:\ z_i=0 \text{ for } i\in\sigma,\ |z_i|=1 \text{ for } i\notin\tau\}.\end{aligned}$$

It follows that if $|\sigma|=i$ and $|\tau|=j$, then $B_{\sigma\subset\tau}\cong(D^2)^{j-i}\times T^{m-j}$, where the disc factors $D^2\subset(D^2)^{j-i}$ are indexed by $\tau\setminus\sigma$, while the circle factors $S^1\subset T^{m-j}$ are indexed by $[m]\setminus\tau$.

Definition 6.10. Let $\mathcal{C}$ be a cubical subcomplex of I^m. The *moment-angle complex* $\mathrm{ma}(\mathcal{C})$ corresponding to $\mathcal{C}$ is the T^m-invariant decomposition of $\rho^{-1}(\mathcal{C})$ into the "moment-angle" blocks $B_{\sigma\subset\tau}$ (6.2) corresponding to the faces $C_{\sigma\subset\tau}$ of $\mathcal{C}$. Thus, $\mathrm{ma}(\mathcal{C})$ is defined from the commutative diagram

$$\begin{tikzcd} \mathrm{ma}(\mathcal{C}) \arrow[r] \arrow[d] & (D^2)^m \arrow[d, "\rho"] \\ \mathcal{C} \arrow[r] & I^m \end{tikzcd}.$$

The torus T^m acts on $\mathrm{ma}(\mathcal{C})$ with orbit space $\mathcal{C}$.

Let K^{n-1} be a simplicial complex on the set $[m]$. In section 4.2 two canonical cubical subcomplexes of I^m, namely $\mathrm{cub}(K)$ (4.3) and $\mathrm{cc}(K)$ (4.4), were associated to K^{n-1}. We denote the corresponding moment-angle complexes by $\mathcal{W}_K$ and $\mathcal{Z}_K$ respectively. Thus, we have

$$(6.3)\qquad \begin{tikzcd} \mathcal{W}_K \arrow[r] \arrow[d, "\rho"'] & (D^2)^m \arrow[d, "\rho"] \\ \mathrm{cub}(K) \arrow[r] & I^m \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathcal{Z}_K \arrow[r] \arrow[d, "\rho"'] & (D^2)^m \arrow[d, "\rho"] \\ \mathrm{cc}(K) \arrow[r] & I^m \end{tikzcd},$$

where the horizontal arrows are embeddings, while the vertical ones are orbit maps for T^m-actions. Note that $\dim\mathcal{Z}_K=m+n$ and $\dim\mathcal{W}_K=m+n-1$.

Remark. Suppose that $K=K_P$ for some simple polytope P. Then it follows from (6.1) that $\mathcal{Z}_K$ is identified with $\mathcal{Z}_P$ (or, more precisely, with $i_e(\mathcal{Z}_P)$). The simple polyhedral complex P_K, used in [**48**] to define $\mathcal{Z}_K$ for general K, now can be interpreted as a certain face decomposition of the cubical complex $\mathrm{cc}(K)$ (see also the proof of Lemma 6.13 below).

Note the complex $\mathcal{Z}_K$ depends on the ambient set $[m]$ of K as well as the complex K. In the case when it is important to emphasize this we will use the notation $\mathcal{Z}_{K,[m]}$. If we assume that K is a simplicial complex *on the vertex set* $[m]$, then $\mathcal{Z}_K$ is determined by K. However, in some situations (see e.g. section 6.4) it is convenient to consider simplicial complexes K on $[m]$ whose vertex sets are proper subsets of $[m]$. Let $\{i\}$ be a *ghost vertex* of K, i.e. $\{i\}$ is a one-element subset of $[m]$ which is not a vertex of K. Then the whole cubical subcomplex $\mathrm{cc}(K)\subset I^m$ is contained in the facet $\{y_i = 1\}$ of I^m (see the remark after Construction 4.9). The following proposition follows easily from (6.3).

PROPOSITION 6.11. *Suppose $\{i_1\}, \ldots, \{i_k\}$ are ghost vertices of K. Then*

$$\mathcal{Z}_{K,[m]} = \mathcal{Z}_{K,[m]\setminus\{i_1,\ldots,i_k\}} \times T^k.$$

We call this easy observation "stabilization of moment-angle complexes via the multiplication by tori". It means that if we embed K into a set larger than its vertex set then the corresponding complex $\mathcal{Z}_K$ is multiplied by the torus of dimension equal to the number of "ghost vertices".

EXAMPLE 6.12. 1. Let K be the boundary of $(m-1)$-simplex. Then $\mathrm{cc}(K)$ is the union of m facets of I^m meeting at the vertex $(1, \ldots, 1)$, and $\mathcal{Z}_K$ is the $(2m-1)$-sphere S^{2m-1} (compare with Example 6.7).

2. Let K be an $(m-1)$-simplex. Then $\mathrm{cc}(K)$ is the whole cube I^m and $\mathcal{Z}_K$ is the m-disc $(D^2)^m$.

LEMMA 6.13. *Suppose K is a simplicial $(n-1)$-sphere. Then $\mathcal{Z}_K$ is an $(m+n)$-dimensional (closed) manifold.*

PROOF. In this proof we identify the polyhedrons $|K|$ and $|\operatorname{cone}(K)|$ with their images $\mathrm{cub}(K) \subset I^m$ and $\mathrm{cc}(K) \subset I^m$ under the map $|\operatorname{cone}(K)| \to I^m$, see Proposition 4.10. For each vertex $\{i\} \in K$ denote by $\widetilde{F}_i$ the union of $(n-1)$-cubes of $\mathrm{cub}(K)$ that contain $\{i\}$. Alternatively, $\widetilde{F}_i$ is $|\operatorname{star}_{K'}\{i\}|$. These $\widetilde{F}_1, \ldots, \widetilde{F}_m$ will play the role of facets of a simple polytope. If $K = K_P$ for some P, then $\widetilde{F}_i$ is the image of a facet of P under the map $i_P : \mathcal{C}(P) \to I^m$ (see Construction 4.5). As in the case of simple polytopes, we define "faces" of $\mathrm{cc}(K)$ as non-empty intersections of "facets" $\widetilde{F}_1, \ldots, \widetilde{F}_m$. Then the "vertices" (i.e. non-empty intersections of n "facets") are the barycenters of $(n-1)$-simplices of $|K|$. For every such barycenter b, denote by U_b the open subset of $\mathrm{cc}(K)$ obtained by deleting all "faces" not containing b. Then U_b is identified with $\mathbb{R}^n_+$, while $\rho^{-1}(U_b)$ is homeomorphic to $T^{m-n} \times \mathbb{R}^{2n}$. This defines a structure of manifold with corners on the n-ball $\mathrm{cc}(K) = |\operatorname{cone}(K)|$, with atlas $\{U_b\}$. Furthermore, $\mathcal{Z}_K = \rho^{-1}(\mathrm{cc}(K))$ is a manifold, with atlas $\{\rho^{-1}(U_b)\}$. □

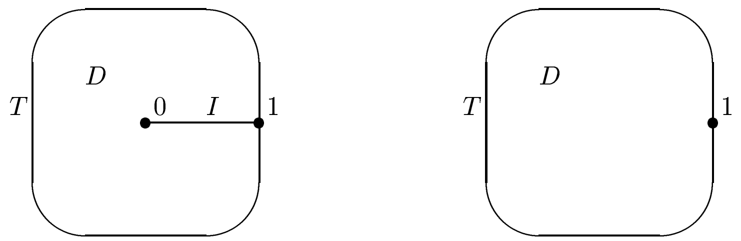
PROBLEM 6.14. *Characterise simplicial complexes K for which $\mathcal{Z}_K$ is a manifold.*

We will see below (Theorem 7.6) that if $\mathcal{Z}_K$ is a manifold, then K is a Gorenstein* complex (see Definition 3.37) for homological reasons. Hence, the answer to the above problem is somewhere between "simplicial spheres" and "Gorenstein* complexes".

6.3. Cell decompositions of moment-angle complexes

Here we consider two cell decompositions of $(D^2)^m$ and apply them to construct cell decompositions for moment-angle complexes. The first one has 5^m cells and descends to a cell complex structure (with 5 types of cells) on any moment-angle complex $\mathrm{ma}(\mathcal{C}) \subset (D^2)^m$. The second cell decomposition of $(D^2)^m$ has only 3^m cells, but it defines a cell complex structure (with 3 types of cells) only on moment-angle complexes $\mathcal{Z}_K$.

Let us consider the cell decomposition of D^2 with one 2-cell D, two 1-cells I, T and two 0-cells 0, 1, shown on Figure 6.1 (a). It defines a cell complex structure



(a) (b)

FIGURE 6.1. Cell decompositions of D^2.

on the poly-disc $(D^2)^m$ with 5^m cells. Each cell of this complex is a product of cells of 5 different types: D_i, I_i, 0_i, T_i and 1_i, $i = 1, \dots, m$. We encode cells in the language of "sign vectors" used in the theory of hyperplane arrangements, see e.g. [**22**]. Each cell of $(D^2)^m$ with respect to our 5^m-cell decomposition will be represented by a sign vector $\mathcal{R} \in \{D, I, 0, T, 1\}^m$. We denote by $\mathcal{R}_D$, $\mathcal{R}_I$, $\mathcal{R}_0$, $\mathcal{R}_T$ and $\mathcal{R}_1$ respectively the D-, I-, 0-, T- and 1-component of $\mathcal{R}$. Each of these components can be seen as a subset of $[m]$, and all five subsets are complementary. This justifies the notations $|\mathcal{R}_D|$, $|\mathcal{R}_I|$, $|\mathcal{R}_0|$, $|\mathcal{R}_T|$ and $|\mathcal{R}_1|$ for the number of D-, I-, 0-, T- and 1-entries of $\mathcal{R}$ respectively. In particular, we see that the closure of a cell $\mathcal{R}$ is homeomorphic to a product of $|\mathcal{R}_D|$ discs, $|\mathcal{R}_I|$ segments and $|\mathcal{R}_T|$ circles. Our first observation is that this cell decomposition of $(D^2)^m$ induces a cell decomposition of any moment-angle complex in $(D^2)^m$:

LEMMA 6.15. *For any cubical subcomplex $\mathcal{C}$ of I^m the corresponding moment-angle complex* $\mathrm{ma}(\mathcal{C})$ *is a cellular subcomplex of* $(D^2)^m$.

PROOF. Indeed, $\mathrm{ma}(\mathcal{C})$ is a union of "moment-angle" blocks $B_{\sigma\subset\tau}$ (6.2), and each $B_{\sigma\subset\tau}$ is the closure of cell $\mathcal{R}$ with $\mathcal{R}_D = \tau \setminus \sigma$, $\mathcal{R}_0 = \sigma$, $\mathcal{R}_T = [m] \setminus \tau$, $\mathcal{R}_I = \mathcal{R}_1 = \varnothing$. □

Now we restrict our attention to the moment-angle complex $\mathcal{Z}_K$ corresponding to cubical complex $\mathrm{cc}(K) \subset I^m$ (see (6.3)). By the definition, $\mathcal{Z}_K$ is the union of moment-angle blocks $B_{\sigma\subset\tau} \subset (D^2)^m$ with $\tau \in K$. Denote

$$B_\tau := B_{\varnothing\subset\tau} = \big\{(z_1, \dots, z_m) \in (D^2)^m \ : \ |z_j| = 1 \text{ for } j \notin \tau\big\}. \tag{6.4}$$

Then $B_\tau = \rho^{-1}(C_\tau)$ (remember our previous notation $C_\tau := C_{\varnothing\subset\tau}$) and $B_{\sigma\subset\tau} \subset B_\tau$ for any $\sigma \subset \tau$. It follows that

$$\mathcal{Z}_K = \bigcup_{\tau\in K} B_\tau \tag{6.5}$$

(compare this with the note after (4.4)).

REMARK. If $K = K_P$ for a simple polytope P and $|\tau| = n$, then B_τ is $i_e(B_v)$ for $v = \bigcap_{j\in\tau} F_j$. Hence, (6.5) reduces to (6.1) in this case.

Note that $B_\tau \cap B_{\tau'} = B_{\tau\cap\tau'}$. This observation allows us to simplify the cell decomposition from Lemma 6.15 in the case $\mathrm{ma}(\mathcal{C}) = \mathcal{Z}_K$. For this we replace the union of cells 0, I, D (see Figure 6.1 (a)) by one 2-dimensional cell (which we keep denoting D for simplicity). The resulting cell decomposition of D^2 with 3 cells is shown on Figure 6.1 (b). It defines a cell decomposition of $(D^2)^m$ with 3^m cells, each of which is a product of 3 different types D_i, T_i and 1_i, $i = 1, \dots, m$. Again we use the sign vector language and encode the new cells of $(D^2)^m$ by sign vectors $\mathcal{T} \in \{D, T, 1\}^m$. The notation $\mathcal{T}_D$, $|\mathcal{T}_T|$ etc. have the same meaning as in the case of 5^m-cell decomposition. The closure of $\mathcal{T}$ is now a product of $|\mathcal{T}_D|$ discs and $|\mathcal{T}_T|$ circles.

LEMMA 6.16. *The moment-angle complex $\mathcal{Z}_K$ is a cellular subcomplex of $(D^2)^m$ with respect to the 3^m-cell decomposition (see Figure 6.1 (b)). Those cells $\mathcal{T} \subset (D^2)^m$ which form $\mathcal{Z}_K$ are determined by the condition $\mathcal{T}_D \in K$.*

PROOF. Since $B_\tau = B_{\varnothing \subset \tau}$ is the closure of cell $\mathcal{T}$ with $\mathcal{T}_D = \tau$, $\mathcal{T}_T = [m] \setminus \tau$ and $\mathcal{T}_1 = \varnothing$, the statement follows from (6.5). $\square$

REMARK. Note that for general $\mathcal{C}$ the moment-angle complex $\mathrm{ma}(\mathcal{C})$ is *not* a cell subcomplex with respect to the 3^m-cell decomposition of $(D^2)^m$.

LEMMA 6.17. *Let $\phi : K_1 \hookrightarrow K_2$ be an inclusion of simplicial complexes on the sets $[m_1]$ and $[m_2]$ respectively. Then it induces an equivariant cellular map $\phi_{\mathrm{ma}} : \mathcal{Z}_{K_1} \to \mathcal{Z}_{K_2}$ of the corresponding moment-angle complexes.*

PROOF. Assign the i-th vector of the standard basis of $\mathbb{C}^{m_1}$ to the element $i \in [m_1]$, and similarly for $\mathbb{C}^{m_2}$ and $[m_2]$. This allows us to extend the map $\phi : [m_1] \to [m_2]$ to an inclusion $\phi_{\mathbb{C}} : \mathbb{C}^{m_1} \to \mathbb{C}^{m_2}$. For any subset $\tau \subset [m_1]$ the map $\phi_{\mathbb{C}}$ takes $B_\tau \subset \mathbb{C}^{m_1}$ (see (6.4)) to $B_{\phi(\tau)} \subset \mathbb{C}^{m_2}$. Since ϕ is a simplicial map, for τ a simplex of K_1 we have $\phi(\tau) \in K_2$ and $\phi_{\mathbb{C}}(B_\tau) \subset \mathcal{Z}_{K_2}$ (see (6.5)). Hence, $\phi_{\mathbb{C}}$ defines an (equivariant) map $\phi_{\mathrm{ma}} : \mathcal{Z}_{K_1} \to \mathcal{Z}_{K_2}$. $\square$

Let us apply the construction of $\mathcal{Z}_K$ (see (6.3)) to the case when $K = \varnothing$ (regarded as a simplicial complex on $[m]$). Then $\mathrm{cc}(K) = (1, \dots, 1) \in I^m$ (the cone over the empty set is just one vertex), and so $\mathcal{Z}_\varnothing = \rho^{-1}(1, \dots, 1) \cong T^m$, where $\mathcal{Z}_\varnothing = \mathcal{Z}_{\varnothing,[m]}$. (Another way to see this is to apply Proposition 6.11 in the case $K = \varnothing$.) We also observe that $\mathcal{Z}_\varnothing$ is contained in $\mathcal{Z}_K$ for any K on $[m]$ as a T^m-stable subset.

LEMMA 6.18. *Let K be a simplicial complex on the vertex set $[m]$. The inclusion $\mathcal{Z}_\varnothing \hookrightarrow \mathcal{Z}_K$ is a cellular embedding homotopical to a map to a point, i.e. the torus $\mathcal{Z}_\varnothing$ is a cellular subcomplex contractible within $\mathcal{Z}_K$.*

PROOF. $\mathcal{Z}_\varnothing \subset \mathcal{Z}_K$ is a cellular subcomplex since it is the closure of the m-dimensional cell $\mathcal{T}$ with $\mathcal{T}_T = [m]$. So it remains to prove that T^m is contractible within $\mathcal{Z}_K$. We do this by induction on m. If $m = 1$ then the only option for K is $K = \Delta^0$ (0-simplex), so $\mathcal{Z}_K = D^2$ (see Example 6.12.2) and $\mathcal{Z}_{\varnothing,[1]} = S^1$ is contractible in $\mathcal{Z}_K$. Now suppose that the vertex set of K is $[m]$. The embedding under the question factors as

$$\mathcal{Z}_{\varnothing,[m]} \hookrightarrow \mathcal{Z}_{K_{[m-1]},[m]} \hookrightarrow \mathcal{Z}_{K,[m]}, \tag{6.6}$$

where $K_{[m-1]}$ is the maximal subcomplex of K on the vertex set $[m-1]$, see (2.1). By Proposition 6.11, $\mathcal{Z}_{\varnothing,[m]} = \mathcal{Z}_{\varnothing,[m-1]} \times S^1$ and $\mathcal{Z}_{K_{[m-1]},[m]} = \mathcal{Z}_{K_{[m-1]},[m-1]} \times S^1$. By the inductive hypothesis we may assume that the embedding $\mathcal{Z}_{\varnothing,[m-1]} \subset \mathcal{Z}_{K_{[m-1]},[m-1]}$ is null-homotopic, so the composite embedding (6.6) is homotopic to the map $\mathcal{Z}_{\varnothing,[m-1]} \times S^1 \to \mathcal{Z}_{K,[m]}$ that sends $\mathcal{Z}_{\varnothing,[m-1]}$ to a point and S^1 to the closure of the cell $(1, \dots, 1, T) \subset \mathcal{Z}_{K,[m]}$ (the latter is understood as a vector of letters $D, T, 1$). But since $\{m\}$ is a vertex of K, the complex $\mathcal{Z}_K$ also contains the cell $(1,\ldots,1,D)$, so a disc D^2 is patched to the closure of $(1,\ldots,1,T)$. It follows that the whole map (6.6) is null-homotopic. ☐

COROLLARY 6.19. *For any simplicial complex K on the vertex set $[m]$ the moment-angle complex $\mathcal{Z}_K$ is simply connected.*

PROOF. Indeed, the 1-skeleton of our cellular decomposition of $\mathcal{Z}_K$ is contained in the torus $\mathcal{Z}_\varnothing$, which is null-homotopic by Lemma 6.18. ☐

6.4. Moment-angle complexes corresponding to joins, connected sums and bistellar moves

Here we study the behavior of moment-angle complexes $\mathcal{Z}_K$ with respect to constructions from section 2.2. In particular, we describe moment-angle complexes corresponding to joins and connected sums of simplicial complexes and interpret bistellar moves (see Definition 2.39) as certain surgery-like operations on moment-angle complexes.

AGREEMENT. Let $\tau = \{j_1,\ldots,j_k\}$ be a subset of $[m]$. In this section we will denote the moment-angle block $B_\tau \cong (D^2)^k \times T^{m-k}$ (6.4) by $D^{2k}_\tau \times T^{m-k}$. The boundary of B_τ is

$$\partial B_\tau = \bigcup_{j\in\tau} D^{2k-2}_{\tau\setminus\{j\}} \times T^{m-k+1} \cong S^{2k-1} \times T^{m-k}$$

(compare with Example 6.7). We denote $\partial B_\tau = S^{2k-1}_\tau \times T^{m-k}$. Furthermore, for any partition $[m] = \sigma\cup\tau\cup\rho$ into three complementary subsets with $|\sigma| = i$, $|\tau| = j$, $|\rho| = r$, we will use the notation $D^{2i}_\sigma \times S^{2j-1}_\tau \times T^r_\rho$ for the corresponding subset of $(D^2)^m$.

CONSTRUCTION 6.20 (moment-angle complex corresponding to join). Let K_1, K_2 be simplicial complexes on the sets $[m_1]$, $[m_2]$ respectively, and $K_1 * K_2$ the join of K_1 and K_2 (see Construction 2.9). Identify the cube $I^{m_1+m_2}$ with $I^{m_1}\times I^{m_2}$. Then, using (4.4), we calculate

$$\begin{aligned}\mathrm{cc}(K_1 * K_2) = \bigcup_{\tau_1\in K_1,\,\tau_2\in K_2} C_{\tau_1\cup\tau_2} &= \bigcup_{\tau_1\in K_1,\,\tau_2\in K_2} C_{\tau_1}\times C_{\tau_2}\\ &= \Bigl(\bigcup_{\tau_1\in K_1} C_{\tau_1}\Bigr)\times\Bigl(\bigcup_{\tau_2\in K_2} C_{\tau_2}\Bigr) = \mathrm{cc}(K_1)\times\mathrm{cc}(K_2).\end{aligned}$$

Hence,

$$\mathcal{Z}_{K_1*K_2} = \mathcal{Z}_{K_1}\times\mathcal{Z}_{K_2}.$$

This can be thought as a generalization of Proposition 6.4 to arbitrary simplicial complexes.

