HIGHER WHITEHEAD PRODUCTS IN MOMENT-ANGLE COMPLEXES AND SUBSTITUTION OF SIMPLICIAL COMPLEXES

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Dedicated to our Teacher Victor Matveevich Buchstaber on the occasion of his 75th birthday

Abstract. We study the question of realisability of iterated higher Whitehead products with a given form of nested brackets by simplicial complexes, using the notion of the moment-angle complex $\mathcal{Z}_\mathcal{K}$. Namely, we say that a simplicial complex $\mathcal{K}$ realises an iterated higher Whitehead product w if w is a nontrivial element of $\pi_*(\mathcal{Z}_\mathcal{K})$. The combinatorial approach to the question of realisability uses the operation of substitution of simplicial complexes: for any iterated higher Whitehead product w we describe a simplicial complex $\partial\Delta_w$ that realises w. Furthermore, for a particular form of brackets inside w, we prove that $\partial\Delta_w$ is the smallest complex that realises w. We also give a combinatorial criterion for the nontriviality of the product w. In the proof of nontriviality we use the Hurewicz image of w in the cellular chains of $\mathcal{Z}_\mathcal{K}$ and the description of the cohomology product of $\mathcal{Z}_\mathcal{K}$. The second approach is algebraic: we use the coalgebraic versions of the Koszul and Taylor complex for the face coalgebra of $\mathcal{K}$ to describe the canonical cycles corresponding to iterated higher Whitehead products w. This gives another criterion for realisability of w.

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1. Introduction

Higher Whitehead products are important invariants of unstable homotopy type. They have been studied since the 1960s in the works of homotopy theorists such as Hardie [Ha], Porter [Po] and Williams [Wi].

The appearance of moment-angle complexes and, more generally, polyhedral products in toric topology at the end of the 1990s brought a completely new perspective on higher homotopy invariants such as higher Whitehead products. The homotopy fibration of

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polyhedral products

$$(D^2,S^1)^{\mathcal K}\to(\mathbb{C}P^\infty)^{\mathcal K}\to(\mathbb{C}P^\infty)^m \tag{1.1}$$

was used as the universal model for studying iterated higher Whitehead products in [PR]. Here $(D^2,S^1)^{\mathcal K}=\mathcal Z_{\mathcal K}$ is the moment-angle complex, and $(\mathbb{C}P^\infty)^{\mathcal K}$ is homotopy equivalent to the Davis–Januszkiewicz space [BP1, BP2]. The form of nested brackets in an iterated higher Whitehead product is reflected in the combinatorics of the simplicial complex $\mathcal K$.

There are two classes of simplicial complexes $\mathcal K$ for which the moment-angle complex is particularly nice. From the geometric point of view, it is interesting to consider complexes $\mathcal K$ for which $\mathcal Z_{\mathcal K}$ is a manifold. This happens, for example, when $\mathcal K$ is a simplicial subdivision of sphere or the boundary of a polytope. The resulting moment-angle manifolds $\mathcal Z_{\mathcal K}$ often have remarkable geometric properties [Pa]. On the other hand, from the homotopy-theoretic point of view, it is important to identify the class of simplicial complexes $\mathcal K$ for which the moment-angle complex $\mathcal Z_{\mathcal K}$ is homotopy equivalent to a wedge of spheres. We denote this class by B_Δ. The spheres in the wedge are usually expressed in terms of iterated higher Whitehead products of the canonical 2-spheres in the polyhedral product $(\mathbb{C}P^\infty)^{\mathcal K}$. We denote by W_Δ the subclass in B_Δ consisting of those $\mathcal K$ for which $\mathcal Z_{\mathcal K}$ is a wedge of iterated higher Whitehead products. The question of describing the class W_Δ was studied in [PR] and formulated explicitly in [BP2, Problem 8.4.5]. It follows from the results of [PR] and [GPTW] that $W_\Delta=B_\Delta$ if we restrict attention to *flag* simplicial complexes only, and a flag complex $\mathcal K$ belongs to W_Δ if and only if its one-skeleton is a chordal graph. Furthermore, it is known that W_Δ contains directed MF-complexes [GT], shifted and totally fillable complexes [IK1, IK2]. On the other hand, it has been recently shown in [Ab] that the class W_Δ is *strictly* contained in B_Δ. There is also a related question of *realisability* of an iterated higher Whitehead product w with a given form of nested brackets: we say that a simplicial complex $\mathcal K$ *realises* an iterated higher Whitehead product w if w is a nontrivial element of $\pi_*(\mathcal Z_{\mathcal K})$ (see Definition 2.2). For example, the boundary of simplex $\mathcal K=\partial\Delta(1,\dots,m)$ realises a single (non-iterated) higher Whitehead product $[\mu_1,\dots,\mu_m]$, which maps $\mathcal Z_{\mathcal K}=S^{2m-1}$ into the fat wedge $(\mathbb{C}P^\infty)^{\mathcal K}$.