CONSTRUCTION 6.21 (moment-angle complexes and connected sums). Suppose we are given two pure $(n-1)$-dimensional simplicial complexes K_1, K_2 on the sets $[m_1]$, $[m_2]$ respectively, and let $K_1 \# K_2$ be their connected sum at some σ_1 and σ_2. (Here $K_1 \# K_2$ is considered as a simplicial complex on $[m_1+m_2-n]$, with suitable identification $\sigma_1=\sigma_2=\sigma$, see Construction 2.12.) If we regard K_1 as a simplicial complex on $[m_1+m_2-n]$, then the corresponding moment-angle complex is $\mathcal{Z}_{K_1} \times T^{m_2-n}$ (Proposition 6.11), where $\mathcal{Z}_{K_1} = \mathcal{Z}_{K_1,[m_1]}$, and similarly for K_2. Denote $\widehat{K}_1 := K_1 \setminus \{\sigma_1\}$ and $\widehat{K}_2 := K_2 \setminus \{\sigma_2\}$. Then

$$\mathcal{Z}_{\widehat{K}_1} = \mathcal{Z}_{K_1} \setminus (T^{m_1-n} \times D^{2n}_{\sigma_1}), \quad \mathcal{Z}_{\widehat{K}_2} = \mathcal{Z}_{K_2} \setminus (D^{2n}_{\sigma_2} \times T^{m_2-n}) \tag{6.7}$$

by (6.5). Now we see that

$$\mathcal{Z}_{K_1 \# K_2} = \mathcal{Z}_{\widehat{K}_1} \times T^{m_2-n} \cup T^{m_1-n} \times \mathcal{Z}_{\widehat{K}_2}, \tag{6.8}$$

where the two pieces are glued along $T^{m_1-n} \times S^{2n-1}_{\sigma_1} \times T^{m_2-n} \cong T^{m_1-n} \times S^{2n-1}_{\sigma_2} \times T^{m_2-n}$, using the identification of σ_1 with σ_2. Equivalently,

$$\mathcal{Z}_{K_1 \# K_2} = \bigcup_{\substack{\tau \in K_1 \text{ or } \tau \in K_2 \\ \tau \neq \sigma}} D^{2|\tau|}_{\tau} \times T^{m_1+m_2-n-|\tau|}.$$

EXAMPLE 6.22. Let $K_1 = K$ be a pure $(n-1)$-dimensional simplicial complex on $[m]$ and $K_2 = \partial\Delta^n$ (the boundary of n-simplex). Choose a maximal simplex $\sigma \in K$ and consider the connected sum $K \#_\sigma \partial\Delta^n$ (the choice of a maximal simplex in $\partial\Delta^n$ is irrelevant). Note that $\mathcal{Z}_{\partial\Delta^n} \cong S^{2n+1}$ can be decomposed as

$$D^{2n}_\sigma \times S^1 \cup_{S^{2n-1} \times S^1} S^{2n-1}_\sigma \times D^2$$

(see examples 6.7 and 6.12), therefore $\mathcal{Z}_{\widehat{\partial\Delta^n}} = S^{2n-1}_\sigma \times D^2$. Now it follows from (6.8) and (6.7) that

$$\begin{aligned}(6.9)\quad & \mathcal{Z}_{K \#_\sigma \partial\Delta^n} \\ &= \big(\mathcal{Z}_K \times S^1 \setminus T^{m-n} \times D^{2n}_\sigma \times S^1\big) \cup_{T^{m-n} \times S^{2n-1} \times S^1} (T^{m-n} \times S^{2n-1}_\sigma \times D^2).\end{aligned}$$

Thus, $\mathcal{Z}_{K \#_\sigma \partial\Delta^n}$ is obtained by removing the "equivariant" handle $T^{m-n} \times D^{2n} \times S^1$ from $\mathcal{Z}_K \times S^1$ and then attaching $T^{m-n} \times S^{2n-1} \times D^2$ along the boundary $T^{m-n} \times S^{2n-1} \times S^1$.

As we mentioned above, the connected sum with the boundary of simplex is a bistellar 0-move. Other bistellar moves also can be interpreted as "equivariant surgery operations" on $\mathcal{Z}_K$.

CONSTRUCTION 6.23 (equivariant surgery operations). Let K be an $(n-1)$-dimensional pure simplicial complex on $[m]$, and let $\sigma \in K$ be an $(n-1-k)$-simplex $(1 \leqslant k \leqslant n-2)$ such that $\operatorname{link} \sigma$ is the boundary $\partial\tau$ of a k-simplex τ that is not a face of K. Let K' be the complex obtained from K by applying the corresponding bistellar k-move, see Definition 2.39:

$$K' = \big(K \setminus (\sigma * \partial\tau)\big) \cup (\partial\sigma * \tau) \tag{6.10}$$

(note that due to our assumptions K' has the same number of vertices as K). The moment-angle complexes corresponding to $\sigma * \partial\tau$ and $\partial\sigma * \tau$ are $D^{2(n-k)}_\sigma \times S^{2k+1}_\tau$ and $S^{2(n-k)-1}_\sigma \times D^{2(k+1)}_\tau$ respectively (this follows from Example 6.12 and Construction 6.20). Using stabilization arguments (Proposition 6.11), we obtain

$$\begin{aligned}&(6.11)\\ &\mathcal{Z}_{K'} = \big(\mathcal{Z}_K \setminus T^{m-n-1} \times D^{2(n-k)}_\sigma \times S^{2k+1}_\tau\big) \cup (T^{m-n-1} \times S^{2(n-k)-1}_\sigma \times D^{2(k+1)}_\tau),\end{aligned}$$

where $T^{m-n-1} \times S^{2(n-k)-1}_\sigma \times D^{2(k+1)}_\tau$ is attached along its boundary $T^{m-n-1} \times S^{2(n-k)-1} \times S^{2k+1}$. This describes the behavior of $\mathcal{Z}_K$ under bistellar k-moves (the cases $k = 0$ and $k = n-1$ are covered by (6.9)).

LEMMA 6.24. *Let $K = K^{n-1}$ be a simplicial sphere and K' the simplicial sphere obtained from K by applying a bistellar k-move* (6.10), $0 < k < n-1$. *Then the corresponding moment-angle manifolds $\mathcal{Z}_K$ and $\mathcal{Z}'_K$ are T^m-equivariantly cobordant. If K' is obtained from K by applying a 0-move, then $\mathcal{Z}_{K'}$ is cobordant to $\mathcal{Z}_K \times S^1$.*

PROOF. We give a proof for k-moves, $k > 1$. The case $k = 0$ is considered similarly. Consider the product $U = \mathcal{Z}_K \times [0,1]$ of $\mathcal{Z}_K$ with a segment. Define $X = T^{m-n-1} \times D_\sigma^{2(n-k)} \times S_\tau^{2k+1}$ and $Y = T^{m-n-1} \times D_\sigma^{2(n-k)} \times D_\tau^{2(k+1)}$ (the latter is a "solid equivariant handle"). Since $X \subset \mathcal{Z}_K$ and $X \subset \partial Y$, we can attach Y to U at $X \times 1 \subset \mathcal{Z}_K \times 1$. Denote the resulting manifold (with boundary) by V, i.e. $V = U \cup_X Y$. Then it follows from (6.11) that $\partial V = \mathcal{Z}_K \cup \mathcal{Z}_{K'}$ (here $\mathcal{Z}_K$ comes from $\mathcal{Z}_K \times 0 \subset U$, while $\mathcal{Z}_K \times 1$ is replaced by $\mathcal{Z}_{K'}$). This concludes the proof. $\square$

Now we have the following topological corollary of Pachner's Theorem 2.40.

THEOREM 6.25. *Let K^{n-1} be a PL sphere. Then for some p the moment-angle manifold $\mathcal{Z}_K \times T^p$ is equivariantly cobordant to $S^{2n+1} \times T^{m+p-n-1}$. This cobordism is realized by a sequence of equivariant surgeries.*

PROOF. By Theorem 2.40, the PL sphere K is taken to $\partial\Delta^n$ by a sequence of bistellar moves. Since $\mathcal{Z}_{\partial\Delta^n} \cong S^{2n+1}$, the statement follows from Lemma 6.24. $\square$

6.5. Borel constructions and Davis–Januszkiewicz space

Here we study basic homotopy properties of $\mathcal{Z}_K$. We also provide necessary arguments for the statements about the cohomology of quasitoric manifolds, which we left unproved in section 5.2.

Let ET^m be the contractible space of the universal principal T^m-bundle over classifying space BT^m. It is well known that BT^m is (homotopy equivalent to) the product of m copies of infinite-dimensional projective space $\mathbb{C}P^\infty$. The cell decomposition of $\mathbb{C}P^\infty$ with one cell in every even dimension determines the *canonical* cell decomposition of BT^m. The cohomology of BT^m (with coefficients in $\mathbf{k}$) is thus the polynomial ring $\mathbf{k}[v_1, \ldots, v_m]$, $\deg v_i = 2$.

DEFINITION 6.26. Let X be a T^m-space. The *Borel construction* (alternatively, *homotopy quotient* or *associated bundle*) is the identification space

$$ET^m \times_{T^m} X := ET^m \times X/\sim,$$

where $(e, x) \sim (eg, g^{-1}x)$ for any $e \in ET^m$, $x \in X$, $g \in T^m$.

The projection $(e,x) \to e$ displays $ET^m \times_{T^m} X$ as the total space of a bundle $ET^m \times_{T^m} X \to BT^m$ with fibre X and structure group T^m. At the same time, there is a principal T^m-bundle $ET^m \times X \to ET^m \times_{T^m} X$.

In the sequel we denote the Borel construction $ET^m \times_{T^m} X$ corresponding to a T^m-space X by $B_T X$. In particular, for any simplicial complex K on m vertices we have the Borel construction $B_T \mathcal{Z}_K$ and the bundle $p : B_T \mathcal{Z}_K \to BT^m$ with fibre $\mathcal{Z}_K$.

For each $i = 1, \ldots, m$ denote by BT_i the i-th factor in $BT^m = (\mathbb{C}P^\infty)^m$. For a subset $\sigma \subset [m]$ we denote by BT_σ the product of BT_i's with $i \in \sigma$. Obviously, BT_σ is a cellular subcomplex of BT^m, and $BT_\sigma \cong BT^k$ if $|\sigma| = k$.

DEFINITION 6.27. Let K be a simplicial complex. We refer to the cellular subcomplex

$$\bigcup_{\sigma\in K} BT_\sigma \subset BT^m$$

as the *Davis–Januszkiewicz space*, and denote it $DJ(K)$.

The following statement is an immediate corollary of the definition of Stanley–Reisner ring $\mathbf{k}(K)$ (Definition 3.1).

PROPOSITION 6.28. *The cellular cochain algebra $C^*(DJ(K))$ and the cohomology algebra $H^*(DJ(K))$ are isomorphic to the face ring $\mathbf{k}(K)$. The cellular inclusion $i : DJ(K) \hookrightarrow BT^m$ induces the quotient epimorphism $i^* : \mathbf{k}[v_1, \dots , v_m] \to \mathbf{k}(K) = \mathbf{k}[v_1, \dots , v_m]/\mathcal{I}_K$ in the cohomology.*

THEOREM 6.29. *The fibration $p : B_T\mathcal{Z}_K \to BT^m$ is homotopy equivalent to the cellular inclusion $i : DJ(K) \hookrightarrow BT^m$. More precisely, there is a deformation retraction $B_T\mathcal{Z}_K \to DJ(K)$ such that the diagram*

$$\begin{tikzcd}
B_T\mathcal{Z}_K \arrow[r, "p"] \arrow[d, squiggly] & BT^m \arrow[d, equal] \\
DJ(K) \arrow[r, "i"] & BT^m
\end{tikzcd}$$

is commutative.

PROOF. Consider the decomposition (6.5). Since each $B_\tau \subset \mathcal{Z}_K$ is T^m-stable, the Borel construction $B_T\mathcal{Z}_K = ET^m \times_{T^m} \mathcal{Z}_K$ is patched from the Borel constructions $ET^m \times_{T^m} B_\tau$ for $\tau \in K$. Suppose $|\tau| = j$; then $B_\tau \cong (D^2)^j \times T^{m-j}$ (see (6.4)). By the definition of Borel construction, $ET^m \times_{T^m} B_\tau \cong (ET^j \times_{T^j} (D^2)^j) \times ET^{m-j}$. The space $ET^j \times_{T^j} (D^2)^j$ is the total space of a $(D^2)^j$-bundle over BT^j. It follows that there is a deformation retraction $ET^m \times_{T^m} B_\tau \to BT_\tau$, which defines a homotopy equivalence between the restriction of $p : B_T\mathcal{Z}_K \to BT^m$ to $ET^m \times_{T^m} B_\tau$ and the cellular inclusion $BT_\tau \hookrightarrow BT^m$. These homotopy equivalences corresponding to different simplices $\tau \in K$ fit together to yield a required homotopy equivalence between $p : B_T\mathcal{Z}_K \to BT^m$ and $i : DJ(K) \hookrightarrow BT^m$. □

COROLLARY 6.30. *The moment-angle complex $\mathcal{Z}_K$ is the homotopy fibre of the cellular inclusion $i : DJ(K) \hookrightarrow BT^m$.*

As a corollary, we get the following statement, firstly proved in [**48**, Theorem 4.8].

COROLLARY 6.31. *The cohomology algebra $H^*(B_T\mathcal{Z}_K)$ is isomorphic to the face ring $\mathbf{k}(K)$. The projection $p : B_T\mathcal{Z}_K \to BT^m$ induces the quotient epimorphism $p^* : \mathbf{k}[v_1, \dots , v_m] \to \mathbf{k}(K) = \mathbf{k}[v_1, \dots , v_m]/\mathcal{I}_K$ in the cohomology.*

COROLLARY 6.32. *The T^m-equivariant cohomology of $\mathcal{Z}_K$ is isomorphic to the Stanley–Reisner ring of K:*

$$H^*_{T^m}(\mathcal{Z}_K) \cong \mathbf{k}(K).$$

The following information about the homotopy groups of $\mathcal{Z}_K$ can be retrieved from the above constructions.

THEOREM 6.33. (a) *The complex* $\mathcal{Z}_K$ *is* 2*-connected (i.e.* $\pi_1(\mathcal{Z}_K) = \pi_2(\mathcal{Z}_K) = 0$*), and* $\pi_i(\mathcal{Z}_K) = \pi_i(B_T\mathcal{Z}_K) = \pi_i(DJ(K))$ *for* $i \geqslant 3$.

(b) *If* $K = K_P$ *and* P *is* q*-neighborly (see Definition* 1.15*), then* $\pi_i(\mathcal{Z}_K) = 0$ *for* $i < 2q+1$*. Moreover,* $\pi_{2q+1}(\mathcal{Z}_P)$ *is a free Abelian group generated by the* $(q+1)$*-element missing faces of* K_P.

PROOF. Note that $BT^m = K(\mathbb{Z}^m, 2)$ and the 3-skeleton of $DJ(K)$ coincides with that of BT^m. If P is q-neighborly, then it follows from Definition 6.27 that the $(2q+1)$-skeleton of $DJ(K_P)$ coincides with that of BT^m. Now, both statements follow easily from the exact homotopy sequence of the map $i : DJ(K) \to BT^m$ with homotopy fibre $\mathcal{Z}_K$ (see Corollary 6.30). □

REMARK. We say that a simplicial complex K on the set $[m]$ is k*-neighborly* if any k-element subset of $[m]$ is a simplex of K. (This definition is an obvious extension of the notion of k-neighborly simplicial polytope to arbitrary simplicial complexes.) Then the second part of Theorem 6.33 holds for arbitrary q-neighborly simplicial complex.

Suppose now that $K = K_P$ for some simple n-polytope P and M^{2n} is a quasitoric manifold over P with characteristic function ℓ (see Definition 5.10). Then we have the subgroup $H(\ell) \subset T^m$ acting freely on $\mathcal{Z}_P$ and the principal T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}$ (Proposition 6.5).

PROPOSITION 6.34. *The Borel construction* $ET^n \times_{T^n} M^{2n}$ *is homotopy equivalent to* $B_T\mathcal{Z}_P$.

PROOF. Since $H(\ell)$ acts freely on $\mathcal{Z}_P$, we have

$$\begin{aligned} B_T\mathcal{Z}_P &= ET^m \times_{T^m} \mathcal{Z}_P \\ &\cong EH(\ell) \times \Big(E\big(T^m/H(\ell)\big) \times_{T^m/H(\ell)} \mathcal{Z}_P/H(\ell)\Big) \simeq ET^n \times_{T^n} M^{2n}. \end{aligned}$$

□

COROLLARY 6.35. *The* T^n*-equivariant cohomology ring of a quasitoric manifold* M^{2n} *over* P^n *is isomorphic to the Stanley–Reisner ring of* P^n*:*

$$H^*_{T^n}(M^{2n}) \cong \mathbf{k}(P^n).$$

PROOF. It follows from Proposition 6.34 and Corollary 6.32. □

THEOREM 6.36 ([**48**, Theorem 4.12]). *The Leray–Serre spectral sequence of the bundle*

$$ET^n \times_{T^n} M^{2n} \to BT^n \tag{6.12}$$

with fibre M^{2n} *collapses at the* E_2 *term, i.e.* $E_2^{p,q} = E_\infty^{p,q}$.

PROOF. Since both BT^n and M^{2n} have only even-dimensional cells (see Proposition 5.16), all the differentials in the spectral sequence are trivial by dimensional reasons. □

COROLLARY 6.37. *Projection* (6.12) *induces a monomorphism* $\mathbf{k}[t_1, \dots, t_n] \to \mathbf{k}(P)$ *in the cohomology. The inclusion of fibre* $M^{2n} \hookrightarrow ET^n \times_{T^n} M^{2n}$ *induces an epimorphism* $\mathbf{k}(P) \to H^*(M^{2n})$.

Now we are ready to give proofs for the statements from section 5.2.

PROOF OF LEMMA 5.17 AND THEOREM 5.18. The monomorphism

$$H^*(BT^n) = \mathbf{k}[t_1, \dots , t_n] \to \mathbf{k}(P) = H^*(ET^n \times_{T^n} M^{2n})$$

takes t_i to θ_i, $i = 1, \dots , n$. By Theorem 6.36, $\mathbf{k}(P)$ is a free $\mathbf{k}[t_1, \dots , t_n]$-module, hence, $\theta_1, \dots , \theta_n$ is a regular sequence. Therefore, the kernel of $\mathbf{k}(P) \to H^*(M^{2n})$ is exactly $\mathcal{J}_\ell = (\theta_1, \dots , \theta_n)$. ☐

6.6. Walk around the construction of $\mathcal{Z}_K$: generalizations, analogues and additional comments

Many of our previous constructions (namely, the cubical complex $\mathrm{cc}(K)$, the moment-angle complex $\mathcal{Z}_K$, the Borel construction $B_T\mathcal{Z}_K$, the Davis–Januszkiewicz space $DJ(K)$, and also the complement $U(K)$ of a coordinate subspace arrangement appearing in section 8.2) admit a unifying combinatorial interpretation in terms of the following construction, which was mentioned to us by N. Strickland (in private communications).

CONSTRUCTION 6.38. Let X be a space, and W a subspace of X. Let K be a simplicial complex on the set $[m]$. Define the following subset in the product of m copies of X:

$$K_\bullet(X, W) = \bigcup_{\sigma\in K} \Bigl(\prod_{i\in\sigma} X \times \prod_{i\notin\sigma} W\Bigr).$$

EXAMPLE 6.39. 1. $\mathrm{cc}(K) = K_\bullet(I^1, 1)$ (see (4.4)).
2. $\mathcal{Z}_K = K_\bullet(D^2, S^1)$ (see (6.5)).
3. $DJ(K) = K_\bullet(\mathbb{C}P^\infty, *)$ (see Definition 6.27).
4. $B_T\mathcal{Z}_K = K_\bullet(ES^1 \times_{S^1} D^2, ES^1 \times_{S^1} S^1)$ (see the proof of Theorem 6.29).

Another unifying description of the above spaces can be achieved using categorical constructions of *limits* and *colimits* of different diagrams over the face category $\mathrm{cat}(K)$ of K. (The objects of $\mathrm{cat}(K)$ are simplices $\sigma \in K$ and the morphisms are inclusions.) For instance, Definition 6.27 is an example of this procedure: the Davis–Januszkiewicz space is the colimit of the diagram of spaces over $\mathrm{cat}(K)$ that assigns BT_σ to a simplex $\sigma \in K$. In the case when K is a flag complex (see Definition 2.18 and Proposition 2.19) the colimit over $\mathrm{cat}(K)$ reduces to the *graph product*, studied in the theory of groups (see e.g. [**43**]). Well-known examples of graph products include right-angled *Coxeter* and *Artin groups* (see e.g. [**48**], [**49**]). The most general categorical setup for the above constructions involves the notion of *homotopy colimit* [**24**], [**138**]. This fundamental algebraic-topological concept has already found combinatorial applications, see [**139**]. The complex $\mathcal{Z}_K$ can be seen as the homotopy colimit of a certain diagram of tori; this interpretation is similar to the homotopy colimit description of toric varieties proposed in [**139**]. For more information on this approach see [**113**].

The combinatorial theory of toric spaces is parallel to some extent to its $\mathbb{Z}/2$-, or "real", counterpart. We say a few words about the $\mathbb{Z}/2$-theory here, referring the reader to [**48**], [**49**] and other papers of R. Charney, M. Davis, T. Januszkiewicz and their co-authors for a more detailed treatment (some further results can be also found in [**113**]). The first step is to pass from the torus T^m to its "real analogue", the group $(\mathbb{Z}/2)^m$. The standard cube $I^m = [0,1]^m$ is the orbit space for the action of $(\mathbb{Z}/2)^m$ on the bigger cube $[-1,1]^m$, which in turn can be regarded as a "real analogue" of the poly-disc $(D^2)^m \subset \mathbb{C}^m$. Now, given a cubical subcomplex

$\mathcal{C} \subset I^m$, one can construct a $(\mathbb{Z}/2)^m$-symmetrical cubical complex embedded into $[-1,1]^m$ just in the same way as it is done in Definition 6.10. In particular, for any simplicial complex K on the vertex set $[m]$ one can introduce the real versions $\mathbb{R}\mathcal{Z}_K$ and $\mathbb{R}\mathcal{W}_K$ of the moment-angle complexes $\mathcal{Z}_K$ and $\mathcal{W}_K$ (6.3). In the notations of Construction 6.38 we have

$$\mathbb{R}\mathcal{Z}_K = K_\bullet\big([-1,1],\{-1,1\}\big).$$

This cubical complex was studied, e.g. in [**10**] under the name *mirroring construction*. If K is a simplicial $(n-1)$-sphere, then $\mathbb{R}\mathcal{Z}_K$ is an n-dimensional manifold (the proof is similar to that of Lemma 6.13). Thereby, for any simplicial sphere K^{n-1} with m vertices we get a $(\mathbb{Z}/2)^m$-symmetric n-manifold with a $(\mathbb{Z}/2)^m$-invariant cubical subdivision. As suggested by the results of [**10**], this class of cubical manifolds may be useful in the combinatorial theory of face vectors of cubical complexes (see section 4.1). The real analogue $\mathbb{R}\mathcal{Z}_P$ of the manifold $\mathcal{Z}_P$ (corresponding to the case of a polytopal simplicial sphere) is the *universal Abelian cover* of the polytope P^n regarded as an orbifold (or manifold with corners), see e.g. [**68**, §4.5]. In [**78**] manifolds $\mathbb{R}\mathcal{Z}_P$ and $\mathcal{Z}_P$ are interpreted as the configuration spaces of equivariant *hinge mechanisms* (or *linkages*) in $\mathbb{R}^2$ and $\mathbb{R}^3$.

EXAMPLE 6.40. Let P^2_m be an m-gon. Then $\mathbb{R}\mathcal{Z}_{P^2_m}$ is a 2-dimensional manifold. It is easy to see that $\mathbb{R}\mathcal{Z}_{P^2_3} = \mathbb{R}\mathcal{Z}_{\Delta^2} \cong S^2$ (a 2-sphere patched from 8 triangles) and $\mathbb{R}\mathcal{Z}_{P^2_4} = \mathbb{R}\mathcal{Z}_{\Delta^1\times\Delta^1} \cong T^2$ (a 2-torus patched from 16 squares). More generally, $\mathbb{R}\mathcal{Z}_{P^2_m}$ is patched from 2^m polygons, meeting by 4 at each vertex. Hence, we have $m2^{m-2}$ vertices and $m2^{m-1}$ edges, so the Euler characteristic is

$$\chi(\mathbb{R}\mathcal{Z}_{P^2_m}) = 2^{m-2}(4-m).$$

Thus, $\mathbb{R}\mathcal{Z}_{P^2_m}$ is a surface of genus $1-2^{m-1}+m2^{m-3}$. This also can be seen directly by decomposing P^2_m into a connected sum of an $(m-1)$-gon and triangle and using the real version of Example 6.22.

Replacing T^n by $(\mathbb{Z}/2)^n$ in Definition 5.10, we obtain a real version of quasitoric manifolds, which was introduced in [**48**] under the name *small covers*. Thereby a small cover of a simple polytope P^n is a $(\mathbb{Z}/2)^n$-manifold M^n with quotient P^n. The name refers to the fact that any branched cover of P^n (as an orbifold) by a smooth manifold has at least 2^n sheets. Small covers were studied in [**48**] along with quasitoric manifolds, and many results on quasitoric manifolds quoted from [**48**] in section 5.2 have analogues in the small cover case. Also, like in the torus case, every small cover is the quotient of the universal cover $\mathbb{R}\mathcal{Z}_P$ by a free action of the group $(\mathbb{Z}/2)^{m-n}$.

An important class of small covers (and quasitoric manifolds) was introduced in [**48**, Example 1.15] under the name *pullbacks from the linear model*. They correspond to simple polytopes P^n whose dual triangulation can be folded onto the $(n-1)$-simplex (more precisely, the polytopal sphere K_P admits a non-degenerate simplicial map onto Δ^{n-1}; note that this is always the case when K_P is a barycentric subdivision of some other polytopal sphere, see Example 2.15). If this condition is satisfied then there exists a special characteristic map (5.4) which assigns to each facet of P^n a coordinate subtorus $T_i \subset T^n$ (or coordinate subgroup $(\mathbb{Z}/2)_i \subset (\mathbb{Z}/2)^n$ in the small cover case). Pullbacks from the linear model have a number of nice properties, in particular, they are all stably parallelizable ([**48**, Corollary 6.10], compare with Theorem 5.33). The existence of a non-degenerate simplicial map from K_P to Δ^{n-1} can be reformulated by saying that the polytope P^n admits a *regular n-paint coloring*. The latter means that the facets of P^n can be colored with n paints in such a way that any two adjacent facets have different color. A simple polytope P^n admits a regular n-paint coloring if and only if every 2-face has an even number of edges. This is a classical result for $n = 3$; the proof in the general case can be found in [**81**]. Some additional results about pullbacks from the linear model in dimension 3 were obtained in [**79**]. It was shown there that any small cover M^3 which is a pullback from the linear model admits an equivariant embedding into $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ with the standard action of $(\mathbb{Z}/2)^3$ on $\mathbb{R}^3$ and the trivial action on $\mathbb{R}$. Another result from [**79**] says that any such M^3 can be obtained from a set of 3-dimensional tori by applying several equivariant connected sums and *equivariant Dehn twists* (compare with section 6.4).

Although not every simple 3-polytope admits a regular 3-paint coloring, a regular 4-paint coloring can always be achieved due to the Four Color Theorem. This argument was used in [**48**, Example 1.21] to prove that there is a small cover (or quasitoric manifold) over every simple 3-polytope. (One can just construct a characteristic map by assigning the coordinate circles in T^3 to the first three colors and the diagonal circle to the fourth one.)

On the other hand, it would be particularly interesting to develop a quaternionic analogue of the theory. Unlike the real case, not much is done here. To begin, of course, we have to replace T^n by the quaternionic torus $Sp(1)^n \cong (S^3)^n$. Developing quaternionic analogues of toric and quasitoric manifolds is quite tricky. R. Scott in [**119**] used the quaternionic analogue of characteristic map to approach this problem. However, the non-commutativity of the quaternionic torus implies that it does not contain sufficiently many subgroups for the resulting quaternionic toric manifolds to have an actual $Sp(1)^n$-action. A polytopal structure also appears in the quotients of some other types of manifolds studied in the quaternionic geometry, see e.g. [**25**]. We also mention that since only coordinate subgroups of T^m are involved in the definition of the moment-angle complex $\mathcal{Z}_K$, this particular construction of a toric space does have a quaternionic analogue which is an $Sp(1)^m$-space.

At the end we give one example which builds on a generalization of the construction of $\mathcal{Z}_K$ to the case of an arbitrary group G.

EXAMPLE 6.41 (classifying space for group G). Let K be a simplicial complex on the vertex set $[m]$. Set $\mathcal{Z}_K(G) := K_\bullet(\mathrm{cone}(G), G)$ (see Construction 6.38), where $\mathrm{cone}(G)$ is the cone over G with the obvious G-action. By the construction, the group G^m acts on $\mathcal{Z}_K(G)$, with quotient $\mathrm{cone}(K)$. It is also easy to observe that the diagonal subgroup in G^m acts freely on $\mathcal{Z}_K(G)$, thus identifying $\mathcal{Z}_K(G)$ as a principal G-space.

Suppose now that $K_1 \subset K_2 \subset \cdots \subset K_i \subset \cdots$ is a sequence of embedded simplicial complexes such that K_i is i-neighborly. The group G acts freely on the contractible space $\varinjlim \mathcal{Z}_{K_i}(G)$, and the corresponding quotient is thus the classifying space BG. Thus, we have the following filtration in the universal fibration $EG \to BG$:

$$\begin{array}{ccccccccc} \mathcal{Z}_{K_1}(G) & \hookrightarrow & \mathcal{Z}_{K_2}(G) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{Z}_{K_i}(G) & \hookrightarrow & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ \mathcal{Z}_{K_1}(G)/G & \hookrightarrow & \mathcal{Z}_{K_2}(G)/G & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{Z}_{K_i}(G)/G & \hookrightarrow & \cdots . \end{array}$$

The well-known *Milnor filtration* in the universal fibration of the group G corresponds to the case $K_i = \Delta^{i-1}$.

CHAPTER 7

Cohomology of moment-angle complexes and combinatorics of triangulated manifolds

7.1. The Eilenberg–Moore spectral sequence

In their paper [**60**] of 1966, Eilenberg and Moore constructed a spectral sequence of great importance for algebraic topology. This spectral sequence can be considered as an extension of Adams' approach to calculating the cohomology of loop spaces [**1**]. In 1960-70s different applications of the Eilenberg–Moore spectral sequence led to many important results on the cohomology of loop spaces and homogeneous spaces for Lie group actions. In this chapter we discuss some new applications of this spectral sequence to combinatorial problems. This section contains the necessary information about the spectral sequence; we follow L. Smith's paper [**121**] in this description.

The following theorem provides an algebraic setup for the Eilenberg–Moore spectral sequence.

THEOREM 7.1 (Eilenberg–Moore [**121**, Theorem 1.2]). *Let A be a commutative differential graded $\mathbf{k}$-algebra, and M, N differential graded A-modules. Then there exists a spectral sequence $\{E_r, d_r\}$ converging to $\operatorname{Tor}_A(M,N)$ and whose E_2-term is*

$$E_2^{-i,j} = \operatorname{Tor}_{H[A]}^{-i,j}\bigl(H[M], H[N]\bigr), \quad i,j \geqslant 0,$$

where $H[\cdot]$ denotes the cohomology algebra (or module).

The above spectral sequence lives in the *second* quadrant and its differentials d_r add $(r, 1-r)$ to bidegree, $r \geqslant 1$. It is called the (algebraic) *Eilenberg–Moore spectral sequence*. For the corresponding decreasing filtration $\{F^{-p}\operatorname{Tor}_A(M,N)\}$ in $\operatorname{Tor}_A(M,N)$ we have

$$E_\infty^{-p,n+p} = F^{-p}\Bigl(\sum_{-i+j=n} \operatorname{Tor}_A^{-i,j}(M,N)\Bigr)\Big/ F^{-p+1}\Bigl(\sum_{-i+j=n} \operatorname{Tor}_A^{-i,j}(M,N)\Bigr).$$

Topological applications of Theorem 7.1 arise in the case when A, M, N are singular (or cellular) cochain algebras of certain topological spaces. The classical situation is described by the commutative diagram

$$(7.1)\qquad \begin{tikzcd} E \arrow[r] \arrow[d, squiggly] & E_0 \arrow[d, squiggly] \\ B \arrow[r] & B_0, \end{tikzcd}$$

where $E_0 \to B_0$ is a Serre fibre bundle with fibre F over a simply connected base B_0, and $E \to B$ is the pullback along a continuous map $B \to B_0$. For any space X, let $C^*(X)$ denote either the singular cochain algebra of X or (in the case when X is a cellular complex) the cellular cochain algebra of X. Obviously, $C^*(E_0)$ and $C^*(B)$ are $C^*(B_0)$-modules. Under these assumptions the following statement holds.

LEMMA 7.2 ([**121**, Proposition 3.4]). *$\mathrm{Tor}_{C^*(B_0)}(C^*(E_0), C^*(B))$ is an algebra in a natural way, and there is a canonical isomorphism of algebras*

$$\mathrm{Tor}_{C^*(B_0)}\big(C^*(E_0), C^*(B)\big) \to H^*(E).$$

Applying Theorem 7.1 in the case $A = C^*(B_0)$, $M = C^*(E_0)$, $N = C^*(B)$ and taking into account Lemma 7.2, we come to the following statement.

THEOREM 7.3 (Eilenberg–Moore). *There exists a spectral sequence of commutative algebras $\{E_r, d_r\}$ with*

(a) $E_r \Rightarrow H^*(E)$;
(b) $E_2^{-i,j} = \mathrm{Tor}^{-i,j}_{H^*(B_0)}\big(H^*(E_0), H^*(B)\big)$.

The spectral sequence of Theorem 7.3 is called the (topological) *Eilenberg–Moore spectral sequence*. The case when B in (7.1) is a point is particularly important for applications, so we state the corresponding result separately.

COROLLARY 7.4. *Let $E \to B$ be a fibration over a simply connected space B with fibre F. Then there exists a spectral sequence of commutative algebras $\{E_r, d_r\}$ with*

(a) $E_r \Rightarrow H^*(F)$;
(b) $E_2 = \mathrm{Tor}_{H^*(B)}\big(H^*(E), \mathbf{k}\big)$.

We refer to the spectral sequence of Corollary 7.4 as the *Eilenberg–Moore spectral sequence of fibration $E \to B$*.

EXAMPLE 7.5. Let M^{2n} be a quasitoric manifold over P^n (see Definition 5.10). Consider the Eilenberg–Moore spectral sequence of the bundle $ET^n \times_{T^n} M^{2n} \to BT^n$ with fibre M^{2n}. By Proposition 6.34, $H^*(ET^n \times_{T^n} M^{2n}) = H^*(B_T\mathcal{Z}_P) \cong \mathbf{k}(P^n)$. The monomorphism

$$\mathbf{k}[t_1, \dots, t_n] = H^*(BT^n) \to H^*(ET^n \times_{T^n} M^{2n}) = \mathbf{k}(P^n)$$

takes t_i to θ_i $(i = 1, \dots, n)$, see (5.6). The E_2 term of the Eilenberg–Moore spectral sequence is

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{H^*(BT^n)}\big(H^*(ET^n \times_{T^n} M^{2n}), \mathbf{k}\big) = \mathrm{Tor}^{*,*}_{\mathbf{k}[t_1,\dots,t_n]}\big(\mathbf{k}(P^n), \mathbf{k}\big).$$

Since $\mathbf{k}(P^n)$ is a free $\mathbf{k}[t_1, \dots, t_n]$-module, we have

$$\begin{aligned}\mathrm{Tor}^{*,*}_{\mathbf{k}[t_1,\dots,t_n]}\big(\mathbf{k}(P^n), \mathbf{k}\big) &= \mathrm{Tor}^{0,*}_{\mathbf{k}[t_1,\dots,t_n]}\big(\mathbf{k}(P^n), \mathbf{k}\big)\\ &= \mathbf{k}(P^n) \otimes_{\mathbf{k}[t_1,\dots,t_n]} \mathbf{k} = \mathbf{k}(P^n)/(\theta_1, \dots, \theta_n).\end{aligned}$$

Therefore, $E_2^{0,*} = \mathbf{k}(P^n)/\mathcal{J}_\ell$ and $E_2^{-p,*} = 0$ for $p > 0$. It follows that the Eilenberg–Moore spectral sequence collapses at the E_2 term and $H^*(M^{2n}) = \mathbf{k}(P^n)/J_\ell$, in accordance with Theorem 5.18.

7.2. Cohomology algebra of $\mathcal{Z}_K$

Here we apply the Eilenberg–Moore spectral sequence to calculating the cohomology algebra of the moment-angle complex $\mathcal{Z}_K$. As an immediate corollary we obtain that the cohomology algebra inherits a canonical *bi*grading from the spectral sequence. The corresponding bigraded Betti numbers coincide with important combinatorial invariants of K introduced by Stanley [**128**].

THEOREM 7.6. *The following isomorphism of algebras holds:*

$$H^*(\mathcal{Z}_K) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr).$$

This formula either can be seen as an isomorphism of graded algebras, where the grading in the right hand side is by the total degree, or used to define a bigraded algebra structure in the left hand side. In particular,

$$H^p(\mathcal{Z}_K) \cong \sum_{-i+2j=p} \operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr).$$

PROOF. Let us consider the Eilenberg–Moore spectral sequence of the commutative square

$$\begin{array}{ccc} E & \longrightarrow & ET^m \\ \downarrow & & \downarrow \\ DJ(K) & \xrightarrow{\ i\ } & BT^m, \end{array} \tag{7.2}$$

where the left vertical arrow is the pullback along i. Corollary 6.30 shows that E is homotopy equivalent to $\mathcal{Z}_K$.

By Proposition 6.28, the map $i : DJ(K) \hookrightarrow BT^m$ induces the quotient epimorphism

$$i^* : C^*(BT^m) = \mathbf{k}[v_1,\dots,v_m] \to \mathbf{k}(K) = C^*(DJ(K)),$$

where $C^*(\cdot)$ denotes the cellular cochain algebra. Since ET^m is contractible, there is a chain equivalence $C^*(ET^m) \simeq \mathbf{k}$. More precisely, $C^*(ET^m)$ can be identified with the Koszul resolution $\Lambda[u_1,\dots,u_m] \otimes \mathbf{k}[v_1,\dots,v_m]$ of $\mathbf{k}$ (see Example 3.24). Therefore, we have an isomorphism

$$\operatorname{Tor}_{C^*(BT^m)}\Bigl(C^*\bigl(DJ(K)\bigr), C^*(ET^m)\Bigr) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr). \tag{7.3}$$

The Eilenberg–Moore spectral sequence of commutative square (7.2) has

$$E_2 = \operatorname{Tor}_{H^*(BT^m)}\Bigl(H^*\bigl(DJ(K)\bigr), H^*(ET^m)\Bigr)$$

and converges to $\operatorname{Tor}_{C^*(BT^m)}(C^*(DJ(K)), C^*(ET^m))$ (Theorem 7.1). Since

$$\operatorname{Tor}_{H^*(BT^m)}\Bigl(H^*\bigl(DJ(K)\bigr), H^*(ET^m)\Bigr) = \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr),$$

it follows from (7.3) that the spectral sequence collapses at the E_2 term, that is, $E_2 = E_\infty$. Lemma 7.2 shows that the module $\operatorname{Tor}_{C^*(BT^m)}(C^*(DJ(K)), C^*(ET^m))$ is an algebra isomorphic to $H^*(\mathcal{Z}_K)$, which concludes the proof. □

Theorem 7.6 displays the cohomology of $\mathcal{Z}_K$ as a *bi*graded algebra and says that the corresponding bigraded Betti numbers $b^{-i,2j}(\mathcal{Z}_K)$ coincide with that of $\mathbf{k}(K)$, see (3.5). The next theorem follows from Lemma 3.29 and Corollary 3.30.

THEOREM 7.7. *The following isomorphism of bigraded algebras holds:*

$$H^{*,*}(\mathcal{Z}_K) \cong H\bigl[\Lambda[u_1,\dots,u_m] \otimes \mathbf{k}(K), d\bigr],$$

where the bigraded structure and the differential in the right hand side are defined by (3.4).

In the sequel, given two subsets $\sigma = \{i_1, \dots, i_p\}$, $\tau = \{j_1, \dots, j_q\}$ of $[m]$, we will denote the square-free monomial

$$u_{i_1} \dots u_{i_p} v_{j_1} \dots v_{j_q} \in \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K)$$

by $u_\sigma v_\tau$. Note that $\operatorname{bideg} u_\sigma v_\tau = (-p, 2(p+q))$.

Remark. Since the differential d in (3.4) does not change the second degree, the differential bigraded algebra $[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K), d]$ splits into the sum of differential subalgebras consisting of elements of fixed second degree.

Corollary 7.8. *The Leray–Serre spectral sequence of the principal T^m-bundle $ET^m \times \mathcal{Z}_K \to B_T\mathcal{Z}_K$ collapses at the E_3 term.*

Proof. The spectral sequence under consideration converges to $H^*(ET^m \times \mathcal{Z}_K) = H^*(\mathcal{Z}_K)$ and has

$$E_2 = H^*(T^m) \otimes H^*(B_T\mathcal{Z}_K) = \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K).$$

The differential in the E_2 term acts as in (3.4). Hence,

$$E_3 = H[E_2, d] = H\big[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K)\big] = H^*(\mathcal{Z}_K),$$

by Theorem 7.7. □

Construction 7.9. Consider the subspace $A^{-q}(K) \subset \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K)$ spanned by monomials u_σ and $u_\sigma v_\tau$ such that τ is a simplex of K, $|\sigma| = q$ and $\sigma \cap \tau = \varnothing$. Define

$$A^*(K) = \bigoplus_{q=0}^{m} A^{-q}(K).$$

Since $d(u_i) = v_i$ and $d(v_i) = 0$, we have $d(A^{-q}(K)) \subset A^{-q+1}(K)$. Therefore, $A^*(K)$ is a cochain subcomplex in $[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K), d]$. Moreover, $A^*(K)$ inherits the bigraded module structure from $\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K)$, with differential d adding $(1,0)$ to bidegree. Hence, we have an additive inclusion (i.e. a monomorphism of bigraded modules) $i_a : A^*(K) \hookrightarrow \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K)$. On the other hand, $A^*(K)$ is an algebra in the obvious way, but is *not* a subalgebra of $\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K)$. (For instance, $v_1^2 = 0$ in $A^*(K)$, but $v_1^2 \neq 0$ in $\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K)$.) Nevertheless, we have multiplicative *projection* (an epimorphism of bigraded algebras) $j_m : \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K) \to A^*(K)$. The additive inclusion i_a and the multiplicative projection j_m satisfy $j_m \cdot i_a = \mathrm{id}$.

Lemma 7.10. *The cochain complexes $[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K), d]$ and $[A^*(K), d]$ are cochain homotopy equivalent and therefore have the same cohomology. This implies the following isomorphism of bigraded $\mathbf{k}$-modules:*

$$H[A^*(K), d] \cong \operatorname{Tor}_{\mathbf{k}[v_1, \dots, v_m]}\big(\mathbf{k}(K), \mathbf{k}\big).$$

Proof. A routine check shows that the cochain homotopy operator s for the Koszul resolution (see the proof of Proposition VII.2.1 in [**92**]) establishes a cochain homotopy equivalence between the maps id and $i_a \cdot j_m$ from the algebra $[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K), d]$ to itself. That is,

$$ds + sd = \mathrm{id} - i_a \cdot j_m.$$

We just illustrate the above identity on few simple examples.

1) $s(u_1v_2) = u_1u_2,\quad ds(u_1v_2) = u_2v_1 - u_1v_2,\quad sd(u_1v_2) = u_1v_2 - u_2v_1,$
hence, $\quad (ds+sd)(u_1v_2) = 0 = (\mathrm{id} - i_a \cdot j_m)(u_1v_2);$

2) $s(u_1v_1) = u_1^2 = 0,\quad ds(u_1v_1) = 0,\quad d(u_1v_1) = v_1^2,\quad sd(u_1v_1) = u_1v_1,$
hence, $\quad (ds+sd)(u_1v_1) = u_1v_1 = (\mathrm{id} - i_a \cdot j_m)(u_1v_1);$

3) $s(v_1^2) = u_1v_1,\quad ds(v_1^2) = v_1^2,\quad d(v_1^2) = 0,$
hence, $\quad (ds+sd)(v_1^2) = v_1^2 = (\mathrm{id} - i_a \cdot j_m)(v_1^2).$

□

Now we recall our cell decomposition of $\mathcal{Z}_K$, see Lemma 6.16. The cells of $\mathcal{Z}_K$ are the sign vectors $\mathcal{T} \in \{D, T, 1\}^m$ with $\mathcal{T}_D \in K$. Assign to each pair σ, τ of disjoint subsets of $[m]$ the vector $\mathcal{T}(\sigma,\tau)$ with $\mathcal{T}(\sigma,\tau)_D = \sigma$, $\mathcal{T}(\sigma,\tau)_T = \tau$. Then $\mathcal{T}(\sigma,\tau)$ is a cell of $\mathcal{Z}_K$ if and only if $\sigma \in K$. Let $C_*(\mathcal{Z}_K)$ and $C^*(\mathcal{Z}_K)$ denote the cellular chain and cochain complexes of $\mathcal{Z}_K$ respectively. Both complexes $C^*(\mathcal{Z}_K)$ and $A^*(K)$ have the same cohomology $H^*(\mathcal{Z}_K)$. The complex $C^*(\mathcal{Z}_K)$ has the canonical additive basis consisting of cochains $\mathcal{T}(\sigma,\tau)^*$. As an algebra, $C^*(\mathcal{Z}_K)$ is generated by the cochains D_i^*, T_j^* (of dimension 2 and 1 respectively) dual to the cells $D_i = \mathcal{T}(\{i\}, \varnothing)$ and $T_j = \mathcal{T}(\varnothing, \{j\})$, $1 \leqslant i, j \leqslant m$. At the same time, $A^*(K)$ is multiplicatively generated by v_i, u_j, $1 \leqslant i, j \leqslant m$.