We suggest two approaches to the questions above. The first approach is combinatorial: using the operation of substitution of simplicial complexes (Section 4), for any iterated higher Whitehead product w we describe a simplicial complex $\partial\Delta_w$ that realises w (Theorem 5.1). Furthermore, for a particular form of brackets inside w, we prove in Theorem 5.2 (a) that $\partial\Delta_w$ is the smallest complex that realises w. We also give a combinatorial criterion for the nontriviality of the product w (Theorem 5.2 (b)). In the proof of nontriviality we use the Hurewicz image of w in the cellular chains of $\mathcal Z_{\mathcal K}$ and the description of the cohomology product of $\mathcal Z_{\mathcal K}$ from [BP1]. Theorems 5.1, 5.2 and further examples not included in this paper lead us to conjecture that $\partial\Delta_w$ is the smallest complex realising w, for any iterated higher Whitehead product (see Problem 5.5).

The second approach is algebraic: we use the coalgebraic versions of the Koszul complex and the Taylor resolution of the face coalgebra of $\mathcal K$ to describe the canonical cycles corresponding to iterated higher Whitehead products w. This gives another criterion for realisability of w in Theorem 7.1.

2. Preliminaries

A *simplicial complex* $\mathcal K$ on the set $[m]=\{1,2,\dots,m\}$ is a collection of subsets $I\subset[m]$ closed under taking any subsets. We refer to $I\in\mathcal K$ as a *simplex* or a *face* of $\mathcal K$, and always assume that $\mathcal K$ contains $\varnothing$ and all singletons $\{i\}$, $i=1,\dots,m$. We do not distinguish between $\mathcal K$ and its geometric realisation when referring to the homotopy or topological type of $\mathcal K$.

We denote by Δ^{m-1} or $\Delta(1,\dots,m)$ the full simplex on the set $[m]$. Similarly, denote by $\Delta(I)$ a simplex with the vertex set $I \subset [m]$ and denote its boundary by $\partial\Delta(I)$. A *missing face*, or a *minimal non-face* of $\mathcal{K}$ is a subset $I \subset [m]$ such that $I \notin \mathcal{K}$, but $\partial\Delta(I) \subset \mathcal{K}$.

Assume we are given a set of m pairs of based cell complexes

$$(\underline{X},\underline{A}) = \{(X_1,A_1),\dots,(X_m,A_m)\}$$

where $A_i \subset X_i$. For each simplex $I \in \mathcal{K}$ we set

$$(\underline{X},\underline{A})^I = \{(x_1,\dots,x_m) \in X_1 \times \dots \times X_m \mid x_j \in A_j \text{ for } j \notin I\}.$$

The *polyhedral product* of $(\underline{X},\underline{A})$ corresponding to $\mathcal{K}$ is the following subset of $X_1 \times \dots \times X_m$:

$$(\underline{X},\underline{A})^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (\underline{X},\underline{A})^I \qquad (\subset X_1 \times \dots \times X_m).$$

In the case when $(X_i,A_i) = (D^2,S^1)$ for each i, we use the notation $\mathcal{Z}_{\mathcal{K}}$ for $(D^2,S^1)^{\mathcal{K}}$, and refer to $\mathcal{Z}_{\mathcal{K}} = (D^2,S^1)^{\mathcal{K}}$ as the *moment-angle complex*. Also, if $(X_i,A_i) = (X,pt)$ for each i, where pt denotes the basepoint, we use the abbreviated notation $X^{\mathcal{K}}$ for $(X,pt)^{\mathcal{K}}$.

Theorem 2.1 ([BP2, Theorem 4.3.2])**.** *The moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is the homotopy fibre of the canonical inclusion $(\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m$.*

There is also the following more explicit description of the fibre inclusion $\mathcal{Z}_{\mathcal{K}} \to (\mathbb{C}P^\infty)^{\mathcal{K}}$ in (1.1). Consider the map of pairs $(D^2,S^1) \to (\mathbb{C}P^\infty,pt)$ sending the interior of the disc homeomorphically onto the complement of the basepoint in $\mathbb{C}P^1$. By the functoriality, we have the induced map of the polyhedral products $\mathcal{Z}_{\mathcal{K}} = (D^2,S^1)^{\mathcal{K}} \to (\mathbb{C}P^\infty)^{\mathcal{K}}$.

The general definition of higher Whitehead products can be found in [Ha]. We only describe Whitehead products in the space $(\mathbb{C}P^\infty)^{\mathcal{K}}$ and their lifts to $\mathcal{Z}_{\mathcal{K}}$. In this case the indeterminacy of higher Whitehead products can be controlled effectively because extension maps can be chosen canonically.

Consider the *ith coordinate map*

$$\mu_i\colon (D^2,S^1) \to S^2 \cong \mathbb{C}P^1 \hookrightarrow (\mathbb{C}P^\infty)^{\vee m} \hookrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Here the second map is the canonical inclusion of $\mathbb{C}P^1$ into the i-th summand of the wedge. The third map is induced by the embedding of m disjoint points into $\mathcal{K}$. The *Whitehead product* (or *Whitehead bracket*) $[\mu_i,\mu_j]$ of μ_i and μ_j is the homotopy class of the map

$$S^3 \cong \partial D^4 \cong \partial(D^2 \times D^2) \cong (D^2 \times S^1) \cup (S^1 \times D^2) \xrightarrow{[\mu_i,\mu_j]} (\mathbb{C}P^\infty)^{\mathcal{K}}$$

where

$$[\mu_i,\mu_j](x,y) = \begin{cases} \mu_i(x) & \text{for } (x,y) \in D^2 \times S^1; \\ \mu_j(y) & \text{for } (x,y) \in S^1 \times D^2. \end{cases}$$

Every Whitehead product $[\mu_i,\mu_j]$ becomes trivial after composing with the embedding $(\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m \simeq K(\mathbb{Z}^m,2)$. This implies that $[\mu_i,\mu_j]\colon S^3 \to (\mathbb{C}P^\infty)^{\mathcal{K}}$ lifts to the fibre $\mathcal{Z}_{\mathcal{K}}$, as shown next:

$$\begin{array}{ccccc} \mathcal{Z}_{\mathcal{K}} & \longrightarrow & (\mathbb{C}P^\infty)^{\mathcal{K}} & \longrightarrow & (\mathbb{C}P^\infty)^m \\ & \nwarrow & \uparrow{\scriptstyle [\mu_i,\mu_j]} & & \\ & & S^3 & & \end{array}$$

We use the same notation $[\mu_i,\mu_j]$ for a lifted map $S^3 \to \mathcal{Z}_{\mathcal{K}}$. Such a lift can be chosen canonically as the inclusion of a subcomplex

$$[\mu_i,\mu_j]\colon S^3 \cong (D^2 \times S^1) \cup (S^1 \times D^2) \hookrightarrow \mathcal{Z}_{\mathcal{K}}.$$

The Whitehead product $[\mu_i,\mu_j]$ is trivial if and only if the map $[\mu_i,\mu_j]\colon S^3\to\mathcal{Z}_\mathcal{K}$ can be extended to a map $D^4\cong D^2_i\times D^2_j\hookrightarrow\mathcal{Z}_\mathcal{K}$. This is equivalent to the condition that $\Delta(i,j)=\{i,j\}$ is a 1-simplex of $\mathcal{K}$.

Higher Whitehead products are defined inductively as follows. Let $\mu_{i_1},\ldots,\mu_{i_n}$ be a collection of maps such that the $(n-1)$-fold product

$$[\mu_{i_1},\ldots,\widehat{\mu_{i_k}},\ldots,\mu_{i_n}]\colon S^{2(n-1)-1}\to(\mathbb{C}P^\infty)^\mathcal{K}$$

is trivial for any k. Then there exists a *canonical* extension $\overline{[\mu_{i_1},\ldots,\widehat{\mu_{i_k}},\ldots,\mu_{i_n}]}$ to a map from $D^{2(n-1)}$ given by the composite

$$\overline{[\mu_{i_1},\ldots,\widehat{\mu_{i_k}},\ldots,\mu_{i_n}]}\colon D^2_{i_1}\times\cdots\times D^2_{i_{k-1}}\times D^2_{i_{k+1}}\times\cdots\times D^2_{i_n}\hookrightarrow\mathcal{Z}_\mathcal{K}\to(\mathbb{C}P^\infty)^\mathcal{K}.$$

Furthermore, all these extensions are compatible on the subproducts corresponding to the vanishing brackets of shorter length. The *n-fold product* $[\mu_{i_1},\ldots,\mu_{i_n}]$ is defined as the homotopy class of the map

$$S^{2n-1}\cong\partial(D^2_{i_1}\times\cdots\times D^2_{i_n})\cong\bigcup_{k=1}^n(D^2_{i_1}\times\cdots\times S^1_{i_k}\times\cdots\times D^2_{i_n})\xrightarrow{[\mu_{i_1},\ldots,\mu_{i_n}]}(\mathbb{C}P^\infty)^\mathcal{K}$$

which is given by

$$[\mu_{i_1},\ldots,\mu_{i_n}](x_1,\ldots,x_n)=\overline{[\mu_{i_1},\ldots,\widehat{\mu}_{i_k},\ldots,\mu_{i_n}]}(x_1,\ldots,\widehat{x}_k,\ldots,x_n)\quad\text{if } x_k\in S^1_{i_k}.$$

In Proposition 3.3 below we show that $[\mu_{i_1},\ldots,\mu_{i_p}]$ is defined in $\pi_{2p-1}((\mathbb{C}P^\infty)^\mathcal{K})$ if and only if $\partial\Delta(i_1,\ldots,i_p)$ is a subcomplex of $\mathcal{K}$, and $[\mu_{i_1},\ldots,\mu_{i_p}]$ is trivial if and only if $\Delta(i_1,\ldots,i_p)$ is a simplex of $\mathcal{K}$.