THEOREM 7.11. *The correspondence $v_\sigma u_\tau \mapsto \mathcal{T}(\sigma,\tau)^*$ establishes a canonical isomorphism between the differential graded algebras $A^*(K)$ and $C^*(\mathcal{Z}_K)$.*

PROOF. It follows directly from the definitions of $A^*(K)$ and $C^*(\mathcal{Z}_K)$ that the proposed map is an isomorphism of graded algebras. So it remains to prove that it commutes with differentials. Let d, d_c and ∂_c denote the differential in $A^*(K)$, $C^*(\mathcal{Z}_K)$ and $C_*(\mathcal{Z}_K)$ respectively. Since $d(v_i) = 0$ and $d(u_i) = v_i$, we need to show that $d_c(D_i^*) = 0$, $d_c(T_i^*) = D_i^*$. We have $\partial_c(D_i) = T_i$, $\partial_c(T_i) = 0$. A 2-cell of $\mathcal{Z}_K$ is either D_j or $T_{jk} = T_j \times T_k$ $(k \neq j)$. Then

$$\langle d_cT_i^*, D_j\rangle = \langle T_i^*, \partial_cD_j\rangle = \langle T_i^*, T_j\rangle = \delta_{ij}, \quad \langle d_cT_i^*, T_{jk}\rangle = \langle T_i^*, \partial_cT_{jk}\rangle = 0,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Hence, $d_c(T_i^*) = D_i^*$. Further, a 3-cell of $\mathcal{Z}_K$ is either D_jT_k or $T_{j_1j_2j_3} = T_{j_1} \times T_{j_2} \times T_{j_3}$. Then

$$\langle d_cD_i^*, D_jT_k\rangle = \langle D_i^*, \partial_c(D_jT_k)\rangle = \langle D_i^*, T_{jk}\rangle = 0,$$
$$\langle d_cD_i^*, T_{j_1j_2j_3}\rangle = \langle D_i^*, \partial_cT_{j_1j_2j_3}\rangle = 0.$$

Hence, $d_c(D_i^*) = 0$. □

The above theorem provides a topological interpretation for the differential algebra $[A^*(K), d]$. In the sequel we will not distinguish the cochain complexes $A^*(K)$ and $C^*(\mathcal{Z}_K)$, and identify u_i with T_i^*, v_i with D_i^*.

Now we can summarise the results of Proposition 3.4, Lemma 3.32, Lemma 6.17, Corollary 6.32 and Theorem 7.11 in the following statement describing the functorial properties of the correspondence $K \mapsto \mathcal{Z}_K$.

PROPOSITION 7.12. *Let us introduce the following functors:*

- *$\mathcal{Z}$, the covariant functor $K \mapsto \mathcal{Z}_K$ from the category of finite simplicial complexes and simplicial inclusions to the category of toric spaces and equivariant maps (the moment-angle complex functor);*

- $\mathbf{k}(\cdot)$, *the contravariant functor* $K \mapsto \mathbf{k}(K)$ *from simplicial complexes to graded* $\mathbf{k}$*-algebras (the Stanley–Reisner functor);*
- Tor-alg, *the contravariant functor*

$$K \mapsto \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr)$$

from simplicial complexes to bigraded $\mathbf{k}$*-algebras (the* Tor*-algebra functor, it coincides with the composition of* $\mathbf{k}(\cdot)$ *and* $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\cdot,\mathbf{k})$*);*
- H_T^*, *the contravariant functor* $X \mapsto H_T^*(X)$ *from the category of toric spaces and equivariant maps to* $\mathbf{k}$*-algebras (the equivariant cohomology functor);*
- H^*, *the contravariant functor* $X \mapsto H^*(X)$ *from spaces to* $\mathbf{k}$*-algebras (the ordinary cohomology functor).*

Then we have the following identities:

$$H_T^* \circ \mathcal{Z} = \mathbf{k}(\cdot), \qquad H^* \circ \mathcal{Z} = \text{Tor-alg}.$$

The later identity implies that for every simplicial inclusion $\phi : K_1 \to K_2$ *the cohomology map* $\phi_{\mathrm{ma}}^* : H^*(\mathcal{Z}_{K_2}) \to H^*(\mathcal{Z}_{K_1})$ *coincides with the induced homomorphism* ϕ_t^* *(3.7) of* Tor*-algebras. In particular,* ϕ *induces a homomorphism* $H^{-q,2p}(\mathcal{Z}_{K_2}) \to H^{-q,2p}(\mathcal{Z}_{K_1})$ *of bigraded cohomology modules.*

In the Cohen–Macaulay case we have the following reduction theorem for the cohomology of $\mathcal{Z}_K$.

Theorem 7.13. *Suppose that* K^{n-1} *is Cohen–Macaulay, and let* $\mathcal{J}$ *be an ideal in* $\mathbf{k}(K)$ *generated by degree-two regular sequence of length* n*. Then the following isomorphism of algebras holds:*

$$H^*(\mathcal{Z}_K) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]/\mathcal{J}}\bigl(\mathbf{k}(K)/\mathcal{J},\mathbf{k}\bigr).$$

Proof. This follows from Theorem 7.6 and Lemma 3.35. □

Note that the $\mathbf{k}$-algebra $\mathbf{k}(K)/\mathcal{J}$ is finite-dimensional (unlike $\mathbf{k}(K)$). In some circumstances (see section 7.4) this helps to calculate the cohomology of $\mathcal{Z}_K$ more efficiently.

7.3. Bigraded Betti numbers of $\mathcal{Z}_K$: the case of general K

The bigraded structure in the algebra $[A^*(K),d]$ defines a bigrading in the cellular chain complex $[C_*(\mathcal{Z}_K),\partial_c]$ via the isomorphism of Theorem 7.11. We have

$$\operatorname{bideg}(D_i) = (0,2), \quad \operatorname{bideg}(T_i) = (-1,2), \quad \operatorname{bideg}(1_i) = (0,0). \tag{7.4}$$

The differential ∂_c adds $(-1,0)$ to bidegree and thus the bigrading descends to the cellular homology of $\mathcal{Z}_K$.

In this section we assume that the ground field $\mathbf{k}$ is of zero characteristic. Define the bigraded Betti numbers

$$b_{-q,2p}(\mathcal{Z}_K) = \dim H_{-q,2p}\bigl[C_*(\mathcal{Z}_K),\partial_c\bigr], \quad q,p = 0,\dots,m. \tag{7.5}$$

Theorem 7.11 and Lemma 7.10 show that

$$b_{-q,2p}(\mathcal{Z}_K) = \dim \operatorname{Tor}^{-q,2p}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr) = \beta^{-q,2p}\bigl(\mathbf{k}(K)\bigr), \tag{7.6}$$

see (3.5). Alternatively, $b_{-q,2p}(\mathcal{Z}_K)$ equals the dimension of $(-q,2p)$-th bigraded component of the cohomology algebra $H[\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}(K),d]$. For the ordinary Betti numbers $b_k(\mathcal{Z}_K)$ we have

$$b_k(\mathcal{Z}_K)=\sum_{-q+2p=k} b_{-q,2p}(\mathcal{Z}_K),\quad k=0,\dots,m+n. \tag{7.7}$$

The lemma below describes some basic properties of bigraded Betti numbers (7.5).

LEMMA 7.14. *Let K^{n-1} be a simplicial complex with $m=f_0$ vertices and f_1 edges, and $\mathcal{Z}_K$ the corresponding moment-angle complex,* $\dim\mathcal{Z}_K=m+n$. *Then*

(a) $b_{0,0}(\mathcal{Z}_K)=b_0(\mathcal{Z}_K)=1$ *and* $b_{0,2p}(\mathcal{Z}_K)=0$ *for* $p>0$;
(b) $b_{-q,2p}=0$ *for* $p>m$ *or* $q>p$;
(c) $b_1(\mathcal{Z}_K)=b_2(\mathcal{Z}_K)=0$;
(d) $b_3(\mathcal{Z}_K)=b_{-1,4}(\mathcal{Z}_K)=\binom{f_0}{2}-f_1$;
(e) $b_{-q,2p}(\mathcal{Z}_K)=0$ *for* $q\geqslant p>0$ *or* $p-q>n$;
(f) $b_{m+n}(\mathcal{Z}_K)=b_{-(m-n),2m}(\mathcal{Z}_K)$.

PROOF. In this proof we calculate the Betti numbers using the cochain subcomplex $A^*(K)\subset\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}(K)$. The module $A^*(K)$ has the basis consisting of monomials $u_\tau v_\sigma$ with $\sigma\in K$ and $\sigma\cap\tau=\varnothing$. Since $\operatorname{bideg} v_i=(0,2)$, $\operatorname{bideg} u_j=(-1,2)$, the bigraded component $A^{-q,2p}(K)$ is spanned by monomials $u_\tau v_\sigma$ with $|\sigma|=p-q$ and $|\tau|=q$. In particular, $A^{-q,2p}(K)=0$ if $p>m$ or $q>p$, whence the assertion (b) follows. To prove (a) we observe that $A^{0,0}(K)$ is generated by 1, while any $v_\sigma\in A^{0,2p}(K)$ $(p>0)$ is a coboundary, whence $H^{0,2p}(\mathcal{Z}_K)=0$ for $p>0$.

Now look at assertion (e). Every $u_\tau v_\sigma\in A^{-q,2p}(K)$ has $\sigma\in K$, while any simplex of K has at most n vertices. It follows that $A^{-q,2p}(K)=0$ for $p-q>n$. By (b), $b_{-q,2p}(\mathcal{Z}_K)=0$ for $q>p$, so it remains to prove that $b_{-q,2q}(\mathcal{Z}_K)=0$ for $q>0$. The module $A^{-q,2q}(K)$ is generated by monomials u_τ with $|\tau|=q$. Since $d(u_i)=v_i$, it follows easily that there are no non-zero cocycles in $A^{-q,2q}(K)$. Hence, $H^{-q,2q}(\mathcal{Z}_K)=0$.

The assertion (c) follows from (e) and (7.7).

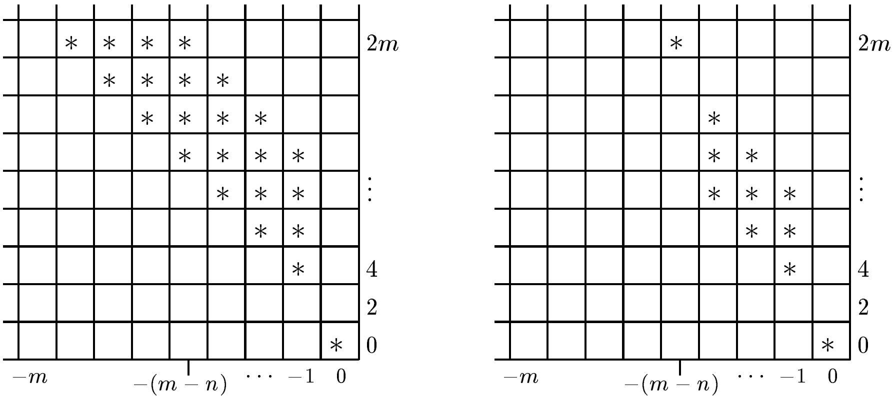
It also follows from (e) that $H^3(\mathcal{Z}_K)=H^{-1,4}(\mathcal{Z}_K)$. The basis for $A^{-1,4}(K)$ consists of monomials u_jv_i, $i\neq j$. We have $d(u_jv_i)=v_iv_j$ and $d(u_iu_j)=u_jv_i-u_iv_j$. Hence, u_jv_i is a cocycle if and only if $\{i,j\}$ is not a 1-simplex in K; in this case two cocycles u_jv_i and u_iv_j represent the same cohomology class. Assertion (d) follows.

The remaining assertion (f) follows from the fact that a monomial $u_\sigma v_\tau\in A^*(K)$ has maximal total degree $(m+n)$ if and only if $|\tau|=n$ and $|\sigma|=m-n$. $\square$

Lemma 7.14 shows that non-zero bigraded Betti numbers $b_{r,2p}(\mathcal{Z}_K)$, $r\neq 0$ appear only in the strip bounded by the lines $p=m$, $r=-1$, $p+r=1$ and $p+r=n$ in the second quadrant, see Figure 7.1 (a).

The homogeneous component $C_{-q,2p}(\mathcal{Z}_K)$ has basis of cellular chains $\mathcal{T}(\sigma,\tau)$ with $\sigma\in K$, $|\sigma|=p-q$ and $|\tau|=q$. It follows that

$$\dim C_{-q,2p}(\mathcal{Z}_K)=f_{p-q-1}\tbinom{m-p+q}{q}, \tag{7.8}$$



(a) arbitrary K^{n-1} (b) $|K| = S^{n-1}$

FIGURE 7.1. Possible locations of non-zero bigraded Betti numbers $b_{-q,2p}(\mathcal{Z}_K)$ (marked by $*$).

where $(f_0, f_1, \dots, f_{n-1})$ is the f-vector of K^{n-1} and $f_{-1} = 1$. The differential ∂_c does not change the second degree:

$$\partial_c : C_{-q,2p}(\mathcal{Z}_K) \to C_{-q-1,2p}(\mathcal{Z}_K).$$

Hence, the chain complex $C_{*,*}(\mathcal{Z}_K)$ splits as follows:

$$[C_{*,*}(\mathcal{Z}_K), \partial_c] = \bigoplus_{p=0}^{m} [C_{*,2p}(\mathcal{Z}_K), \partial_c].$$

REMARK. The similar decomposition holds also for the cellular cochain complex $[C^{*,*}(\mathcal{Z}_K), d_c] \cong [A^{*,*}(K), d]$.

Let us consider the Euler characteristic of complex $[C_{*,2p}(\mathcal{Z}_K), \partial_c]$:

$$\chi_p(\mathcal{Z}_K) := \sum_{q=0}^{m} (-1)^q \dim C_{-q,2p}(\mathcal{Z}_K) = \sum_{q=0}^{m} (-1)^q b_{-q,2p}(\mathcal{Z}_K). \tag{7.9}$$

Define the generating polynomial $\chi(\mathcal{Z}_K; t)$ by

$$\chi(\mathcal{Z}_K; t) = \sum_{p=0}^{m} \chi_p(\mathcal{Z}_K) t^{2p}.$$

The following theorem calculates this polynomial in terms of the h-vector of K.

THEOREM 7.15. *For every $(n-1)$-dimensional simplicial complex K with m vertices it holds that*

$$\chi(\mathcal{Z}_K; t) = (1 - t^2)^{m-n}(h_0 + h_1 t^2 + \dots + h_n t^{2n}) = (1 - t^2)^m F\big(\mathbf{k}(K); t\big), \tag{7.10}$$

where $(h_0, h_1, \dots, h_n)$ is the h-vector of K.

PROOF. It follows from (7.9) and (7.8) that

$$\chi_p(\mathcal{Z}_K) = \sum_{j=0}^{m} (-1)^{p-j} f_{j-1} \binom{m-j}{p-j}, \tag{7.11}$$

Then

$$\begin{aligned}(7.12)\quad \chi(\mathcal{Z}_K;t) &= \sum_{p=0}^{m}\chi_p(K)t^{2p} = \sum_{p=0}^{m}\sum_{j=0}^{m} t^{2j}t^{2(p-j)}(-1)^{p-j}f_{j-1}\binom{m-j}{p-j}\\ &= \sum_{j=0}^{m} f_{j-1}t^{2j}(1-t^2)^{m-j} = (1-t^2)^m\sum_{j=0}^{n} f_{j-1}(t^{-2}-1)^{-j}.\end{aligned}$$

Denote $h(t) = h_0 + h_1t + \cdots + h_nt^n$. From (1.7) we get

$$t^nh(t^{-1}) = (t-1)^n\sum_{i=0}^{n} f_{i-1}(t-1)^{-i}.$$

Substituting t^{-2} for t above, we finally rewrite (7.12) as

$$\frac{\chi(\mathcal{Z}_K;t)}{(1-t^2)^m} = \frac{t^{-2n}h(t^2)}{(t^{-2}-1)^n} = \frac{h(t^2)}{(1-t^2)^n},$$

which is equivalent to the first identity from (7.10). The second identity follows from Lemma 3.8. □

The formula from the above theorem can be used to express the face vector of a simplicial complex in terms of the bigraded Betti numbers of the corresponding moment-angle complex $\mathcal{Z}_K$.

COROLLARY 7.16. *The Euler characteristic of $\mathcal{Z}_K$ is zero.*

PROOF. We have

$$\chi(\mathcal{Z}_K) = \sum_{p,q=0}^{m}(-1)^{-q+2p}b_{-q,2p}(\mathcal{Z}_K) = \sum_{p=0}^{m}\chi_p(\mathcal{Z}_K) = \chi(\mathcal{Z}_K;1),$$

so the statement follows from (7.10). □

REMARK. Another proof of the above corollary follows from the observation that the diagonal subgroup $S^1 \subset T^m$ always acts freely on $\mathcal{Z}_K$ (see section 7.5). Hence, there exists a principal S^1-bundle $\mathcal{Z}_K \to \mathcal{Z}_K/S^1$, which implies $\chi(\mathcal{Z}_K) = 0$.

The torus $\mathcal{Z}_\varnothing = \rho^{-1}(1,\dots,1) \cong T^m$ is a cellular subcomplex of $\mathcal{Z}_K$, see Lemma 6.18. The cellular cochain subcomplex $C^*(\mathcal{Z}_\varnothing) \subset C^*(\mathcal{Z}_K) \cong A^*(K)$ has the basis consisting of cochains $\mathcal{T}(\varnothing,\tau)^*$ and is mapped to the exterior algebra $\Lambda[u_1,\dots,u_m] \subset A^*(K)$ under the isomorphism of Theorem 7.11. It follows that there is an isomorphism of $\mathbf{k}$-modules

$$(7.13)\qquad C^*(\mathcal{Z}_K,\mathcal{Z}_\varnothing) \cong A^*(K)/\Lambda[u_1,\dots,u_m].$$

We introduce relative bigraded Betti numbers

$$(7.14)\qquad b_{-q,2p}(\mathcal{Z}_K,\mathcal{Z}_\varnothing) = \dim H^{-q,2p}\big[C^*(\mathcal{Z}_K,\mathcal{Z}_\varnothing),d\big],\quad q,p = 0,\dots,m,$$

define the p-th relative Euler characteristic $\chi_p(\mathcal{Z}_K,\mathcal{Z}_\varnothing)$ by

$$(7.15)\quad \chi_p(\mathcal{Z}_K,\mathcal{Z}_\varnothing) = \sum_{q=0}^{m}(-1)^q\dim C^{-q,2p}(\mathcal{Z}_K,\mathcal{Z}_\varnothing) = \sum_{q=0}^{m}(-1)^q b_{-q,2p}(\mathcal{Z}_K,\mathcal{Z}_\varnothing),$$

and define the corresponding generating polynomial:

$$\chi(\mathcal{Z}_K,\mathcal{Z}_\varnothing;t) = \sum_{p=0}^{m}\chi_p(\mathcal{Z}_K,\mathcal{Z}_\varnothing)t^{2p}.$$

THEOREM 7.17. *For any $(n-1)$-dimensional simplicial complex K with m vertices it holds that*

$$\chi(\mathcal{Z}_K, \mathcal{Z}_\varnothing; t) = (1-t^2)^{m-n}(h_0 + h_1t^2 + \cdots + h_nt^{2n}) - (1-t^2)^m. \tag{7.16}$$

PROOF. Since $C^*(\mathcal{Z}_\varnothing) = \Lambda[u_1, \ldots, u_m]$ and $\operatorname{bideg} u_i = (-1, 2)$, we have

$$\dim C^{-q}(\mathcal{Z}_\varnothing) = \dim C^{-q,2q}(\mathcal{Z}_\varnothing) = \tbinom{m}{q}.$$

Combining (7.13), (7.9) and (7.15), we get

$$\chi_p(\mathcal{Z}_K, \mathcal{Z}_\varnothing) = \chi_p(\mathcal{Z}_K) - (-1)^p \dim C^{-p,2p}(\mathcal{Z}_\varnothing).$$

Hence,

$$\begin{aligned}\chi(\mathcal{Z}_K, \mathcal{Z}_\varnothing; t) &= \chi(\mathcal{Z}_K; t) - \sum_{p=0}^{m}(-1)^p\tbinom{m}{p}t^{2p} \\ &= (1-t^2)^{m-n}(h_0 + h_1t^2 + \cdots + h_nt^{2n}) - (1-t^2)^m,\end{aligned}$$

by (7.10). $\square$

We will use the above theorem in section 7.6.

7.4. Bigraded Betti numbers of $\mathcal{Z}_K$: the case of spherical K

If K is a simplicial sphere then $\mathcal{Z}_K$ is a manifold (Lemma 6.13). This imposes additional conditions on the cohomology of $\mathcal{Z}_K$ and leads to some interesting interpretations of combinatorial results and problems from chapters 2 and 3.

THEOREM 7.18. *Let K be an $(n-1)$-dimensional simplicial sphere, and $\mathcal{Z}_K$ the corresponding moment-angle manifold, $\dim \mathcal{Z}_K = m+n$. Then the fundamental cohomology class of $\mathcal{Z}_K$ is represented by any monomial $\pm v_\sigma u_\tau \in A^*(K)$ of bidegree $(-(m-n), 2m)$ such that σ is an $(n-1)$-simplex of K and $\sigma \cap \tau = \varnothing$. The sign depends on a choice of orientation for $\mathcal{Z}_K$.*

PROOF. Lemma 7.14 (f) shows that $H^{m+n}(\mathcal{Z}_K) = H^{-(m-n),2m}(\mathcal{Z}_K)$. The module $A^{-(m-n),2m}(K)$ is spanned by the monomials $v_\sigma u_\tau$ such that $\sigma \in K^{n-1}$, $|\sigma| = n$, $\tau = [m] \setminus \sigma$. Every such monomial is a cocycle. Suppose that σ, σ' are two $(n-1)$-simplices of K^{n-1} sharing a common $(n-2)$-face. We claim that the corresponding cocycles $v_\sigma u_\tau$, $v_{\sigma'} u_{\tau'}$ (where $\tau = [m] \setminus \sigma$, $\tau' = [m] \setminus \sigma'$) represent the same cohomology class up to a sign. Indeed, let

$$\begin{aligned} v_\sigma u_\tau &= v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}}, \\ v_{\sigma'} u_{\tau'} &= v_{i_1} \cdots v_{i_{n-1}} v_{j_1} u_{i_n} u_{j_2} \cdots u_{j_{m-n}}. \end{aligned}$$

Since every $(n-2)$-face of K is contained in exactly two $(n-1)$-faces, the identity

$$\begin{aligned} &d(v_{i_1} \cdots v_{i_{n-1}} u_{i_n} u_{j_1} u_{j_2} \cdots u_{j_{m-n}}) \\ &\qquad = v_{i_1} \cdots v_{i_n} u_{j_1} \cdots u_{j_{m-n}} - v_{i_1} \cdots v_{i_{n-1}} v_{j_1} u_{i_n} u_{j_2} \cdots u_{j_{m-n}} \end{aligned}$$

holds in $A^*(K) \subset \Lambda[u_1, \ldots, u_m] \otimes \mathbf{k}(K)$. Hence, $[v_\sigma u_\tau] = [v_{\sigma'} u_{\tau'}]$ (as cohomology classes). Since K^{n-1} is a simplicial sphere, every two $(n-1)$-simplices can be connected by a chain of simplices in such a way that any two successive simplices share a common $(n-2)$-face. Thus, all monomials $v_\sigma u_\tau$ in $A^{-(m-n),2m}(K)$ represent the same cohomology class (up to a sign). This class is a generator of $H^{m+n}(\mathcal{Z}_K)$, i.e. the fundamental cohomology class of $\mathcal{Z}_K$. $\square$

REMARK. In the above proof we have used two combinatorial properties of K^{n-1}. The first one is that every $(n-2)$-face is contained in exactly two $(n-1)$-faces, and the second is that every two $(n-1)$-simplices can be connected by a chain of simplices with any two successive simplices sharing a common $(n-2)$-face. Simplicial complexes satisfying these two conditions are called *pseudomanifolds*. In particular, every triangulated manifold is a pseudomanifold. Hence, for any triangulated manifold K^{n-1} we have $b_{m+n}(\mathcal{Z}_K) = b_{-(m-n),2m}(\mathcal{Z}_K) = 1$, and the generator of $H^{m+n}(\mathcal{Z}_K)$ can be chosen as described in Theorem 7.18.