Alongside with higher Whitehead products of canonical coordinate maps μ_i we consider *general iterated* higher Whitehead products, i. e. higher Whitehead products in which arguments can be higher Whitehead products. For example,

$$\Bigl[\mu_1,\mu_2,[\mu_3,\mu_4,\mu_5],\bigl[\mu_6,\mu_{13},[\mu_7,\mu_8,\mu_9],\mu_{10}\bigr],[\mu_{11},\mu_{12}]\Bigr].$$

Among general iterated higher Whitehead products we distinguish *nested* products, which have the form

$$w=\Bigl[\bigl[\ldots\bigl[[\mu_{i_{11}},\ldots,\mu_{i_{1p_1}}],\mu_{i_{21}},\ldots,\mu_{i_{2p_2}}\bigr],\ldots\bigr],\mu_{i_{n1}},\ldots,\mu_{i_{np_n}}\Bigr]\colon S^{d(w)}\to(\mathbb{C}P^\infty)^\mathcal{K}.$$

Here $d(w)$ denotes the dimension of w. Sometimes we refer to $[\mu_{i_1},\ldots,\mu_{i_p}]$ as a *single* (noniterated) higher Whitehead product.

As in the case of ordinary Whitehead products any iterated higher Whitehead product lifts to a map $S^{d(w)}\to\mathcal{Z}_\mathcal{K}$ for dimensional reasons.

Definition 2.2. We say that a simplicial complex $\mathcal{K}$ *realises* a higher iterated Whitehead product w if w is a nontrivial element of $\pi_*(\mathcal{Z}_\mathcal{K})$.

Example 2.3. The complex $\partial\Delta(i_1,\ldots,i_p)$ realises the single higher Whitehead product $[\mu_{i_1},\ldots,\mu_{i_p}]$.

Construction 2.4 (cell decomposition of $\mathcal{Z}_\mathcal{K}$)**.** Following [BP2, §4.4], we decompose the disc D^2 into 3 cells: the point $1\in D^2$ is the 0-cell; the complement to 1 in the boundary circle is the 1-cell, which we denote by S; and the interior of D^2 is the 2-cell, which we denote by D. These cells are canonically oriented as subsets of $\mathbb{R}^2$. By taking products we obtain a cellular decomposition of $(D^2)^m$, in which cells are encoded by pairs of subsets

$J, I \subset [m]$ with $J \cap I = \varnothing$: the set J encodes the S-cells in the product and I encodes the D-cells. We denote the cell of $(D^2)^m$ corresponding to a pair J, I by $\varkappa(J, I)$:

$$\begin{aligned}\varkappa(J,I) &= \prod_{i\in I} D_i \times \prod_{j\in J} S_j \\ &= \big\{(x_1,\ldots,x_m)\in (D^2)^m \mid \\ &\qquad\qquad x_i \in D \text{ for } i\in I,\ x_j\in S \text{ for } j\in J \text{ and } x_l = 1 \text{ for } l\notin J\cup I\big\}.\end{aligned}$$

Then $\mathcal{Z}_\mathcal{K}$ is a cellular subcomplex in $(D^2)^m$; we have $\varkappa(J, I) \subset \mathcal{Z}_\mathcal{K}$ whenever $I \in \mathcal{K}$.

Given a subset $J \subset [m]$, we denote by $\mathcal{K}_J$ the *full subcomplex* of $\mathcal{K}$ on J, that is,

$$\mathcal{K}_J = \{I \in \mathcal{K} | \, I \subset J\}.$$

Let $C_{p-1}(\mathcal{K}_J)$ denote the group of $(p-1)$-dimensional simplicial chains of $\mathcal{K}_J$; its basis consists of simplices $L \in \mathcal{K}_J$, $|L| = p$. We also denote by $\mathcal{C}_q(\mathcal{Z}_\mathcal{K})$ the group of q-dimensional cellular chains of $\mathcal{Z}_\mathcal{K}$ with respect to the cell decomposition described above.

Theorem 2.5 (see [BP2, Theorems 4.5.7, 4.5.8])**.** *The homomorphisms*

$$C_{p-1}(\mathcal{K}_J) \longrightarrow \mathcal{C}_{p+|J|}(\mathcal{Z}_\mathcal{K}), \quad L \mapsto \operatorname{sign}(L,J)\varkappa(J\backslash L, L)$$

induce injective homomorphisms

$$\widetilde{H}_{p-1}(\mathcal{K}_J) \hookrightarrow H_{p+|J|}(\mathcal{Z}_\mathcal{K}),$$

which are functorial with respect to simplicial inclusions. Here $L \in \mathcal{K}_J$ is a simplex, and $\operatorname{sign}(L, J)$ *is the sign of the shuffle (L, J). The inclusions above induce an isomorphism of abelian groups*

$$\bigoplus_{J\subset[m]} \widetilde{H}_*(\mathcal{K}_J) \xrightarrow{\cong} H_*(\mathcal{Z}_\mathcal{K}).$$

The cohomology versions of these isomorphisms combine to form a ring isomorphism

$$\bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J) \xrightarrow{\cong} H^*(\mathcal{Z}_\mathcal{K}).$$

where the ring structure on the left hand side is given by the maps

$$H^{k-|I|-1}(\mathcal{K}_I) \otimes H^{\ell-|J|-1}(\mathcal{K}_J) \to H^{k+\ell-|I|-|J|-1}(\mathcal{K}_{I\cup J})$$

which are induced by the canonical simplicial inclusions $\mathcal{K}_{I\cup J} \to \mathcal{K}_I * \mathcal{K}_J$ for $I \cap J = \varnothing$ and are zero for $I \cap J \neq \varnothing$.