COROLLARY 7.19. *The Poincaré duality for the moment angle manifold $\mathcal{Z}_K$ corresponding to a simplicial sphere K^{n-1} respects the bigraded structure in the (co)homology, i.e.*

$$H^{-q,2p}(\mathcal{Z}_K) \cong H_{-(m-n)+q,2(m-p)}(\mathcal{Z}_K).$$

In particular,

$$b_{-q,2p}(\mathcal{Z}_K) = b_{-(m-n)+q,2(m-p)}(\mathcal{Z}_K). \qquad \square \tag{7.17}$$

COROLLARY 7.20. *Let K^{n-1} be an $(n-1)$-dimensional simplicial sphere, and $\mathcal{Z}_K$ the corresponding moment-angle complex,* $\dim \mathcal{Z}_K = m+n$. *Then*

(a) $b_{-q,2p}(\mathcal{Z}_K) = 0$ *for $q \geqslant m-n$, with only exception $b_{-(m-n),2m} = 1$;*
(b) $b_{-q,2p}(\mathcal{Z}_K) = 0$ *for $p-q \geqslant n$, with only exception $b_{-(m-n),2m} = 1$.*

It follows that if K^{n-1} is a simplicial sphere, then non-zero bigraded Betti numbers $b_{r,2p}(\mathcal{Z}_K)$ with $r \neq 0$ and $r \neq m-n$ appear only in the strip bounded by the lines $r = -(m-n-1)$, $r = -1$, $p+r = 1$ and $p+r = n-1$ in the second quadrant, see Figure 7.1 (b). Compare this with Figure 7.1 (a) corresponding to the case of general K.

EXAMPLE 7.21. Let $K = \partial\Delta^{m-1}$. Then $\mathbf{k}(K) = \mathbf{k}[v_1, \dots, v_m]/(v_1 \cdots v_m)$, see Example 3.9. A direct calculation shows that the cohomology $H[\mathbf{k}(K) \otimes \Lambda[u_1, \dots, u_m], d]$ (see Theorem 7.7) is additively generated by the classes 1 and $[v_1 v_2 \cdots v_{m-1} u_m]$. We have $\deg(v_1 v_2 \cdots v_{m-1} u_m) = 2m-1$, and Theorem 7.18 says that $v_1 v_2 \cdots v_{m-1} u_m$ represents the fundamental cohomology class of $\mathcal{Z}_K \cong S^{2m-1}$.

EXAMPLE 7.22. Let K be the boundary complex of an m-gon P^2 with $m \geqslant 4$. We have $\mathbf{k}(K) = \mathbf{k}[v_1, \dots, v_m]/\mathcal{I}_P$, where $\mathcal{I}_P$ is generated by the monomials $v_i v_j$, $i - j \neq 0, 1 \mod m$. The complex $\mathcal{Z}_K = \mathcal{Z}_P$ is a manifold of dimension $m+2$. The Betti numbers of these manifolds were calculated in [**31**]. Namely,

$$\dim H^k(\mathcal{Z}_P) = \begin{cases} 1 & \text{for } k = 0, m+2; \\ 0 & \text{for } k = 1, 2, m, m+1; \\ (m-2)\binom{m-2}{k-2} - \binom{m-2}{k-1} - \binom{m-2}{k-3} & \text{for } 3 \leqslant k \leqslant m-1. \end{cases} \tag{7.18}$$

For example, in the case $m = 5$ the group $H^3(\mathcal{Z}_P)$ has 5 generators represented by the cocycles $v_i u_{i+2} \in \mathbf{k}(K) \otimes \Lambda[u_1, \dots, u_5]$, $i = 1, \dots, 5$, while the group $H^4(\mathcal{Z}_P)$ has 5 generators represented by the cocycles $v_j u_{j+2} u_{j+3}$, $j = 1, \dots, 5$. As it follows from Theorem 7.18, the product of cocycles $v_i u_{i+2}$ and $v_j u_{j+2} u_{j+3}$ represents a non-zero cohomology class in $H^7(\mathcal{Z}_P)$ if and only if all the indices $i, i+2, j, j+2, j+3$ are different. Thus, for each of the 5 cohomology classes $[v_i u_{i+2}]$ there is a unique (Poincaré dual) cohomology class $[v_j u_{j+2} u_{j+3}]$ such that the product $[v_i u_{i+2}] \cdot [v_j u_{j+2} u_{j+3}]$ is non-zero. This observation has the following generalization, which describes the multiplicative structure in the cohomology of $\mathcal{Z}_{P^2}$ for any m-gon P^2.

PROPOSITION 7.23 (Cohomology ring of $\mathcal{Z}_{P^2}$). *Let P^2 be an m-gon, $m \geqslant 4$.*

(a) *The only non-zero bigraded cohomology groups of $\mathcal{Z}_{P^2}$ are $H^{0,0}(= H^0)$, $H^{-p,2(p+1)}(= H^{p+2})$ for $p = 1, \ldots, m-3$, and $H^{-m+2,2m}(= H^{m+2})$.*

(b) *The group $H^{-p,2(p+1)}$ is free and is generated by the cohomology classes $[v_i u_\tau]$ such that $|\tau| = p$, $i \notin \tau$ and $i \pm 1 \notin \tau$. These cohomology classes are subject to relations of the form $du_{\tau'} = 0$ for $|\tau'| = p+1$. The corresponding Betti numbers are given by (7.18).*

(c) *The group $H^{-m+2,2m}$ is one-dimensional with generator $[v_1 v_2 u_3 \cdots u_m]$.*

(d) *The product of two cohomology classes $[v_{i_1} u_{\tau_1}] \in H^{-p_1,2(p_1+1)}$ and $[v_{i_2} u_{\tau_2}] \in H^{-p_2,2(p_2+1)}$ equals $[v_1 v_2 u_3 \cdots u_m]$ (up to a sign) if $\{\{i_1\}, \{i_2\}, \tau_1, \tau_2\}$ is a partition of $[m]$, and zero otherwise.*

Therefore, the only non-trivial products in the ring $H^*(\mathcal{Z}_{P^2})$ are those which give a multiple of the fundamental class.

PROOF. Statement (a) follows from Corollary 7.20, (b) is obvious, and (c) follows from Theorem 7.18. In order to prove (d) we mention that the product of two classes $a_1 \in H^{-p_1,2(p_1+1)}$ and $a_2 \in H^{-p_2,2(p_2+1)}$ has bidegree $(-p, 2q)$ with $q-p = 2$, whence it can be non-zero only if it belongs to $H^{-m+2,2m}$, by Corollary 7.20. □

It follows from (7.9) and (7.17) that for any simplicial sphere K the following holds:

$$\chi_p(\mathcal{Z}_K) = (-1)^{m-n} \chi_{m-p}(\mathcal{Z}_K).$$

From this and (7.10) we get

$$\begin{aligned}
\frac{h_0 + h_1 t^2 + \cdots + h_n t^{2n}}{(1-t^2)^n} &= (-1)^{m-n} \frac{\chi_m + \chi_{m-1} t^2 + \cdots + \chi_0 t^{2m}}{(1-t^2)^m} \\
&= (-1)^n \frac{\chi_0 + \chi_1 t^{-2} + \cdots + \chi_m t^{-2m}}{(1-t^{-2})^m} = (-1)^n \frac{h_0 + h_1 t^{-2} + \cdots + h_n t^{-2n}}{(1-t^{-2})^n} \\
&= \frac{h_0 t^{2n} + h_1 t^{2(n-1)} + \cdots + h_n}{(1-t^2)^n}.
\end{aligned}$$

Hence, $h_i = h_{n-i}$. Thus, the Dehn–Sommerville equations are a corollary of the bigraded Poincaré duality (7.17).

The identity (7.10) also allows us to interpret different inequalities for the f-vectors of simplicial spheres or triangulated manifolds in terms of topological invariants (the bigraded Betti numbers) of the corresponding moment-angle manifolds (or complexes) $\mathcal{Z}_K$.

EXAMPLE 7.24. Using the expansion

$$\left(\frac{1}{1-t^2}\right)^{m-n} = \sum_{i=0}^{\infty} \binom{m-n+i-1}{i} t^{2i}$$

together with the UBC for simplicial spheres (Corollary 3.19) and identity (7.10), we deduce that the inequality $\chi(\mathcal{Z}_K; t) \leqslant 1$ holds coefficient-wise for any simplicial sphere K^{n-1}. That is,

$$\chi_i(\mathcal{Z}_K; t) \leqslant 0 \quad \text{for } i > 0.$$

EXAMPLE 7.25. Using Lemma 7.14 we calculate

$$\begin{aligned}
&\chi_0(\mathcal{Z}_K) = 1, && \chi_1(\mathcal{Z}_K) = 0,\\
&\chi_2(\mathcal{Z}_K) = -b_{-1,4}(\mathcal{Z}_K) = -b_3(\mathcal{Z}_K), && \chi_3(\mathcal{Z}_K) = b_{-2,6}(\mathcal{Z}_K) - b_{-1,6}(\mathcal{Z}_K)
\end{aligned}$$

(note that $b_4(\mathcal{Z}_K) = b_{-2,6}(\mathcal{Z}_K)$ and $b_5(\mathcal{Z}_K) = b_{-1,6}(\mathcal{Z}_K) + b_{-3,8}(\mathcal{Z}_K)$). Now, identity (7.10) shows that

$$\begin{aligned}
h_0 &= 1,\\
h_1 &= m-n,\\
h_2 &= \tbinom{m-n+1}{2} - b_3(\mathcal{Z}_K),\\
h_3 &= \tbinom{m-n+2}{3} - (m-n)b_{-1,4}(\mathcal{Z}_K) + b_{-2,6}(\mathcal{Z}_K) - b_{-1,6}(\mathcal{Z}_K).
\end{aligned}$$

It follows that the inequality $h_1 \leqslant h_2$ $(n \geqslant 4)$ from the GLBC (1.14) for simplicial spheres is equivalent to the following:

$$b_3(\mathcal{Z}_K) \leqslant \tbinom{m-n}{2}. \tag{7.19}$$

(Note that this inequality is not valid for $n = 2$, see e.g. Example 7.22, and becomes identity for $n = 3$.) The next inequality $h_2 \leqslant h_3$ $(n \geqslant 6)$ from (1.14) is equivalent to the following:

$$\tbinom{m-n+1}{3} - (m-n-1)b_{-1,4}(\mathcal{Z}_K) + b_{-2,6}(\mathcal{Z}_K) - b_{-1,6}(\mathcal{Z}_K) \geqslant 0. \tag{7.20}$$

We see that the combinatorial GLBC inequalities are interpreted as "topological" inequalities for the (bigraded) Betti numbers of a manifold. This might open a possibility to use topological methods (such as the equivariant topology or Morse theory) for proving inequalities like (7.19) or (7.20). Such a topological approach to problems like g-conjecture or GLBC has an advantage of being independent on whether the simplicial sphere K is polytopal or not. Indeed, as we have already mentioned, all known proofs for the necessity condition in the g-theorem for simplicial polytopes (including the original one by Stanley given in section 5.1, McMullen's proof [**97**], and the recent proof by Timorin [**133**]) follow the same scheme. Namely, the numbers h_i, $i = 1, \dots, n$, are interpreted as the dimensions of graded components A^i of a certain algebra A satisfying the Hard Lefschetz Theorem. The latter means that there is an element $\omega \in A^1$ such that the multiplication by ω defines a *monomorphism* $A^i \to A^{i+1}$ for $i < \left[\frac{n}{2}\right]$. This implies $h_i \leqslant h_{i+1}$ for $i < \left[\frac{n}{2}\right]$ (see section 5.1). However, such an element ω is lacking for non-polytopal K, which means that a new technique has to be developed in order to prove the g-conjecture for simplicial spheres.

As it was mentioned in section 3.5, simplicial spheres are Gorenstein* complexes. Using Theorems 3.38, 3.39 and our Theorem 7.6 we obtain the following answer to a weaker version of Problem 6.14.

PROPOSITION 7.26. *The complex $\mathcal{Z}_K$ is a Poincaré duality complex (over $\mathbf{k}$) if and only if K is Gorenstein*, i.e., for any simplex $\sigma \in K$ (including $\sigma = \varnothing$) the subcomplex* $\operatorname{link}\sigma$ *has the homology of a sphere of dimension* $\dim(\operatorname{link}\sigma)$.

7.5. Partial quotients of $\mathcal{Z}_P$

Here we return to the case of polytopal K (i.e. $K = K_P$ for some simple polytope P) and study quotients of $\mathcal{Z}_P$ by freely acting subgroups $H \subset T^m$.

For any combinatorial simple polytope P^n, define $s = s(P^n)$ to be the maximal dimension for which there exists a subgroup $H \cong T^s$ in T^m acting freely on $\mathcal{Z}_P$. The number $s(P^n)$ is obviously a combinatorial invariant of P^n.

PROBLEM 7.27 (V.M. Buchstaber). *Provide an efficient way to calculate the number $s(P^n)$, e.g. in terms of known combinatorial invariants of P^n.*

PROPOSITION 7.28. *If P^n has m facets, then $s(P^n) \leqslant m - n$.*

PROOF. Every subtorus of T^m of dimension $> m - n$ intersects non-trivially with any n-dimensional isotropy subgroup, and therefore cannot act freely on $\mathcal{Z}_P$. □

PROPOSITION 7.29. *The diagonal circle subgroup $S_d := \{(e^{2\pi i\varphi}, \dots, e^{2\pi i\varphi}) \in T^m\}$, $\varphi \in \mathbb{R}$, acts freely on any $\mathcal{Z}_P$. Thus, $s(P^n) \geqslant 1$.*

PROOF. By Definition 6.1, every isotropy subgroup for $\mathcal{Z}_P$ is coordinate, and therefore intersects S_d only at the unit. □

An alternative lower bound for the number $s(P^n)$ was proposed in [**79**]. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be the set of facets of P^n. We generalize the definition of a regular coloring from section 6.6 as follows. A surjective map $\varrho : \mathcal{F} \to [k]$ (where $[k] = \{1, \dots, k\}$) is called a *regular k-paint coloring* of P^n if $\varrho(F_i) \neq \varrho(F_j)$ whenever $F_i \cap F_j \neq \varnothing$. The *chromatic number* $\gamma(P^n)$ is the minimal k for which there exists a regular k-paint coloring of P^n. Then $\gamma(P^n) \geqslant n$ and, due to the result mentioned in section 6.6, the equality is achieved if and only if every 2-face of P^n is an evengon. Note also that $\gamma(P^3) \leqslant 4$ by the Four Color Theorem.

EXAMPLE 7.30. Suppose P^n is a 2-neighborly simple polytope with m facets. Then $\gamma(P^n) = m$.

PROPOSITION 7.31 ([**79**]). *The following inequality holds:*

$$s(P^n) \geqslant m - \gamma(P^n).$$

PROOF. The map $\varrho : \mathcal{F} \to [k]$ defines an epimorphism of tori $\tilde{\varrho} : T^m \to T^k$. It is easy to see that if ϱ is a regular coloring, then $\operatorname{Ker} \tilde{\varrho} \cong T^{m-k}$ acts freely on $\mathcal{Z}_P$. □

For more results on colorings and their relations with Problem 7.27 see [**81**].

Let $H \subset T^m$ be a subgroup of dimension $r \leqslant m - n$. Choosing a basis, we can write it in the form

$$H = \bigl\{(e^{2\pi i(s_{11}\varphi_1 + \dots + s_{1r}\varphi_r)}, \dots, e^{2\pi i(s_{m1}\varphi_1 + \dots + s_{mr}\varphi_r)}) \in T^m\bigr\}, \tag{7.21}$$

where $\varphi_i \in \mathbb{R}$, $i = 1, \dots, r$. The integer $m \times r$-matrix $S = (s_{ij})$ defines a monomorphism $\mathbb{Z}^r \to \mathbb{Z}^m$ whose image is a direct summand in $\mathbb{Z}^m$. For any subset $\{i_1, \dots, i_n\} \subset [m]$ denote by $S_{\hat{i}_1, \dots, \hat{i}_n}$ the $(m - n) \times r$ submatrix of S obtained by deleting the rows $i_1, \dots, i_n$. Write each vertex $v \in P^n$ as an intersection of n facets, as in (5.10). The following criterion of freeness for the action of H on $\mathcal{Z}_P$ holds.

LEMMA 7.32. *Subgroup (7.21) acts freely on $\mathcal{Z}_P$ if and only if for every vertex $v = F_{i_1} \cap \dots \cap F_{i_n}$ of P^n the $(m-n) \times r$-submatrix $S_{\hat{i}_1, \dots, \hat{i}_n}$ defines a monomorphism $\mathbb{Z}^r \hookrightarrow \mathbb{Z}^{m-n}$ to a direct summand.*

PROOF. It follows from Definition 6.1 that the orbits of T^m-action on $\mathcal{Z}_P$ corresponding to the vertices of P^n have maximal (rank n) isotropy subgroups. The isotropy subgroup corresponding to a vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ is the coordinate subtorus $T^n_{i_1,\ldots,i_n} \subset T^m$. Subgroup (7.21) acts freely on $\mathcal{Z}_P$ if and only if it intersects each isotropy subgroup only at the unit. This is equivalent to the condition that the map $H \times T^n_{i_1,\ldots,i_n} \to T^m$ is injective for any $v = F_{i_1} \cap \ldots \cap F_{i_n}$. This map is given by the integer $m \times (n+r)$-matrix obtained by adding n columns $(0,\ldots,0,1,0,\ldots,0)^t$ (with 1 at the place i_j, $j = 1,\ldots,n$) to S. The map is injective if and only if this enlarged matrix defines a direct summand in $\mathbb{Z}^m$. The latter holds if and only if each $S_{\hat{i}_1,\ldots,\hat{i}_n}$ defines a direct summand. □

In particular, for subgroups of rank $m-n$ we get the following statement.

COROLLARY 7.33. *The subgroup* (7.21) *of rank $r = m - n$ acts freely on $\mathcal{Z}_P$ if and only if for any vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P^n holds* $\det S_{\hat{i}_1 \ldots \hat{i}_n} = \pm 1$.

PROPOSITION 7.34. *A simple polytope P^n admits a characteristic map if and only if $s(P^n) = m - n$.*

PROOF. Proposition 6.5 shows that if P^n admits a characteristic map ℓ, then the $(m-n)$-dimensional subgroup $H(\ell)$ acts freely on $\mathcal{Z}_P$, whence $s(P^n) = m - n$. Now suppose $s(P^n) = m - n$, i.e. there exists a subgroup (7.21) of rank $r = m - n$ that acts freely on $\mathcal{Z}_P$. The corresponding $m \times (m-n)$-matrix S defines a monomorphism $\mathbb{Z}^{m-n} \to \mathbb{Z}^m$ whose image is a direct summand. It follows that there is an $n \times m$-matrix Λ such that the sequence

$$0 \longrightarrow \mathbb{Z}^{m-n} \xrightarrow{S} \mathbb{Z}^m \xrightarrow{\Lambda} \mathbb{Z}^n \longrightarrow 0$$

is exact. Since S satisfies the condition of Corollary 7.33, the matrix Λ satisfies (5.5), thus defining a characteristic map for P^n. □

Suppose M^{2n} is a quasitoric manifold over P^n with characteristic map ℓ. Write the subgroup $H(\ell)$ in the form (7.21). Now define the following linear forms in $\mathbf{k}[v_1,\ldots,v_m]$:

$$w_i = s_{1i}v_1 + \cdots + s_{mi}v_m, \quad i = 1,\ldots,m-n. \tag{7.22}$$

Under these assumptions the following statement holds.

LEMMA 7.35. *There is the following isomorphism of algebras:*

$$H^*(\mathcal{Z}_P) \cong \operatorname{Tor}_{\mathbf{k}[w_1,\ldots,w_{m-n}]}\bigl(H^*(M^{2n}),\mathbf{k}\bigr),$$

where the $\mathbf{k}[w_1,\ldots,w_{m-n}]$-module structure in $H^*(M^{2n}) = \mathbf{k}[v_1,\ldots,v_m]/\mathcal{I}_P + \mathcal{J}_\ell$ is defined by (7.22).

PROOF. By Theorem 7.13,

$$H^*(\mathcal{Z}_K) \cong \operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]/\mathcal{J}_\ell}\bigl(\mathbf{k}(K)/\mathcal{J}_\ell,\mathbf{k}\bigr).$$

The quotient $\mathbf{k}[v_1,\ldots,v_m]/\mathcal{J}_\ell$ is identified with $\mathbf{k}[w_1,\ldots,w_{m-n}]$. □

THEOREM 7.36. *The Leray–Serre spectral sequence of the T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}$ collapses at the E_3 term. Furthermore, the following isomorphism of algebras holds:*

$$H^*(\mathcal{Z}_P) \cong H\bigl[\Lambda[u_1,\ldots,u_{m-n}] \otimes \bigl(\mathbf{k}(P)/\mathcal{J}_\ell\bigr), d\bigr],$$

where

$$\operatorname{bideg} v_i = (0,2), \quad \operatorname{bideg} u_i = (-1,2);$$
$$d(u_i) = w_i, \quad d(v_i) = 0.$$

PROOF. Since $H^*(T^{m-n}) = \Lambda[u_1, \dots, u_{m-n}]$ and $H^*(M^{2n}) = \mathbf{k}(P)/\mathcal{J}_\ell$, we have

$$E_3 \cong H\big[(\mathbf{k}(P)/\mathcal{J}_\ell) \otimes \Lambda[u_1, \dots, u_{m-n}], d\big].$$

By Lemma 3.29,

$$H\big[(\mathbf{k}(P)/\mathcal{J}_\ell) \otimes \Lambda[u_1, \dots, u_{m-n}], d\big] \cong \operatorname{Tor}_{\mathbf{k}[w_1, \dots, w_{m-n}]}\big(H^*(M^{2n}), \mathbf{k}\big).$$

Combining the above two identities with Lemma 7.35 we get $E_3 = H^*(\mathcal{Z}_P)$, which concludes the proof. □

Our next aim is to calculate the cohomology of the quotient $\mathcal{Z}_P/H$ for arbitrary freely acting subgroup H. First, we write H in the form (7.21) and choose an $(m-r) \times m$-matrix $T = (t_{ij})$ of rank $(m-r)$ satisfying $T \cdot S = 0$. This is done in the same way as in the proof of Proposition 7.34. In particular, if $r = m - n$ then T is the characteristic matrix for the quasitoric manifold $\mathcal{Z}_P/H$.

THEOREM 7.37. *The following isomorphism of algebras holds:*

$$H^*(\mathcal{Z}_P/H) \cong \operatorname{Tor}_{\mathbf{k}[t_1, \dots, t_{m-r}]}\big(\mathbf{k}(P), \mathbf{k}\big),$$

where the $\mathbf{k}[t_1, \dots, t_{m-r}]$*-module structure on* $\mathbf{k}(P) = \mathbf{k}[v_1, \dots, v_m]/\mathcal{I}_P$ *is given by the map*

$$\begin{array}{rcl} k[t_1, \dots, t_{m-r}] & \to & k[v_1, \dots, v_m] \\ t_i & \to & t_{i1}v_1 + \dots + t_{im}v_m. \end{array}$$

REMARK. Theorem 7.37 reduces to Theorem 7.6 in the case $r = 0$ and to Example 7.5 in the case $r = m - n$.

PROOF OF THEOREM 7.37. The inclusion $T^r \cong H \hookrightarrow T^m$ defines the map $h : BT^r \to BT^m$ of the classifying spaces. Let us consider the commutative square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B_TP \arrow[d, "p"] \\ BT^r \arrow[r, "h"] & BT^m, \end{tikzcd}$$

where the left vertical arrow is the pullback along h. The space E is homotopy equivalent to the quotient $\mathcal{Z}_P/H$. Hence, the Eilenberg–Moore spectral sequence of the above square converges to the cohomology of $\mathcal{Z}_P/H$. Its E_2-term is

$$E_2 = \operatorname{Tor}_{\mathbf{k}[v_1, \dots, v_m]}\big(\mathbf{k}(P), \mathbf{k}[w_1, \dots, w_r]\big),$$

where the $\mathbf{k}[v_1, \dots, v_m]$-module structure in $\mathbf{k}[w_1, \dots, w_r]$ is defined by the matrix S, i.e. by the map $v_i \to s_{i1}w_1 + \ldots + s_{ir}w_r$. In the same way as in the proof of Theorem 7.6 we show that the spectral sequence collapses at the E_2 term and the following isomorphism of algebras holds:

$$H^*(\mathcal{Z}_P/H) = \operatorname{Tor}_{\mathbf{k}[v_1, \dots, v_m]}\big(\mathbf{k}(P), \mathbf{k}[w_1, \dots, w_r]\big). \tag{7.23}$$

Now put $\Lambda = \mathbf{k}[v_1, \dots, v_m]$, $? = \mathbf{k}[t_1, \dots, t_{m-r}]$, $A = \mathbf{k}[w_1, \dots, w_r]$ and $C = \mathbf{k}(P)$ in Theorem 3.36. Since Λ is a free ?-module and $\Omega = \Lambda//? \cong \mathbf{k}[w_1, \dots, w_r]$, a spectral sequence $\{\widetilde{E}_s, \widetilde{d}_s\}$ arises. Its E_2 term is

$$\widetilde{E}_2 = \operatorname{Tor}_{\mathbf{k}[w_1,\dots,w_r]}\Big(\mathbf{k}[w_1, \dots, w_r], \operatorname{Tor}_{\mathbf{k}[t_1,\dots,t_{m-r}]}\big(\mathbf{k}(P), \mathbf{k}\big)\Big),$$

and it converges to $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(P), \mathbf{k}[w_1, \dots, w_r])$. Obviously, $\mathbf{k}[w_1, \dots, w_r]$ is a free $\mathbf{k}[w_1, \dots, w_r]$-module, so we have

$$\widetilde{E}_2^{p,q} = 0 \ \text{ for } p \neq 0, \quad \widetilde{E}_2^{0,*} = \operatorname{Tor}_{\mathbf{k}[t_1,\dots,t_{m-r}]}\big(\mathbf{k}(P), \mathbf{k}\big).$$

Thus, the spectral sequence collapses at the E_2 term, and the following isomorphism of algebras holds:

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(P), \mathbf{k}[w_1, \dots, w_r]\big) \cong \operatorname{Tor}_{\mathbf{k}[t_1,\dots,t_{m-r}]}\big(\mathbf{k}(P), \mathbf{k}\big),$$

which together with (7.23) concludes the proof. ☐

COROLLARY 7.38. *$H^*(\mathcal{Z}_P/H) \cong H\big[\Lambda[u_1, \dots, u_{m-r}] \otimes \mathbf{k}(P^n), d\big]$, where $du_i = (t_{i1}v_1 + \ldots + t_{im}v_m)$, $dv_i = 0$, $\operatorname{bideg} v_i = (0, 2)$, $\operatorname{bideg} u_i = (-1, 2)$.*

EXAMPLE 7.39. Let $H = S_d$ is the diagonal subgroup. Then the matrix S is a column of m units. By Theorem 7.37,

$$H^*(\mathcal{Z}_P/S_d) \cong \operatorname{Tor}_{\mathbf{k}[t_1,\dots,t_{m-1}]}\big(\mathbf{k}(P), \mathbf{k}\big), \tag{7.24}$$

where the $\mathbf{k}[t_1, \dots, t_{m-1}]$-module structure in $\mathbf{k}(P) = \mathbf{k}[v_1, \dots, v_m]/I$ is defined by

$$t_i \longrightarrow v_i - v_m, \quad i = 1, \dots, m-1.$$

Suppose that the S^1-bundle $\mathcal{Z}_P \to \mathcal{Z}_P/S_d$ is classified by a map $c : \mathcal{Z}_P/S_d \to BT^1 \cong \mathbb{C}P^\infty$. Since $H^*(\mathbb{C}P^\infty) \cong \mathbf{k}[w]$, the element $c^*(w) \in H^2(\mathcal{Z}_P/S_d)$ is defined.

LEMMA 7.40. *P^n is q-neighborly if and only if $(c^*(w))^q \neq 0$.*

PROOF. The map c^* takes the cohomology ring $H^*(BT^1) \cong \mathbf{k}[w]$ to the subalgebra

$$\mathbf{k}(P) \otimes_{\mathbf{k}[t_1,\dots,t_{m-1}]} \mathbf{k} = \operatorname{Tor}^0_{\mathbf{k}[t_1,\dots,t_{m-1}]}\big(\mathbf{k}(P), \mathbf{k}\big) \subset H^*(\mathcal{Z}_P/H).$$

This subalgebra is isomorphic to the quotient $\mathbf{k}(P)/(v_1 = \cdots = v_m)$. Now the assertion follows from the fact that a polytope P^n is q-neighborly if and only if the ideal $\mathcal{I}_P$ does not contain monomials of degree $< q + 1$. ☐

7.6. Bigraded Poincaré duality and Dehn–Sommerville equations

Here we assume that K^{n-1} is a triangulated manifold. In this case the corresponding moment-angle complex $\mathcal{Z}_K$ is not a manifold in general, however, its singularities can be easily treated. Indeed, the cubical complex $\operatorname{cc}(K)$ (Construction 4.9) is homeomorphic to $|\operatorname{cone}(K)|$ and the vertex of the cone is $p = (1, \dots, 1) \in \operatorname{cc}(K) \subset I^m$. Let $U_\varepsilon(p) \subset \operatorname{cc}(K)$ be a small neighborhood of p in $\operatorname{cc}(K)$. The closure of $U_\varepsilon(p)$ is also homeomorphic to $|\operatorname{cone}(K)|$. It follows from the definition of $\mathcal{Z}_K$ (see (6.3)) that $U_\varepsilon(\mathcal{Z}_\varnothing) := \rho^{-1}(U_\varepsilon(p)) \subset \mathcal{Z}_K$ is a small invariant neighborhood of the torus $\mathcal{Z}_\varnothing = \rho^{-1}(p) \cong T^m$ in $\mathcal{Z}_K$. For small ε the closure of $U_\varepsilon(\mathcal{Z}_\varnothing)$ is homeomorphic to $|\operatorname{cone}(K)| \times T^m$. Removing $U_\varepsilon(\mathcal{Z}_\varnothing)$ from $\mathcal{Z}_K$ we obtain a manifold with boundary, which we denote W_K. Thus, we have

$$W_K = \mathcal{Z}_K \setminus U_\varepsilon(\mathcal{Z}_\varnothing), \quad \partial W_K \cong |K| \times T^m.$$

Note that since $U_\varepsilon(\mathcal{Z}_\varnothing)$ is a T^m-stable subset, the torus T^m acts on W_K.

THEOREM 7.41. *The manifold (with boundary) W_K is equivariantly homotopy equivalent to the moment-angle complex $\mathcal{W}_K$ (see (6.3)). Also, there is a canonical relative homeomorphism of pairs $(W_K, \partial W_K) \to (\mathcal{Z}_K, \mathcal{Z}_\varnothing)$.*

PROOF. To prove the first assertion we construct homotopy equivalence $\mathrm{cc}(K) \setminus U_\varepsilon(p) \to \mathrm{cub}(K)$ as it is shown on Figure 7.2. This map is covered by an equivariant homotopy equivalence $W_K = \mathcal{Z}_K \setminus U_\varepsilon(\mathcal{Z}_\varnothing) \to \mathcal{W}_K$. The second assertion follows easily from the definition of W_K. □

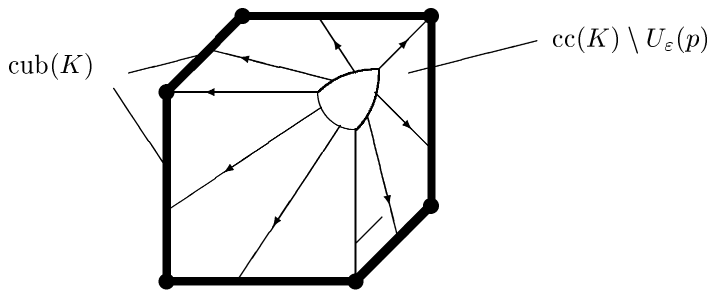


FIGURE 7.2. Homotopy equivalence $\mathrm{cc}(K) \setminus U_\varepsilon(p) \to \mathrm{cub}(K)$.

By Lemma 6.15, the moment-angle complex $\mathcal{W}_K \subset (D^2)^m$ has a cellular structure with 5 different cell types D_i, I_i, 0_i, T_i, 1_i, $i = 1, \dots, m$ (see Figure 6.1). The homology of $\mathcal{W}_K$ (and therefore of W_K) can be calculated from the corresponding cellular chain complex, which we denote $[\mathcal{C}_*(\mathcal{W}_K), \partial_c]$. Although $\mathcal{W}_K$ has more types of cells than $\mathcal{Z}_K$ (5 instead of 3), its cellular chain complex $[\mathcal{C}_*(\mathcal{W}_K), \partial_c]$ also has a natural *bi*grading. Namely, the following statement holds (compare with (7.4)).

LEMMA 7.42. *Put*

$$\operatorname{bideg} D_i = (0,2), \quad \operatorname{bideg} T_i = (-1,2), \quad \operatorname{bideg} I_i = (1,0),$$
$$\operatorname{bideg} 0_i = \operatorname{bideg} 1_i = (0,0), \quad i = 1, \dots, m. \tag{7.25}$$

This turns the cellular chain complex $[\mathcal{C}_*(\mathcal{W}_K), \partial_c]$ into a bigraded differential module with differential ∂_c adding $(-1,0)$ to bidegree. The original grading of $\mathcal{C}_*(\mathcal{W}_K)$ by the dimension of cells corresponds to the total degree (i.e. the dimension of a cell equals the sum of its two degrees).

PROOF. The only thing we need to check is that the differential ∂_c adds $(-1,0)$ to bidegree. This follows from (7.25) and the following formulae:

$$\partial_c D_i = T_i, \quad \partial_c I_i = 1_i - 0_i, \quad \partial_c T_i = \partial_c 1_i = \partial_c 0_i = 0.$$

□

Unlike the bigraded structure in $\mathcal{C}_*(\mathcal{Z}_K)$, elements of $\mathcal{C}_{*,*}(\mathcal{W}_K)$ may have *positive* first degree (due to the positive first degree of I_i). The differential ∂_c does not change the second degree (as in the case of $\mathcal{Z}_K$), which allows us to split the bigraded complex $\mathcal{C}_{*,*}(\mathcal{W}_K)$ into the sum of complexes $\mathcal{C}_{*,2p}(\mathcal{W}_K)$, $p = 0, \dots, m$.

In the same way as we did this for $\mathcal{Z}_K$ and $(\mathcal{Z}_K, \mathcal{Z}_\varnothing)$ we define

$$b_{q,2p}(\mathcal{W}_K) = \dim H_{q,2p}\bigl[\mathcal{C}_{*,*}(\mathcal{W}_K), \partial_c\bigr], \quad -m \leqslant q \leqslant m,\ 0 \leqslant p \leqslant m; \tag{7.26}$$

$$\chi_p(\mathcal{W}_K) = \sum_{q=-m}^{m} (-1)^q \dim \mathcal{C}_{q,2p}(\mathcal{W}_K) = \sum_{q=-m}^{m} (-1)^q b_{q,2p}(\mathcal{W}_K); \tag{7.27}$$

$$\chi(\mathcal{W}_K; t) = \sum_{p=0}^{m} \chi_p(\mathcal{W}_K) t^{2p};$$

(note that q above may be negative).

The following theorem gives a formula for the generating polynomial $\chi(\mathcal{W}_K; t)$ and is analogous to theorems 7.15 and 7.17.

THEOREM 7.43. *For any simplicial complex K^{n-1} with m vertices it holds that*

$$\begin{aligned}\chi(\mathcal{W}_K; t) &= (1-t^2)^{m-n}(h_0 + h_1 t^2 + \cdots + h_n t^{2n}) + \bigl(\chi(K) - 1\bigr)(1-t^2)^m \\ &= (1-t^2)^{m-n}(h_0 + h_1 t^2 + \cdots + h_n t^{2n}) + (-1)^{n-1} h_n (1-t^2)^m,\end{aligned}$$

where $\chi(K) = f_0 - f_1 + \ldots + (-1)^{n-1} f_{n-1} = 1 + (-1)^{n-1} h_n$ is the Euler characteristic of K.

PROOF. By the definition of $\mathcal{W}_K$ (see (6.3)), the vector $\mathcal{R} \in \{D, I, 0, T, 1\}^m$ (see section 6.3) represents a cell of $\mathcal{W}_K$ if and only if the following two conditions are satisfied:

(a) The set $\mathcal{R}_D \cup \mathcal{R}_I \cup \mathcal{R}_0$ is a simplex of K^{n-1}.
(b) $|\mathcal{R}_0| \geqslant 1$.

Let $c_{ijlpq}(\mathcal{W}_K)$ denote the number of cells $\mathcal{R} \subset \mathcal{W}_K$ with $|\mathcal{R}_D| = i$, $|\mathcal{R}_I| = j$, $|\mathcal{R}_0| = l$, $|\mathcal{R}_T| = p$, $|\mathcal{R}_1| = q$, $i + j + l + p + q = m$. It follows that

$$c_{ijlpq}(\mathcal{W}_K) = f_{i+j+l-1}\tbinom{i+j+l}{i}\tbinom{j+l}{l}\tbinom{m-i-j-l}{p}, \tag{7.28}$$

where $(f_0, \ldots, f_{n-1})$ is the f-vector of K (we also assume $f_{-1} = 1$ and $f_k = 0$ for $k < -1$ or $k > n-1$). By (7.25),

$$\operatorname{bideg} \mathcal{R} = \bigl(|\mathcal{R}_I| - |\mathcal{R}_T|, 2(|\mathcal{R}_D| + |\mathcal{R}_T|)\bigr) = \bigl(j - p, 2(i+p)\bigr).$$

Now we calculate $\chi_r(\mathcal{W}_K)$ using (7.27) and (7.28):

$$\chi_r(\mathcal{W}_K) = \sum_{\substack{i,j,l,p \\ i+p=r, l\geqslant 1}} (-1)^{j-p} f_{i+j+l-1}\tbinom{i+j+l}{i}\tbinom{j+l}{l}\tbinom{m-i-j-l}{p}.$$

Substituting $s = i + j + l$ above we obtain

$$\begin{aligned}\chi_r(\mathcal{W}_K) &= \sum_{\substack{l,s,p \\ l\geqslant 1}} (-1)^{s-r-l} f_{s-1}\tbinom{s}{r-p}\tbinom{s-r+p}{l}\tbinom{m-s}{p} \\ &= \sum_{s,p}\Bigl((-1)^{s-r} f_{s-1}\tbinom{s}{r-p}\tbinom{m-s}{p}\sum_{l\geqslant 1}(-1)^l\tbinom{s-r+p}{l}\Bigr)\end{aligned}$$

Since

$$\sum_{l\geqslant 1}(-1)^l\tbinom{s-r+p}{l} = \begin{cases} -1, & s > r-p, \\ \ \ 0, & s \leqslant r-p, \end{cases}$$

we get

$$\begin{aligned}\chi_r(\mathcal{W}_K) &= -\sum_{\substack{s,p\\ s>r-p}}(-1)^{s-r}f_{s-1}\tbinom{s}{r-p}\tbinom{m-s}{p}\\ &= -\sum_{s,p}(-1)^{r-s}f_{s-1}\tbinom{s}{r-p}\tbinom{m-s}{p}+\sum_{s}(-1)^{r-s}f_{s-1}\tbinom{m-s}{r-s}.\end{aligned}$$

The second sum in the above formula is exactly $\chi_r(\mathcal{Z}_K)$ (see (7.11)). To calculate the first sum we observe that

$$\sum_p \tbinom{s}{r-p}\tbinom{m-s}{p}=\tbinom{m}{r}.$$

This follows from calculating the coefficient of α^r in the two sides of the identity $(1+\alpha)^s(1+\alpha)^{m-s}=(1+\alpha)^m$. Hence,

$$\chi_r(\mathcal{W}_K)=-\sum_s(-1)^{r-s}f_{s-1}\tbinom{m}{r}+\chi_r(\mathcal{Z}_K)=(-1)^r\tbinom{m}{r}\bigl(\chi(K)-1\bigr)+\chi_r(\mathcal{Z}_K),$$

since $-\sum_s(-1)^sf_{s-1}=\chi(K)-1$. Finally, using (7.10), we calculate

$$\begin{aligned}\chi(\mathcal{W}_K;t)&=\sum_{r=0}^m\chi_r(\mathcal{W}_K)t^{2r}=\sum_{r=0}^m(-1)^r\tbinom{m}{r}\bigl(\chi(K)-1\bigr)t^{2r}+\sum_{r=0}^m\chi_r(\mathcal{Z}_K)t^{2r}\\ &=\bigl(\chi(K)-1\bigr)(1-t^2)^m+(1-t^2)^{m-n}(h_0+h_1t^2+\dots+h_nt^{2n}).\end{aligned}$$

□

Suppose that K is an orientable triangulated manifold. It is easy to see that then W_K is also orientable. Hence, there are relative Poincaré duality isomorphisms:

$$H_k(W_K)\cong H^{m+n-k}(W_K,\partial W_K),\quad k=0,\dots,m. \tag{7.29}$$

THEOREM 7.44 (Dehn–Sommerville equations for triangulated manifolds). *The following relations hold for the h-vector $(h_0,h_1,\dots,h_n)$ of any triangulated manifold K^{n-1}:*

$$h_{n-i}-h_i=(-1)^i\bigl(\chi(K^{n-1})-\chi(S^{n-1})\bigr)\tbinom{n}{i},\quad i=0,1,\dots,n,$$

where $\chi(S^{n-1})=1+(-1)^{n-1}$ is the Euler characteristic of an $(n-1)$-sphere.

PROOF. Suppose first that K is orientable. By Theorem 7.41, $H_k(W_K)=H_k(\mathcal{W}_K)$ and $H^{m+n-k}(W_K,\partial_cW_K)=H^{m+n-k}(\mathcal{Z}_K,\mathcal{Z}_\varnothing)$. Moreover, it can be seen in the same way as in Corollary 7.19 that relative Poincaré duality isomorphisms (7.29) regard the bigraded structures in the (co)homology of $\mathcal{W}_K$ and $(\mathcal{Z}_K,\mathcal{Z}_\varnothing)$. Hence,

$$\begin{aligned}b_{-q,2p}(\mathcal{W}_K)&=b_{-(m-n)+q,2(m-p)}(\mathcal{Z}_K,\mathcal{Z}_\varnothing),\\ \chi_p(\mathcal{W}_K)&=(-1)^{m-n}\chi_{m-p}(\mathcal{Z}_K,\mathcal{Z}_\varnothing),\\ \chi(\mathcal{W}_K;t)&=(-1)^{m-n}t^{2m}\chi(\mathcal{Z}_K,\mathcal{Z}_\varnothing;\tfrac{1}{t}).\end{aligned} \tag{7.30}$$

Using (7.16), we calculate

$$
\begin{aligned}
(-1)^{m-n}&t^{2m}\chi(\mathcal{Z}_K,\mathcal{Z}_\varnothing;\tfrac{1}{t})\\
&=(-1)^{m-n}t^{2m}(1-t^{-2})^{m-n}(h_0+h_1t^{-2}+\dots+h_nt^{-2n})\\
&\qquad\qquad-(-1)^{m-n}t^{2m}(1-t^{-2})^{m}\\
&\qquad=(1-t^2)^{m-n}(h_0t^{2n}+h_1t^{2n-2}+\dots+h_n)+(-1)^{n-1}(1-t^2)^m.
\end{aligned}
$$

Substituting the formula for $\chi(\mathcal{W}_K;t)$ from Theorem 7.43 and the above expression into (7.30) we obtain

$$
\begin{aligned}
(1-t^2)^{m-n}&(h_0+h_1t^2+\dots+h_nt^{2n})+\bigl(\chi(K)-1\bigr)(1-t^2)^m\\
&=(1-t^2)^{m-n}(h_0t^{2n}+h_1t^{2n-2}+\dots+h_n)+(-1)^{n-1}(1-t^2)^m.
\end{aligned}
$$

Calculating the coefficient of t^{2i} in both sides after dividing the above identity by $(1-t^2)^{m-n}$, we get $h_{n-i}-h_i=(-1)^i(\chi(K^{n-1})-\chi(S^{n-1}))\binom{n}{i}$, as required.