3. The Hurewicz image of a higher Whitehead product

Here we consider the Hurewicz homomorphism $h\colon \pi_*(\mathcal{Z}_\mathcal{K}) \to H_*(\mathcal{Z}_\mathcal{K})$. The canonical cellular chain representing the Hurewicz image $h(w) \in H_*(\mathcal{Z}_\mathcal{K})$ of a *nested* higher Whitehead product w was described in [Ab].

Lemma 3.1 ([Ab, Lemma 4.1])**.** *The Hurewicz image*

$$h\Big(\Big[\big[\ldots\big[[\mu_{i_{11}},\ldots,\mu_{i_{1p_1}}],\mu_{i_{21}},\ldots,\mu_{i_{2p_2}}\big],\ldots\big],\mu_{i_{n1}},\ldots,\mu_{i_{np_n}}\Big]\Big) \in H_{2(p_1+\cdots+p_n)-n}(\mathcal{Z}_\mathcal{K})$$

is represented by the cellular chain

$$h_c(w) = \prod_{k=1}^{n}\left(\sum_{j=1}^{p_k} D_{i_{k1}}\cdots D_{i_{k(j-1)}} S_{i_{kj}} D_{i_{k(j+1)}} \cdots D_{i_{kp_k}}\right).$$

A more general version of this lemma is presented next. It gives a simple recursive formula describing the canonical cellular chain $h_c(w)$ which represents the Hurewicz image of a *general* iterated higher Whitehead product $w \in \pi_*(\mathcal{Z}_\mathcal{K})$, therefore providing an effective method of identifying nontrivial Whitehead products in the homotopy groups of a moment-angle complex $\mathcal{Z}_\mathcal{K}$. Some applications are also given below.

Lemma 3.2. *Let w be a general iterated higher Whitehead product*

$$w = [w_1, \ldots, w_q, \mu_{i_1}, \ldots, \mu_{i_p}] \in \pi_*(\mathcal{Z}_\mathcal{K}).$$

Here w_k is a (general iterated) higher Whitehead product for $k = 1, \ldots, q$. Then the Hurewicz image $h(w) \in H_*(\mathcal{Z}_\mathcal{K})$ is represented by the following canonical cellular chain:

$$h_c(w) = h_c(w_1) \cdots h_c(w_q) \Big(\sum_{k=1}^{p} D_{i_1} \cdots D_{i_{k-1}} S_{i_k} D_{i_{k+1}} \cdots D_{i_p} \Big).$$

We shall refer to $h_c(w)$ as the *canonical* cellular chain for an interated higher Whitehead product w. In the case of nested products, Lemma 3.2 reduces to Lemma 3.1.

Proof of Lemma 3.2. Let $d, d_1, \ldots, d_q$ be the dimensions of $w, w_1, \ldots, w_q$, respectively. The Whitehead product w is represented by the composite map

$$\begin{aligned}
(3.1)\quad S^d &\cong \partial(D^{d_1} \times \cdots \times D^{d_q} \times D^2_{i_1} \times \cdots \times D^2_{i_p}) \\
&\cong \Big(D^{d_1} \times \cdots \times D^{d_q} \times \Big(\bigcup_{k=1}^{p} D^2_{i_1} \times \cdots \times S^1_{i_k} \times \cdots \times D^2_{i_p} \Big) \Big) \\
&\cup \Big(\Big(\bigcup_{l=1}^{q} D^{d_1} \times \cdots \times S^{d_l - 1} \times \cdots \times D^{d_q} \Big) \times D^2_{i_1} \times \cdots \times D^2_{i_p} \Big) \\
&\xrightarrow{\gamma} \Big(S^{d_1} \times \cdots \times S^{d_q} \times \Big(\bigcup_{k=1}^{p} D^2_{i_1} \times \cdots \times S^1_{i_k} \times \cdots \times D^2_{i_p} \Big) \Big) \\
&\qquad \cup \Big(\Big(\bigcup_{l=1}^{q} S^{d_1} \times \cdots \times pt \times \cdots \times S^{d_q} \Big) \times D^2_{i_1} \times \cdots \times D^2_{i_p} \Big) \to \mathcal{Z}_\mathcal{K}.
\end{aligned}$$

The map γ above contracts the boundary of each D^{d_l}, $l = 1, \ldots, q$. Note that the whole cartesian product in the last row above has dimension less than d, so its Hurewicz image is trivial.

Using the same argument for the spheres $S^{d_1}, \ldots, S^{d_q}$, we obtain that w factors through a map from S^d to a union of products of discs and circles, which embeds as a subcomplex in $\mathcal{Z}_\mathcal{K}$. By the induction hypothesis each sphere $S^{d_k}, k = 1, \ldots, q$, maps to the subcomplex of $\mathcal{Z}_\mathcal{K}$ corresponding to the cellular chain $h_c(w_k)$. Therefore, by (3.1), the Hurewicz image of w is represented by the subcomplex corresponding to the product of $h_c(w_1), \ldots, h_c(w_q)$ and $\sum_{k=1}^{p} D_{i_1} \cdots D_{i_{k-1}} S_{i_k} D_{i_{k+1}} \cdots D_{i_p}$. □

As a first corollary we obtain a combinatorial criterion for the nontriviality of a single higher Whitehead product.

Proposition 3.3. *A single higher Whitehead product $[\mu_{i_1}, \ldots, \mu_{i_p}]$ is*

(*a*) *defined in $\pi_{2p-1}((\mathbb{C}P^\infty)^\mathcal{K})$ (and lifts to $\pi_{2p-1}(\mathcal{Z}_\mathcal{K})$) if and only if $\partial\Delta(i_1, \ldots, i_p)$ is a subcomplex of $\mathcal{K}$;*

(*b*) *trivial if and only if $\Delta(i_1, \ldots, i_p)$ is a simplex of $\mathcal{K}$.*

Proof. If the Whitehead product $[\mu_{i_1}, \ldots, \mu_{i_p}]$ is defined, then each $(p-1)$-fold product $[\mu_{i_1}, \ldots, \widehat{\mu}_{i_k} \ldots, \mu_{i_p}]$ is trivial. By the induction hypothesis, this implies that $\partial\Delta(i_1, \ldots, i_p)$ is a subcomplex of $\mathcal{K}$.

Suppose that $\Delta(i_1,\ldots,i_p)$ is not a simplex of $\mathcal{K}$. Then, by Lemma 3.2, the Hurewicz image $h([\mu_{i_1},\ldots,\mu_{i_p}])$ gives a nontrivial homology class in $H_*(\mathcal{Z}_\mathcal{K})$ corresponding to $[\partial\Delta(i_1,\ldots,i_p)]\in\widetilde{H}_*(\mathcal{K}_{i_1,\ldots,i_p})$ via the isomorphism of Theorem 2.5. Thus, $[\mu_{i_1},\ldots,\mu_{i_p}]$ is itself nontrivial. □

This proposition will be generalised to iterated higher Whitehead products in Section 5.

Lemmata 3.1, 3.2 and Theorem 2.5 can be used to detect simplicial complexes $\mathcal{K}$ for which $\mathcal{Z}_\mathcal{K}$ is a wedge of iterated higher Whitehead products. We recall the following definition.

Definition 3.4. A simplicial complex $\mathcal{K}$ belongs to the class W_Δ if $\mathcal{Z}_\mathcal{K}$ is a wedge of spheres, and each sphere in the wedge is a lift of a linear combination of iterated higher Whitehead products.

As a first example of application of our method we deduce the results of Iriye and Kishimoto that shifted and totally fillable complexes belong to the class W_Δ.

Example 3.5. A simplicial complex $\mathcal{K}$ is called *shifted* if its vertices can be ordered in such way that the following condition is satisfied: whenever $I\in\mathcal{K}$, $i\in I$ and $j>i$, we have $(I-i)\cup j\in\mathcal{K}$.

Let $\mathrm{MF}_m(\mathcal{K})$ be the set of missing faces of $\mathcal{K}$ containing the maximal vertex m, i. e.

$$\mathrm{MF}_m(\mathcal{K})=\{I\subset[m]\,|\,I\notin\mathcal{K},\ \partial\Delta(I)\subset\mathcal{K}\text{ and }m\in I\}.$$

As observed in [IK1], for a shifted complex $\mathcal{K}$ there is a homotopy equivalence

$$\mathcal{K}\simeq\bigvee_{I\in\mathrm{MF}_m(\mathcal{K})}\partial\Delta(I) \tag{3.2}$$

(the reason is that the quotient $\mathcal{K}/\operatorname{star}_m\mathcal{K}$ is homeomorphic to the wedge on the right hand side of (3.2), by definition of a shifted complex). Note that a full subcomplex of a shifted complex is again shifted. Then Theorem 2.5 together with (3.2) implies that $H_*(\mathcal{Z}_\mathcal{K})$ is a free abelian group generated by the homology classes of cellular chains of the form

$$\Big(\sum_{l=1}^{p}D_{i_1}\cdots D_{i_{l-1}}S_{i_l}D_{i_{l+1}}\cdots D_{i_p}\Big)S_{j_1}\cdots S_{j_q} \tag{3.3}$$

where $I=\{i_1,\ldots,i_p\}\in\mathrm{MF}_m(\mathcal{K}_{i_1,\ldots,i_p,j_1,\ldots,j_q})$. Lemma 3.1 implies that (3.3) is the canonical cellular chain for the nested Whitehead product

$$w=\Big[\Big[\Big[\ldots[[\mu_{i_1},\ldots,\mu_{i_p}],\mu_{j_1}],\ldots\Big],\mu_{j_{q-1}}\Big],\mu_{j_q}\Big].$$

Hence, the following wedge of the Whitehead products

$$\bigvee_{J\subset[m]}\ \bigvee_{\substack{I=\{i_1,\ldots,i_p\}\in\mathrm{MF}_m(\mathcal{K}_J)\\ J\setminus I=\{j_1,\ldots,j_q\}}}\Big[\Big[\Big[\ldots[[\mu_{i_1},\ldots,\mu_{i_p}],\mu_{j_1}],\ldots\Big],\mu_{j_{q-1}}\Big],\mu_{j_q}\Big]\colon\bigvee_{\substack{J\subset[m]\\ I\in\mathrm{MF}_m(\mathcal{K}_J)}}S^{d(w)}_{J,I}\to\mathcal{Z}_\mathcal{K}$$

induces an isomorphism in homology, so it is a homotopy equivalence. Thus, we obtain the following.