Now suppose that K is non-orientable. Then there exist an orientable triangulated manifold L of the same dimension and a 2-sheet covering $L\to K$. Then we obviously have $f_i(L)=2f_i(K)$, $i=0,1,\dots,n-1$. It follows from (1.7) that

$$
\sum_{i=0}^{n}h_i(L)t^{n-i}-(t-1)^n=2\Bigl(\sum_{i=0}^{n}h_i(K)t^{n-i}-(t-1)^n\Bigr).
$$

Hence,

$$
h_i(L)=2h_i(K)-(-1)^i\tbinom{n}{i},\quad i=0,1,\dots,n.
$$

Since L is orientable, we have $h_{n-i}(L)-h_i(L)=(-1)^i(\chi(L)-\chi(S^{n-1}))\binom{n}{i}$. Therefore,

$$
2\bigl(h_{n-i}(K)-h_i(K)\bigr)-(-1)^{n-i}\tbinom{n}{n-i}+(-1)^i\tbinom{n}{i}=(-1)^i\bigl(\chi(L)-\chi(S^{n-1})\bigr)\tbinom{n}{i}.
$$

Since $\chi(L)=2\chi(K)$, we get

$$
\begin{aligned}
2\bigl(h_{n-i}(K)-h_i(K)\bigr)&=(-1)^i\bigl(2\chi(K)-\chi(S^{n-1})+(-1)^n-1\bigr)\\
&=2\cdot(-1)^i\bigl(\chi(K)-\chi(S^{n-1})\bigr),
\end{aligned}
$$

as required. □

If $|K|=S^{n-1}$ or $n-1$ is odd then Corollary 7.44 gives the classical equations $h_{n-i}=h_i$.

COROLLARY 7.45. *Suppose K^{n-1} is a triangulated manifold with the h-vector $(h_0,\dots,h_n)$. Then*

$$
h_{n-i}-h_i=(-1)^i(h_n-1)\tbinom{n}{i},\quad i=0,1,\dots,n.
$$

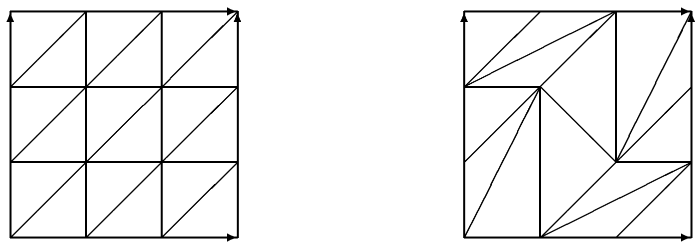
PROOF. Since $\chi(K^{n-1})=1+(-1)^{n-1}h_n$ and $\chi(S^{n-1})=1+(-1)^{n-1}$, we have

$$
\chi(K^{n-1})-\chi(S^{n-1})=(-1)^{n-1}(h_n-1)=(h_n-1)
$$

(the coefficient $(-1)^{n-1}$ can be dropped since for odd $n-1$ the left hand side is zero). □

COROLLARY 7.46. *For any $(n-1)$-dimensional triangulated manifold the numbers $h_{n-i}-h_i$, $i=0,1,\dots,n$, are homotopy invariants. In particular, they do not depend on a triangulation.*

In the case of PL-manifolds the topological invariance of numbers $h_{n-i} - h_i$ was observed by Pachner in [**110**, (7.11)].



(a) $\boldsymbol{f} = (9, 27, 18)$, $\boldsymbol{h} = (1, 6, 12, -1)$ (b) $\boldsymbol{f} = (7, 21, 14)$, $\boldsymbol{h} = (1, 4, 10, -1)$

FIGURE 7.3. "Symmetric" and "minimal" triangulation of T^2

EXAMPLE 7.47 (Triangulations of 2-manifolds). Consider triangulations of the 2-torus T^2. We have $n = 3$, $\chi(T^2) = 0$. From $\chi(K^{n-1}) = 1 + (-1)^{n-1} h_n$ we deduce $h_3 = -1$. Corollary 7.44 gives

$$h_3 - h_0 = -2, \quad h_2 - h_1 = 6.$$

For instance, the triangulation on Figure 7.3 (a) has $f_0 = 9$ vertices, $f_1 = 27$ edges and $f_2 = 18$ triangles. (Note that this triangulation is the canonical triangulation of $\partial\Delta^2 \times \partial\Delta^2$, as described in Construction 2.11.) The corresponding h-vector is $(1, 6, 12, -1)$.

On the other hand, it is well known that a triangulation of T^2 with only 7 vertices can be achieved, see Figure 7.3 (b). Note that this triangulation is neighborly, i.e. its 1-skeleton is a complete graph on 7 vertices. It turns out that no triangulation of T^2 with smaller number of vertices exists.

Suppose now K^2 is a 2-dimensional triangulated manifold with m vertices. Let $\chi = \chi(K^2)$ be its Euler characteristic. Using Corollary 7.44 we may express the f-vector of K^2 via χ and m, namely,

$$\boldsymbol{f}(K^2) = \big(m, 3(m - \chi), 2(m - \chi)\big).$$

Since the number of edges in a triangulation does not exceed the number of pairs of vertices, we get the inequality

$$6(m - \chi) \leqslant m(m - 1), \tag{7.31}$$

from which a lower bound for the number of vertices in a triangulation of K^2 can be deduced. For instance, in the case of torus T^2 we have $\chi = 0$ and (7.31) gives $m \geqslant 7$. Note that a minimal triangulation of K^2 is neighborly (has a complete graph as its 1-skeleton) only if (7.31) turns to equality. We have seen that this is the case for T^2 ($\chi = 0$, $m = 7$). Other examples are the sphere S^2 ($\chi = 2$, $m = 4$) and the real projective plane $\mathbb{R}P^2$ ($\chi = 1$, $m = 6$). A neighborly triangulation of $\mathbb{R}P^2$ is shown in Figure 7.4. However, for most values of χ there is no m which makes (7.31) an equality. For example, minimal triangulations of *orientable* surfaces of genus 1 to 5 are not neighborly. A genus 6 surface (having $\chi = -10$ and $m = 12$) has neighbourly triangulations (which are automatically minimal). These triangulations are important in the problem of polyhedral embeddability of

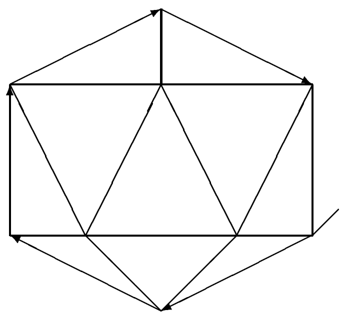


FIGURE 7.4. Neighborly triangulation of $\mathbb{R}P^2$, with $\boldsymbol{f} = (6, 15, 10)$.

orientable triangulated surfaces in $\mathbb{R}^3$ ("polyhedral" here means with flat triangles and no self-intersections). It was shown in [**4**] that there are in total 59 different neighborly triangulations of a genus 6 surface with 12 vertices. Later, using an algorithm for generating oriented matroids, Bokowski and Guedes de Oliveira proved in [**23**] that one of these triangulations cannot be embedded into $\mathbb{R}^3$ with flat triangles. Furthermore, they proved that one triangle can be removed from the triangulation while retaining non-embeddability, so an arbitrary number of handles can be attached at this triangle to get a non-embeddable triangulated surface of any genus $\geqslant 6$. A number of results on minimal triangulations were obtained by Lutz in [**91**] using his computer program BISTELLAR (which we already mentioned in section 2.3).

CHAPTER 8

Cohomology rings of subspace arrangement complements

8.1. General arrangements and their complements

DEFINITION 8.1. An *arrangement* is a finite set $\mathcal{A} = \{L_1, \dots, L_r\}$ of affine subspaces in some affine space (either real or complex). An arrangement $\mathcal{A}$ is called a *subspace arrangement* (or *central arrangement*) if all its subspaces are linear (i.e., contain 0). Given an arrangement $\mathcal{A} = \{L_1, \dots, L_r\}$ in $\mathbb{C}^m$, define its *support* (or *union*) $|\mathcal{A}|$ as

$$|\mathcal{A}| := \bigcup_{i=1}^{r} L_i \subset \mathbb{C}^m,$$

and its *complement* $U(\mathcal{A})$ as

$$U(\mathcal{A}) := \mathbb{C}^m \setminus |\mathcal{A}|,$$

and similarly for arrangements in $\mathbb{R}^m$.

Let $\mathcal{A} = \{L_1, \dots, L_r\}$ be an arrangement. The intersections

$$v = L_{i_1} \cap \dots \cap L_{i_k}$$

form a poset $(\mathcal{L}, <)$ with respect to the inclusion, called the *intersection poset* of the arrangement. The poset $\mathcal{L}$ is assumed to have a unique maximal element T corresponding to the ambient space of the arrangement. The *rank function* d on $\mathcal{L}$ is defined by $d(v) = \dim v$. The complex $\operatorname{ord}(\mathcal{L})$ (see Example 2.17) is called the *order complex of arrangement* $\mathcal{A}$. Define intervals

$$\mathcal{L}_{(v,w)} = \{x \in \mathcal{L} \,:\, v < x < w\}, \quad \mathcal{L}_{>v} = \{x \in \mathcal{L} \,:\, x > v\}.$$

Arrangements and their complements play a pivotal rôle in many constructions of combinatorics, algebraic and symplectic geometry etc.; they also arise as configuration spaces for different classical mechanical systems. In the study of arrangements it is very important to get a sufficiently detailed description of the topology of complements $U(\mathcal{A})$ (this includes number of connected components, homotopy type, homology groups, cohomology ring, etc.). A host of elegant results in this direction appeared during the last three decades, however, the whole picture is far from being complete. The theory ascends to work of Arnold [**6**], in which the classifying space for the colored braid group is described as the complement of the arrangement of all diagonal hyperplanes $\{z_i = z_j\}$, $1 \leqslant i < j \leqslant n$, in $\mathbb{C}^n$. The latter complement can be thought as the configuration space of n ordered points in $\mathbb{C}$. Its cohomology ring was also calculated in [**6**]. This result was generalized by Brieskorn [**29**] and motivated the further development of the theory of *complex hyperplane arrangements* (i.e. arrangements of codimension-one complex affine subspaces). One of the main results here is the following.

THEOREM 8.2 ([**6**], [**29**]). *Let $\mathcal{A} = \{L_1, \dots, L_r\}$ be an arrangement of complex hyperplanes in $\mathbb{C}^m$, where the hyperplane L_j is the zero set of linear function l_j, $j = 1, \dots, r$. Then the integer cohomology algebra of the complement $\mathbb{C}^m \setminus |\mathcal{A}|$ is isomorphic to the algebra generated by closed differential 1-forms $\frac{1}{2\pi i}\frac{dl_j}{l_j}$.*

Relations between the forms $\omega_j = \frac{1}{2\pi i}\frac{dl_j}{l_j}$, $j = 1, \dots, r$, were explicitly described by Orlik and Solomon [**107**]. We give their result in the central case, i.e. when all the hyperplanes are vector subspaces. Then there is one relation

$$\sum_{k=1}^{p} (-1)^k \omega_{j_1} \wedge \dots \wedge \widehat{\omega_{j_k}} \wedge \dots \wedge \omega_{j_p} = 0,$$

for any minimal subset $\{L_{j_1}, \dots, L_{j_p}\}$ of hyperplanes of $\mathcal{A}$ such that $\operatorname{codim} L_{j_1} \cap \dots \cap L_{j_p} = p - 1$ (such subsets are called *circuits* of $\mathcal{L}$).

EXAMPLE 8.3. Let $\mathcal{A}$ be the arrangement of diagonal hyperplanes $\{z_j = z_k\}$, $1 \leqslant j < k \leqslant n$, in $\mathbb{C}^n$. Then we have the forms $\omega_{jk} = \frac{1}{2\pi i}\frac{d(z_j - z_k)}{z_j - z_k}$, satisfying the identities

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0,$$

known as the *Arnold relations*.

The theory of complex hyperplane arrangements is probably the most well understood part of the whole study. Several surveys and monographs are available; we mention just [**108**], [**136**] and [**143**], where further references can be found. Relationships between *real* hyperplane arrangements, polytopes and *oriented matroids* are discussed in [**145**, Lecture 7]. Another interesting related class of arrangements is known as 2-*arrangements* in $\mathbb{R}^{2n}$. A 2-arrangement is an arrangement of real subspaces of codimension 2 with even-dimension intersections. In particular, any complex hyperplane arrangement is a 2-arrangement. The relationships between 2-arrangements and complex hyperplane arrangements are studied in [**144**].

In the case of general arrangement $\mathcal{A}$, the celebrated Goresky–MacPherson theorem [**66**, Part III] expresses the cohomology groups $H^i(U(\mathcal{A}))$ (without ring structure) as a sum of homology groups of subcomplexes of a certain simplicial complex.

THEOREM 8.4 (Goresky and MacPherson [**66**, Part III]). *The (co)homology of a subspace arrangement complement $U(\mathcal{A})$ in $\mathbb{R}^n$: is given by*

$$H_i\big(U(\mathcal{A}); \mathbb{Z}\big) = \bigoplus_{v \in \mathcal{P}} H^{n-d(v)-i-1}\big(\operatorname{ord}(\mathcal{L}_{>v}), \operatorname{ord}(\mathcal{L}_{(v,T)}); \mathbb{Z}\big);$$
$$H^i\big(U(\mathcal{A}); \mathbb{Z}\big) = \bigoplus_{v \in \mathcal{P}} H_{n-d(v)-i-1}\big(\operatorname{ord}(\mathcal{L}_{>v}), \operatorname{ord}(\mathcal{L}_{(v,T)}); \mathbb{Z}\big),$$

(see Definition 8.1*), where we assume that $H_{-1}(\varnothing, \varnothing) = H^{-1}(\varnothing, \varnothing) = \mathbb{Z}$.*

The original proof of this theorem used the *stratified Morse theory*, developed in [**66**].

REMARK. Observing that $\operatorname{ord}(\mathcal{L}_{>v})$ is the cone over $\operatorname{ord}(\mathcal{L}_{(v,T)})$, we may rewrite the formula from Theorem 8.4 as

$$\widetilde{H}_i\big(U(\mathcal{A}); \mathbb{Z}\big) = \bigoplus_{v \in \mathcal{P}} \widetilde{H}^{n-d(v)-i-2}\big(\operatorname{ord}(\mathcal{L}_{(v,T)}); \mathbb{Z}\big), \tag{8.1}$$

and similarly for the cohomology.

REMARK. The homology groups of a complex arrangement in $\mathbb{C}^n$ can be calculated by regarding it as a real arrangement in $\mathbb{R}^{2n}$.

A comprehensive survey of general arrangements is given in [**20**]. Monograph [**136**] gives an alternative approach to homology and homotopy computations, via the *Anderson spectral sequence*. A method of describing the homotopy types of subspace arrangements, using diagrams of spaces over *poset categories*, was proposed in [**146**]. Using this method, a new, elementary, proof of Goresky–MacPherson Theorem 8.4 was found there. The approach of [**146**] was developed later in [**139**] by incorporating homotopy colimits techniques.

The cohomology *rings* of arrangement complements are much more subtle. In general, the integer cohomology ring of $U(\mathcal{A})$ *is not* determined by the intersection poset $\mathcal{L}$ (this is false even for 2-arrangements, as shown in [**144**]). An approach to calculating the cohomology algebra of the complement $U(\mathcal{A})$, based on the results of De Concini and Procesi [**51**], was proposed by Yuzvinsky in [**142**]. Recently, the combinatorial description of the product of any two cohomology classes in a complex subspace arrangement complement $U(\mathcal{A})$, conjectured by Yuzvinsky, has been obtained independently in [**53**] and [**55**]. This description is given in terms of the intersection poset $\mathcal{L}(\mathcal{A})$, the dimension function, and additional orientation data.

8.2. Coordinate subspace arrangements and the cohomology of $\mathcal{Z}_K$.

An arrangement $\mathcal{A} = \{L_1, \dots, L_r\}$ is called *coordinate* if every L_i, $i = 1, \dots, r$, is a coordinate subspace. In this section we apply the results of chapter 7 to cohomology algebras of complex coordinate subspace arrangement complements. The case of real coordinate arrangements is also discussed at the end of this section.

A coordinate subspace of $\mathbb{C}^m$ can be written as

$$L_\sigma = \{(z_1, \dots, z_m) \in \mathbb{C}^m \ : \ z_{i_1} = \dots = z_{i_k} = 0\}, \tag{8.2}$$

where $\sigma = \{i_1, \dots, i_k\}$ is a subset of $[m]$. Obviously, $\dim L_\sigma = m - |\sigma|$.

CONSTRUCTION 8.5. For each simplicial complex K on the set $[m]$ define the complex coordinate subspace arrangement $\mathcal{CA}(K)$ by

$$\mathcal{CA}(K) = \{L_\sigma \ : \ \sigma \notin K\}.$$

Denote the complement of $\mathcal{CA}(K)$ by $U(K)$, that is

$$U(K) = \mathbb{C}^m \setminus \bigcup_{\sigma \notin K} L_\sigma. \tag{8.3}$$

Note that if $K' \subset K$ is a subcomplex, then $U(K') \subset U(K)$.

PROPOSITION 8.6. *The assignment $K \mapsto U(K)$ defines a one-to-one order-preserving correspondence between the set of simplicial complexes on $[m]$ and the set of coordinate subspace arrangement complements in $\mathbb{C}^m$ (or $\mathbb{R}^m$).*

PROOF. Suppose $\mathcal{CA}$ is a coordinate subspace arrangement in $\mathbb{C}^m$. Define

$$K(\mathcal{CA}) := \{\sigma \subset [m] \ : \ L_\sigma \not\subset |\mathcal{CA}|\}. \tag{8.4}$$

Obviously, $K(\mathcal{CA})$ is a simplicial complex. By the definition, $K(\mathcal{CA})$ depends only on $|\mathcal{CA}|$ (or $U(\mathcal{CA})$) and $U(K(\mathcal{CA})) = U(\mathcal{CA})$, whence the proposition follows. $\square$

If $\mathcal{CA}$ contains a hyperplane, say $\{z_i = 0\}$, then its complement $U(\mathcal{CA})$ is factored as $U(\mathcal{CA}_0) \times \mathbb{C}^*$, where $\mathcal{CA}_0$ is a coordinate subspace arrangement in the hyperplane $\{z_i = 0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Thus, for any coordinate subspace arrangement $\mathcal{CA}$, the complement $U(\mathcal{CA})$ decomposes as

$$U(\mathcal{CA}) = U(\mathcal{CA}') \times (\mathbb{C}^*)^k,$$

were $\mathcal{CA}'$ is a coordinate arrangement in $\mathbb{C}^{m-k}$ that does not contain hyperplanes. On the other hand, (8.4) shows that $\mathcal{CA}$ contains the hyperplane $\{z_i = 0\}$ if and only if $\{i\}$ is not a vertex of $K(\mathcal{CA})$. It follows that $U(K)$ is the complement of a coordinate arrangement without hyperplanes if and only if the vertex set of K is the whole $[m]$. Keeping in mind these remarks, we restrict our attention to coordinate subspace arrangements without hyperplanes and simplicial complexes on the vertex set $[m]$.

REMARK. In the notations of Construction 6.38 we have $U(K) = K_\bullet(\mathbb{C}, \mathbb{C}^*)$.

EXAMPLE 8.7. 1. If $K = \Delta^{m-1}$ then $U(K) = \mathbb{C}^m$.
2. If $K = \partial\Delta^{m-1}$ (boundary of simplex) then $U(K) = \mathbb{C}^m \setminus \{0\}$.
3. If K is a disjoint union of m vertices, then $U(K)$ is the complement in $\mathbb{C}^m$ of the set of all codimension-two coordinate subspaces $z_i = z_j = 0,\ 1 \leqslant i < j \leqslant m$.

The diagonal action of algebraic torus $(\mathbb{C}^*)^m$ on $\mathbb{C}^m$ descends to $U(K)$. In particular, there is the standard action of T^m on $U(K)$. The quotient $U(K)/T^m$ can be identified with $U(K) \cap \mathbb{R}^m_+$, where $\mathbb{R}^m_+$ is regarded as a subset of $\mathbb{C}^m$.

LEMMA 8.8. *$\mathrm{cc}(K) \subset U(K) \cap \mathbb{R}^m_+$ and $\mathcal{Z}_K \subset U(K)$ (see Construction 4.9 and (6.3)).*

PROOF. Take $y = (y_1, \dots, y_m) \in \mathrm{cc}(K)$. Let $\sigma = \{i_1, \dots, i_k\}$ be the set of zero coordinates of y, i.e. the maximal subset of $[m]$ such that $y \in L_\sigma \cap \mathbb{R}^n_+$. Then it follows from the definition of $\mathrm{cc}(K)$ (see (4.4)) that σ is a simplex of K. Hence, $L_\sigma \notin \mathcal{CA}(K)$ and $y \in U(K)$, which implies the first statement. The second assertion follows from the fact that $\mathrm{cc}(K)$ is the quotient of $\mathcal{Z}_K$. □

THEOREM 8.9. *There is an equivariant deformation retraction $U(K) \to \mathcal{Z}_K$.*

PROOF. First, we construct a deformation retraction $r : U(K) \cap \mathbb{R}^m_+ \to \mathrm{cc}(K)$. This is done inductively. We start from the boundary complex of an $(m-1)$-simplex and remove simplices of positive dimensions until we obtain K. On each step we construct a deformation retraction, and the composite map will be the required retraction r.

If $K = \partial\Delta^{m-1}$ is the boundary complex of an $(m-1)$-simplex, then $U(K) \cap \mathbb{R}^m_+ = \mathbb{R}^m_+ \setminus \{0\}$. In this case the retraction r is shown on Figure 8.1. Now suppose that K is obtained from K' by removing one $(k-1)$-dimensional simplex $\tau = \{j_1, \dots, j_k\}$, that is $K \cup \tau = K'$. By the inductive hypothesis, we may assume that there is a deformation retraction $r' : U(K') \cap \mathbb{R}^m_+ \to \mathrm{cc}(K')$. Let $a \in \mathbb{R}^m_+$ be the point with coordinates $y_{j_1} = \ldots = y_{j_k} = 0$ and $y_i = 1$ for $i \notin \tau$. Since τ is not a simplex of K, we have $a \notin U(K) \cap \mathbb{R}^m_+$. At the same time, $a \in C_\tau$ (see (4.1)). Hence, we can apply the retraction shown on Figure 8.1 on the face $C_\tau \subset I^m$, with center at a. Denote this retraction by r_τ. Then $r = r_\tau \circ r'$ is the required deformation retraction.

The deformation retraction $r : U(K) \cap \mathbb{R}^m_+ \to \mathrm{cc}(K)$ is covered by an equivariant deformation retraction $U(K) \to \mathcal{Z}_K$, which concludes the proof. □

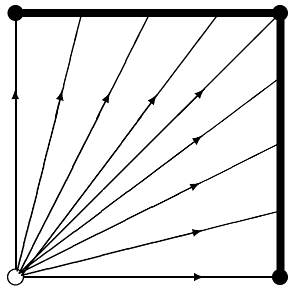


FIGURE 8.1. The retraction $r : U(K) \cap \mathbb{R}^m_+ \to \mathrm{cc}(K)$ for $K = \partial\Delta^{m-1}$.

In the case $K = K_P$ (i.e. K is a polytopal simplicial sphere corresponding to a simple polytope P^n) the deformation retraction $U(K_P) \to \mathcal{Z}_P$ from Theorem 8.9 can be realized as the orbit map for an action of a contractible group. We denote $U(P^n) := U(K_P)$. Set

$$\mathbb{R}^m_> = \{(y_1, \dots, y_m) \in \mathbb{R}^m \ : \ y_i > 0, \ i = 1, \dots, m\} \subset \mathbb{R}^m_+.$$

Then $\mathbb{R}^m_>$ is a group with respect to the multiplication, and it acts on $\mathbb{R}^m$, $\mathbb{C}^m$ and $U(P^n)$ by coordinatewise multiplications. There is the isomorphism $\exp : \mathbb{R}^m \to \mathbb{R}^m_>$ between the additive and the multiplicative groups taking $(y_1, \dots, y_m) \in \mathbb{R}^m$ to $(e^{y_1}, \dots, e^{y_m}) \in \mathbb{R}^m_>$.

Let us consider the $m \times (m-n)$-matrix W introduced in Construction 1.8 for every simple polytope (1.1).

PROPOSITION 8.10. *For any vertex $v = F_{i_1} \cap \dots \cap F_{i_n}$ of P^n the maximal minor $W_{\hat{i}_1 \dots \hat{i}_n}$ of W obtained by deleting n rows $i_1, \dots, i_n$ is non-degenerate:* $\det W_{\hat{i}_1 \dots \hat{i}_n} \neq 0$.

PROOF. If $\det W_{\hat{i}_1 \dots \hat{i}_n} = 0$ then the vectors $\boldsymbol{l}_{i_1}, \dots, \boldsymbol{l}_{i_n}$ (see (1.1)) are linearly dependent, which is impossible. □

The matrix W defines the subgroup

$$R_W = \big\{(e^{w_{11}\tau_1 + \dots + w_{1,m-n}\tau_{m-n}}, \dots, e^{w_{m1}\tau_1 + \dots + w_{m,m-n}\tau_{m-n}})\big\} \subset \mathbb{R}^m_>, \tag{8.5}$$

where $(\tau_1, \dots, \tau_{m-n})$ is running through $\mathbb{R}^{m-n}$. Obviously, $R_W \cong \mathbb{R}^{m-n}_>$.

THEOREM 8.11 ([**33**, Theorem 2.3] and [**38**, §3]). *The subgroup R_W acts freely on $U(P^n) \subset \mathbb{C}^m$. The composition $\mathcal{Z}_P \hookrightarrow U(P^n) \to U(P^n)/R_W$ of the embedding i_e (Lemma* 6.6*) and the orbit map is an equivariant diffeomorphism (with respect to the corresponding T^m-actions).*

Suppose now that P^n is a lattice simple polytope, and let M_P be the corresponding toric variety (Construction 5.4). Along with the real subgroup $R_W \subset \mathbb{R}^m_>$ (8.5) define its complex analogue

$$C_W = \big\{(e^{w_{11}\phi_1 + \dots + w_{1,m-n}\phi_{m-n}}, \dots, e^{w_{m1}\phi_1 + \dots + w_{m,m-n}\phi_{m-n}})\big\} \subset (\mathbb{C}^*)^m,$$

where $(\phi_1, \dots, \phi_{m-n})$ is running through $\mathbb{C}^{m-n}$. Obviously, $C_W \cong (\mathbb{C}^*)^{m-n}$. It is shown in [**45**] (see also [**9**], [**16**]) that C_W acts freely on $U(P^n)$ and the toric variety

M_P can be identified with the orbit space (or *geometric quotient*) $U(P^n)/C_W$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} U(P^n) & \xrightarrow{R_W \cong \mathbb{R}_>^{m-n}} & \mathcal{Z}_P \\ {\scriptstyle C_W \cong (\mathbb{C}^*)^{m-n}}\big\downarrow & & \big\downarrow{\scriptstyle T^{m-n}} \\ M_P & = & M_P. \end{array} \tag{8.6}$$

REMARK. It can be shown [**45**, Theorem 2.1] that *any* toric variety M_Σ corresponding to a fan $\Sigma \subset \mathbb{R}^n$ with m one-dimensional cones can be identified with the universal categorical quotient $U(\mathcal{CA}_\Sigma)/G$, where $U(\mathcal{CA}_\Sigma)$ a certain coordinate arrangement complement (determined by the fan Σ) and $G \cong (\mathbb{C}^*)^{m-n}$. The categorical quotient becomes the geometric quotient if and only if the fan Σ is simplicial. In this case $U(\mathcal{CA}_\Sigma) = U(K_\Sigma)$.

On the other hand, if the projective toric variety M_P is non-singular then M_P is a symplectic manifold of dimension $2n$, and the action of T^n on it is Hamiltonian (see e.g. [**9**] or [**45**, §4]). In this case the diagram (8.6) displays M_P as the result of a *symplectic reduction*. Namely, let $H_W \cong T^{m-n}$ be the maximal compact subgroup in C_W, and $\mu : \mathbb{C}^m \to \mathbb{R}^{m-n}$ the *moment map* for the Hamiltonian action of H_W on $\mathbb{C}^m$. Then for any regular value $a \in \mathbb{R}^{m-n}$ of the map μ there is the following diffeomorphism:

$$\mu^{-1}(a)/H_W \longrightarrow U(P^n)/C_W = M_P$$

(details can be found in [**9**]). In this situation $\mu^{-1}(a)$ is exactly our manifold $\mathcal{Z}_P$. This gives us another interpretation of the manifold $\mathcal{Z}_P$ as the level surface for the moment map (in the case when P^n can be realized as the quotient of a non-singular projective toric variety).

EXAMPLE 8.12. Let $P^n = \Delta^n$ (the n-simplex). Then $m = n+1$, $U(P^n) = \mathbb{C}^{n+1} \setminus \{0\}$. Moreover, $R_W \cong \mathbb{R}_>$, $C_W \cong \mathbb{C}^*$ and $H_W \cong S^1$ are the diagonal subgroups in $\mathbb{R}_>^{n+1}$, $(\mathbb{C}^*)^{n+1}$ and T^{m+1} respectively (see Example 1.9). Hence,

$$\mathcal{Z}_P \cong S^{2n+1} = \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{R}_>, \quad M_P = \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^* = \mathbb{C}P^n.$$

The moment map $\mu : \mathbb{C}^m \to \mathbb{R}$ takes $(z_1, \dots, z_m) \in \mathbb{C}^m$ to $\frac{1}{2}(|z_1|^2 + \ldots + |z_m|^2)$, and for $a \neq 0$ we have $\mu^{-1}(a) \cong S^{2n+1} \cong \mathcal{Z}_K$.

Now we have the following result for the cohomology of subspace arrangement complements.

THEOREM 8.13. *The following isomorphism of graded algebras holds:*

$$\begin{aligned} H^*\big(U(K)\big) &\cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}(K), \mathbf{k}\big) \\ &= H\big[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}(K), d\big]. \end{aligned}$$

PROOF. This follows from Theorems 8.9, 7.6 and 7.7. □

Theorem 8.13 provides an effective way to calculate the cohomology algebra of the complement of any complex coordinate subspace arrangement. The Koszul complex was also used by De Concini and Procesi [**51**] and Yuzvinsky [**142**] for constructing rational models of the cohomology algebra of an arrangement complement. As we see, in the case of coordinate subspace arrangements calculations become shorter and more effective as soon as the Stanley–Reisner ring is brought into the picture.

PROBLEM 8.14. *Calculate the integer cohomology algebra of a coordinate subspace arrangement complement and compare it with the corresponding* Tor-*algebra* $\mathrm{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}(K),\mathbb{Z})$.

EXAMPLE 8.15. Let K be a disjoint union of m vertices. Then $U(K)$ is the complement to the set of all codimension-two coordinate subspaces $z_i = z_j = 0$, $1 \leqslant i < j \leqslant m$, in $\mathbb{C}^m$ (see Example 8.7). The face ring is $\mathbf{k}(K) = \mathbf{k}[v_1,\dots,v_m]/\mathcal{I}_K$, where $\mathcal{I}_K$ is generated by the monomials v_iv_j, $i \neq j$. An easy calculation using Corollary 8.13 shows that the subspace of cocycles in $\mathbf{k}(K) \otimes \Lambda[u_1,\dots,u_m]$ has the basis consisting of monomials $v_{i_1}u_{i_2}u_{i_3}\cdots u_{i_k}$ with $k \geqslant 2$ and $i_p \neq i_q$ for $p \neq q$. Since $\deg(v_{i_1}u_{i_2}u_{i_3}\cdots u_{i_k}) = k+1$, the space of $(k+1)$-dimensional cocycles has dimension $m\binom{m-1}{k-1}$. The space of $(k+1)$-dimensional coboundaries is $\binom{m}{k}$-dimensional (it is spanned by the coboundaries of the form $d(u_{i_1}\cdots u_{i_k})$). Hence,

$$\dim H^0\bigl(U(K)\bigr) = 1, \quad H^1\bigl(U(K)\bigr) = H^2\bigl(U(K)\bigr) = 0,$$
$$\dim H^{k+1}\bigl(U(K)\bigr) = m\tbinom{m-1}{k-1} - \tbinom{m}{k} = (k-1)\tbinom{m}{k}, \quad 2 \leqslant k \leqslant m,$$

and the multiplication in the cohomology is trivial.

In particular, for $m = 3$ we have 6 three-dimensional cohomology classes $[v_iu_j]$, $i \neq j$, subject to 3 relations $[v_iu_j] = [v_ju_i]$, and 3 four-dimensional cohomology classes $[v_1u_2u_3]$, $[v_2u_1u_3]$, $[v_3u_1u_2]$ subject to one relation

$$[v_1u_2u_3] - [v_2u_1u_3] + [v_3u_1u_2] = 0.$$

Hence, $\dim H^3(U(K)) = 3$, $\dim H^4(U(K)) = 2$, and the multiplication is trivial. It can be shown that $U(K)$ in this case has a homotopy type of a wedge of spheres:

$$U(K) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4.$$

EXAMPLE 8.16. Let K be the boundary of an m-gon, $m > 3$. Then

$$U(K) = \mathbb{C}^m \setminus \bigcup_{i-j \neq 0,1 \mod m} \{z_i = z_j = 0\}.$$

By Theorem 8.13, the cohomology ring of $H^*(U(K);\mathbf{k})$ is isomorphic to the ring described in Example 7.22 (note that the multiplication is non-trivial here).

As it is shown in [**65**], in the case of arrangements of *real* coordinate subspaces only *additive* analogue of our Theorem 8.13 holds. Namely, let us consider the polynomial ring $\mathbf{k}[x_1,\dots,x_m]$ with $\deg x_i = 1$, $i = 1,\dots,m$. Then the graded structure in the face ring $\mathbf{k}(K)$ changes accordingly. The Betti numbers of the real coordinate subspace arrangement $U_{\mathbb{R}}(K)$ can be calculated by means of the following result.

THEOREM 8.17 ([**65**, Theorem 3.1]). *The following isomorphism holds:*

$$H^p\bigl(U_{\mathbb{R}}(K)\bigr) \cong \sum_{-i+j=p} \mathrm{Tor}^{-i,j}_{\mathbf{k}[x_1,\dots,x_m]}\bigl(\mathbf{k}(K),\mathbf{k}\bigr) = H^{-i,j}\bigl[\Lambda[u_1,\dots,u_m] \otimes \mathbf{k}(K), d\bigr],$$

where $\operatorname{bideg} u_i = (-1,1)$, $\operatorname{bideg} v_i = (0,1)$, $du_i = x_i$, $dx_i = 0$.

As it was observed in [**65**], there is *no* multiplicative isomorphism analogous to Theorem 8.13 in the case of real arrangements, that is, the algebras $H^*(U_{\mathbb{R}}(K))$ and $\mathrm{Tor}_{\mathbf{k}[x_1,\dots,x_m]}(\mathbf{k}(K),\mathbf{k})$ are not isomorphic in general. The paper [**65**] also contains the formulation of the first multiplicative isomorphism of our Theorem 8.13 for complex coordinate subspace arrangements (see [**65**, Theorem 3.6]), with reference to a paper by Babson and Chan (unpublished).

Up to this point we have used the description of coordinate subspaces by means of equations (see (8.2)). On the other hand, a coordinate subspace can be defined as the linear span of a subset of the standard basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$. This leads to the dual approach to coordinate subspace arrangements, which corresponds to the passage from simplicial complex K to the dual complex $\widehat{K}$ (Example 2.26). Namely, we have

$$\mathcal{CA}(K) = \bigl\{\operatorname{span}\{e_{i_1}, \dots, e_{i_k}\} \,:\, \{i_1, \dots, i_k\} \in \widehat{K}\bigr\}$$

(see Construction 8.5). We may observe further that in the coordinate subspace arrangement case the intersection poset $(\mathcal{L}, <)$ is the inclusion poset of simplices of $\widehat{K}$ with added maximal element (or equivalently, the inclusion poset of cone $\widehat{K}$). Hence, $\operatorname{ord}(\mathcal{L}_{(v,T)})$ is the barycentric subdivision of $\operatorname{link}_{\widehat{K}} v$, where v is regarded as a simplex of $\widehat{K}$. Thus, we may rewrite the Goresky–MacPherson formula (8.1) in the complex subspace arrangement case as

$$\widetilde{H}_i\bigl(U(K)\bigr) = \bigoplus_{\sigma \in \widehat{K}} \widetilde{H}^{2m-2|\sigma|-i-2}\bigl(\operatorname{link}_{\widehat{K}} \sigma\bigr), \tag{8.7}$$

(note that $d(\sigma) = |\sigma|$ and the dimensions are doubled since we are in the complex arrangement case).

The above observations were used in [**54**] to describe the product of two cohomology classes of a coordinate subspace arrangement complement (either real or complex) in terms of the combinatorics of links of simplices in $\widehat{K}$ (see [**54**, Theorem 1.1]).

On the other hand, the isomorphism of algebras established in Theorem 8.13 allows us to connect two seemingly unrelated results, namely, the Goresky–MacPherson theorem for the cohomology of an arrangement complement and the Hochster theorem from the commutative algebra.

Proposition 8.18. *After identification of the cohomology $H^*(U(K))$ with the* Tor*-algebra* $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K), \mathbf{k})$ *established by Theorem* 8.13*, the Hochster Theorem* 3.27 *becomes equivalent to the Goresky–MacPherson Theorem* 8.4 *in the case of coordinate subspace arrangements.*

Proof. Using Theorem 8.13 to identify $\beta^{-i,2j}(\mathbf{k}(K))$ with $\dim_{\mathbf{k}} H^{-i,2j}(U(K))$, we get the following formula from Hochster's Theorem 3.27:

$$H_p\bigl(U(K)\bigr) = \bigoplus_{\tau \subset [m]} \widetilde{H}_{p-|\tau|-1}(K_\tau).$$

Non-empty simplices $\tau \in K$ do not contribute to the above sum since the corresponding full subcomplexes K_τ are contractible. Since $\widetilde{H}_{-1}(\varnothing) = \mathbf{k}$, the empty subset of $[m]$ only contributes $\mathbf{k}$ to $H_0(U(K))$. Hence, we may rewrite the above formula as

$$\widetilde{H}_p\bigl(U(K)\bigr) = \bigoplus_{\tau \notin K} \widetilde{H}_{p-|\tau|-1}(K_\tau). \tag{8.8}$$

Using the Alexander duality (Proposition 2.29), we calculate

$$\widetilde{H}_{p-|\tau|-1}(K_\tau) = \widetilde{H}^{m-3-p+|\tau|+1-|\widehat{\tau}|}\bigl(\operatorname{link}_{\widehat{K}} \widehat{\tau}\bigr) = \widetilde{H}^{2m-2|\widehat{\tau}|-p-2}\bigl(\operatorname{link}_{\widehat{K}} \widehat{\tau}\bigr),$$

where $\widehat{\tau} = [m] \setminus \tau$ is a simplex of $\widehat{K}$. Now we observe that (8.8) is equivalent to (8.7). □

8.3. Diagonal subspace arrangements and the cohomology of $\Omega\mathcal{Z}_K$.

Another interesting particular class of subspace arrangements is diagonal arrangements. A classical example of a diagonal subspace arrangement is given by the arrangement of all diagonal hyperplanes $\{z_i = z_j\}$ in $\mathbb{C}^m$, mentioned in section 8.1 (see Example 8.3). Some further particular examples of diagonal arrangements, the so-called *k-equal arrangements* were considered, e.g. in [**20**], while the cohomology of general diagonal arrangement complements was studied in [**114**]. In this section we establish certain relationships between this cohomology and the cohomology of the loop spaces $\Omega(B_T\mathcal{Z}_K)$ (see section 6.5) and $\Omega\mathcal{Z}_K$.

DEFINITION 8.19. For each subset $\sigma = \{i_1, \ldots, i_k\} \subset [m]$ define the *diagonal subspace* D_σ in $\mathbb{R}^m$ by

$$D_\sigma = \{(y_1, \ldots, y_m) \in \mathbb{R}^m \ : \ y_{i_1} = \cdots = y_{i_k}\}.$$

Diagonal subspaces in $\mathbb{C}^m$ are defined similarly. An arrangement $\mathcal{A} = \{L_1, \ldots, L_r\}$ is called *diagonal* if all L_i, $i = 1, \ldots, r$, are diagonal subspaces.

CONSTRUCTION 8.20. Given a simplicial complex K on the vertex set $[m]$, introduce the diagonal subspace arrangement $\mathcal{DA}(K)$ as the set of subspaces D_σ such that σ is not a simplex of K:

$$\mathcal{DA}(K) = \{D_\sigma \ : \ \sigma \notin K\}.$$

Denote the complement of the arrangement $\mathcal{DA}(K)$ by $M(K)$.

The following statement is proved in the similar way as the corresponding statement (Proposition 8.6) for coordinate subspace arrangements.

PROPOSITION 8.21. *The assignment $K \mapsto M(K)$ defines a one-to-one order-preserving correspondence between the set of simplicial complexes on the vertex set $[m]$ and the set of diagonal subspace arrangement complements in $\mathbb{R}^m$.*

Here we still assume that $\mathbf{k}$ is a field. The multigraded (or $\mathbb{N}^m$-graded) structure in the ring $\mathbf{k}[v_1, \ldots, v_m]$ (Construction 3.33) defines an $\mathbb{N}^m$-grading in the Stanley–Reisner ring $\mathbf{k}(K)$. The monomial $v_1^{i_1} \cdots v_m^{i_m}$ acquires the multidegree $(2i_1, \ldots, 2i_m)$. Let us consider the modules $\operatorname{Tor}_{\mathbf{k}(K)}(\mathbf{k}, \mathbf{k})$. They can be calculated by means of the minimal free resolution (Example 3.23) of $\mathbf{k}$ (regarded as a $\mathbf{k}(K)$-module). The minimal resolution also carries a natural $\mathbb{N}^m$-grading, and we denote the subgroup of elements of multidegree $(2i_1, \ldots, 2i_m)$ in $\operatorname{Tor}_{\mathbf{k}(K)}(\mathbf{k}, \mathbf{k})$ by $\operatorname{Tor}_{\mathbf{k}(K)}(\mathbf{k}, \mathbf{k})_{(2i_1, \ldots, 2i_m)}$.

THEOREM 8.22 ([**114**, Theorem 1.3]). *The following isomorphism holds for the cohomology groups of a real diagonal subspace arrangement complement $M(K)$:*

$$H^i\bigl(M(K); \mathbf{k}\bigr) \cong \operatorname{Tor}_{\mathbf{k}(K)}^{-(m-i)}(\mathbf{k}, \mathbf{k})_{(2, \ldots, 2)}.$$

REMARK. Instead of simplicial complexes K on the vertex set $[m]$ the authors of [**114**] considered square-free monomial ideals $\mathcal{I} \subset \mathbf{k}[v_1, \ldots, v_m]$. Proposition 3.3 shows that the two approaches are equivalent.

THEOREM 8.23. *The following additive isomorphism holds:*

$$H^*\big(\Omega(B_T\mathcal{Z}_K);\mathbf{k}\big) \cong \mathrm{Tor}_{\mathbf{k}(K)}(\mathbf{k},\mathbf{k}).$$

PROOF. Let us consider the Eilenberg–Moore spectral sequence of the Serre fibration $P \to DJ(K)$ with fibre $\Omega\, DJ(K)$, where $DJ(K)$ is the Davis–Januszkiewicz space (Definition 6.27) and P is the path space over $DJ(K)$. By Corollary 7.4,

$$E_2 = \mathrm{Tor}_{H^*(DJ(K))}\big(H^*(P),\mathbf{k}\big) \cong \mathrm{Tor}_{\mathbf{k}(K)}(\mathbf{k},\mathbf{k}), \tag{8.9}$$

and the spectral sequence converges to $\mathrm{Tor}_{C^*(DJ(K))}\big(C^*(P),\mathbf{k}\big) \cong H^*(\Omega\, DJ(K))$. Since P is contractible, there is a cochain equivalence $C^*(P) \simeq \mathbf{k}$. We have $C^*(DJ(K)) \cong \mathbf{k}(K)$. Therefore,

$$\mathrm{Tor}_{C^*(DJ(K))}\big(C^*(P),\mathbf{k}\big) \cong \mathrm{Tor}_{\mathbf{k}(K)}(\mathbf{k},\mathbf{k}),$$

which together with (8.9) shows that the spectral sequence collapses at the E_2 term. Hence, $H^*(\Omega\, DJ(K)) \cong \mathrm{Tor}_{\mathbf{k}(K)}(\mathbf{k},\mathbf{k})$. Finally, Theorem 6.29 shows that $H^*(\Omega\, DJ(K)) \cong H^*(\Omega B_T\mathcal{Z}_K)$, which concludes the proof. $\square$

PROPOSITION 8.24. *The following isomorphism of algebras holds*

$$H^*\big(\Omega(B_T\mathcal{Z}_K)\big) \cong H^*(\Omega\mathcal{Z}_K) \otimes \Lambda[u_1,\ldots,u_m].$$

PROOF. Consider the bundle $B_T\mathcal{Z}_K \to BT^m$ with fibre $\mathcal{Z}_K$. It is easy to see that the corresponding loop bundle $\Omega B_T\mathcal{Z}_K \to T^m$ with fibre $\Omega\mathcal{Z}_K$ is trivial (note that $\Omega BT^m \simeq T^m$). To finish the proof it remains to mention that $H^*(T^m) \cong \Lambda[u_1,\ldots,u_m]$. $\square$

Theorems 8.9 and 8.13 give an application of the theory of moment-angle complexes to calculating the cohomology ring of a coordinate subspace arrangement complement. Likewise, Theorems 8.22, 8.23 and Proposition 8.24 establish a connection between the cohomology of a diagonal subspace arrangement complement and the cohomology of the loop space over the moment-angle complex $\mathcal{Z}_K$. However, the latter relationships are more subtle than those in the case of coordinate subspace arrangements. For instance, we do not have an analogue of the *multiplicative* isomorphism from Theorem 8.13. It would be very interesting to get any statement of such kind, or discover other new applications of the theory of moment-angle complexes to diagonal (or maybe even general) subspace arrangements.

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Index