Theorem 3.6 ([IK1])**.** *Every shifted complex $\mathcal{K}$ belongs to W_Δ.*

Here is another result which can be proved using Lemma 3.2.

Example 3.7. A simplicial complex $\mathcal{K}$ is called *fillable* if there is a collection $\mathrm{MF}_{\mathrm{fill}}(\mathcal{K})$ of missing faces $I_1,\ldots,I_k$ such that $\mathcal{K}\cup I_1\cup\cdots\cup I_k$ is contractible. If any full subcomplex of $\mathcal{K}$ is fillable, then $\mathcal{K}$ is called *totally fillable*.

Note that homology of any full subcomplex $\mathcal{K}_J$ in a totally fillable complex $\mathcal{K}$ is generated by the cycles $\partial\Delta(I)$ for $I \in \mathrm{MF}_{\mathrm{fill}}(\mathcal{K}_J)$. As in Example 3.5, $H_*(\mathcal{Z}_\mathcal{K})$ is a free abelian group generated by the homology classes of cellular chains

$$\Big(\sum_{l=1}^{p} D_{i_1}\ldots D_{i_{l-1}}S_{i_l}D_{i_{l+1}}\ldots D_{i_p}\Big)S_{j_1}\ldots S_{j_q},$$

where $\Delta(i_1,\ldots,i_q) \in \mathrm{MF}_{\mathrm{fill}}(\mathcal{K}_{j_1,\ldots,j_p,i_1,\ldots,i_q})$. Again, the map

$$\bigvee_{J\subset[m]}\ \bigvee_{\substack{I\in\mathrm{MF}_{\mathrm{fill}}(\mathcal{K}_J)\\ J\setminus I=\{j_1,\ldots,j_q\}}} \Big[\Big[\Big[\cdots\big[[\mu_{i_1},\ldots,\mu_{i_p}],\mu_{j_1}\big],\ldots\Big],\mu_{j_{q-1}}\Big],\mu_{j_q}\Big]\colon \bigvee_{\substack{J\subset[m]\\ I\in\mathrm{MF}_{\mathrm{fill}}(\mathcal{K}_J)}} S^{d(w)}_{J,I} \to \mathcal{Z}_\mathcal{K}$$

is a homotopy equivalence, by the same reasons. We obtain the following.

Theorem 3.8 ([IK2]). *Every totally fillable complex $\mathcal{K}$ belongs to W_Δ.*

4. Substitution of simplicial complexes

The combinatorial construction presented here is similar to the one described in [Ay1] and [BBCG], although the resulting complexes are different. An analogous construction for building sets was suggested by N. Erokhovets (see [BP2, Construction 1.5.19]).

Definition 4.1. Let $\mathcal{K}$ be a simplicial complex on the set $[m]$, and let $\mathcal{K}_1,\ldots,\mathcal{K}_m$ be a set of m simplicial complexes. We refer to the simplicial complex

$$\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m) = \{I_{j_1}\sqcup\cdots\sqcup I_{j_k} \mid I_{j_l}\in\mathcal{K}_{j_l},\ l=1,\ldots,k \quad\text{and}\quad \{j_1,\ldots,j_k\}\in\mathcal{K}\} \tag{4.1}$$

as the *substitution* of $\mathcal{K}_1,\ldots,\mathcal{K}_m$ into $\mathcal{K}$.

The set of missing faces $\mathrm{MF}(\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m))$ of a substitution complex can be described as follows. First, every missing face of each $\mathcal{K}_i$ is the missing face of $\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m)$. Second, for every missing face $\Delta(i_1,\ldots,i_k)$ of $\mathcal{K}$ we have the following set of missing faces of the substitution complex:

$$\mathrm{MF}_{i_1,\ldots,i_k}\big(\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m)\big) = \big\{\Delta(j_1,\ldots,j_k) \mid j_l\in\mathcal{K}_{i_l},\ l=1,\ldots,k\big\}.$$

It is easy to see that there are no other missing faces in $\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m)$, so we have

$$\mathrm{MF}\big(\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m)\big) = \mathrm{MF}(\mathcal{K}_1)\sqcup\cdots\sqcup\mathrm{MF}(\mathcal{K}_m)\sqcup \bigsqcup_{\Delta(i_1,\ldots,i_k)\in\mathrm{MF}(\mathcal{K})} \mathrm{MF}_{i_1,\ldots,i_k}\big(\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m)\big).$$

Example 4.2. If each $\mathcal{K}_i$ is a point $\{i\}$, then $\mathcal{K}(\mathcal{K}_1,\ldots,\mathcal{K}_m) = \mathcal{K}$. In particular, $\partial\Delta^{m-1}(1,\ldots,m) = \partial\Delta^{m-1}$. In the case of substitution into a simplex Δ^{m-1} or its boundary $\partial\Delta^{m-1}$ we shall omit the dimension, so we have $\partial\Delta(1,\ldots,m) = \partial\Delta^{m-1}$, which is compatible with the previous notation.

The next example is our starting point for further generalisations.

Example 4.3. Let $\mathcal{K} = \partial\Delta^{m-1}$ and each $\mathcal{K}_i$ is a point, except for $\mathcal{K}_1$. We have $\partial\Delta(\mathcal{K}_1, i_2,\ldots,i_m) = \mathcal{J}_{m-2}(\mathcal{K}_1)$, where $\mathcal{J}_n(\mathcal{L})$ is the operation defined in [Ab, Theorem 5.2]. By [Ab, Theorem 6.1], the iterated substitution

$$\partial\Delta\big(\partial\Delta(j_1,\ldots,j_q), i_1,\ldots,i_p\big)$$

is the smallest simplicial complex that realises the Whitehead product

$$\big[[\mu_{j_1},\ldots,\mu_{j_q}],\mu_{i_1},\ldots,\mu_{i_p}\big].$$

The case $q=3$, $p=2$ is shown in Figure 1.

The next example will be used in Theorem 5.2.

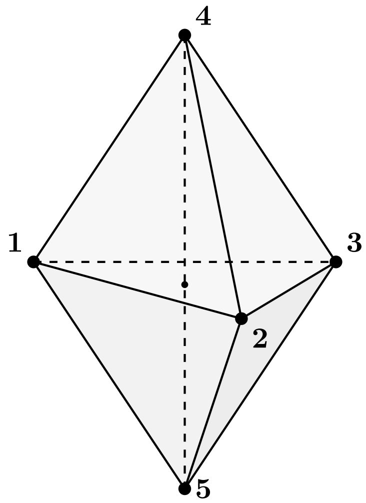

Figure 1. Substitution complex $\partial\Delta(\partial\Delta(1,2,3),4,5)$

Construction 4.4. Here we inductively describe the canonical simplicial complex $\partial\Delta_w$ associated with a general iterated higher Whitehead product w.

We start with the boundary of simplex $\partial\Delta(i_1,\dots,i_m)$ corresponding to a single higher Whitehead product $[\mu_{i_1},\dots,\mu_{i_m}]$. Now we write a general iterated higher Whitehead product recursively as

$$w=[w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}]\in\pi_*(\mathcal{Z}_\mathcal{K}),$$

where $w_1,\dots,w_q$ are nontrivial general iterated higher Whitehead products, $q\geqslant 0$. We assign to w the substitution complex

$$\partial\Delta_w\overset{\text{def}}{=}\partial\Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q},i_1,\dots,i_p).$$

We also define recursively the following subcomplex of $\partial\Delta_w$:

$$\partial\Delta_w^{\text{sph}}=\partial\Delta_{w_1}^{\text{sph}}*\dots*\partial\Delta_{w_q}^{\text{sph}}*\partial\Delta(i_1,\dots,i_p).$$

By definition, $\partial\Delta_w^{\text{sph}}$ is a join of boudaries of simplices, so it is homeomorphic to a sphere. Furthermore, $\dim\partial\Delta_w^{\text{sph}}=\dim\partial\Delta_w$.

We refer to the subcomplex $\partial\Delta_w^{\text{sph}}$ as the *top sphere of* $\partial\Delta_w$.

For example, the top sphere of $\partial\Delta(\partial\Delta(1,2,3),4,5)$ is obtained by deleting the edge $\Delta(4,5)$, see Figure 1.

Proposition 4.5. *The complex $\partial\Delta_w$ is homotopy equivalent to a wedge of spheres, and the top sphere $\partial\Delta_w^{\text{sph}}$ represents the sum of top-dimensional spheres in the wedge.*

Proof. By construction, $\partial\Delta_w$ is obtained from a sphere $\partial\Delta_w^{\text{sph}}$ by attaching simplices of dimension at most $\dim\partial\Delta_w^{\text{sph}}$. It follows that the attaching maps are null-homotopic, which implies both statements. □

5. Realisation of higher Whitehead products

Given an iterated higher Whitehead product w, we show that the substitution complex $\partial\Delta_w$ realises w. Furthermore, for a particular form of brackets inside w, we prove that $\partial\Delta_w$ is the smallest complex that realises w. We also give a combinatorial criterion for the nontriviality of the product w.

Recall from Proposition 3.3 that a single higher Whitehead product $[\mu_{i_1},\dots,\mu_{i_p}]$ is realised by the complex $\partial\Delta(i_1,\dots,i_p)$.

Theorem 5.1. *Let $w_1, \dots, w_q$ be nontrivial iterated higher Whitehead products. The complex $\partial\Delta_w$ described in Construction 4.4 realises the iterated higher Whitehead product*

$$w = [w_1, \dots, w_q, \mu_{i_1}, \dots, \mu_{i_p}]. \tag{5.1}$$

Proof. To see that product (5.1) is defined in $\mathcal{Z}_{\partial\Delta_w}$ we need to construct the corresponding map $S^{d(w)} \to \mathcal{Z}_{\partial\Delta_w}$. This is done precisely as described in the proof of Lemma 3.2. Furthermore, Lemma 3.2 gives the cellular chain $h_c(w) \in \mathcal{C}_*(\mathcal{Z}_{\partial\Delta_w})$ representing the Hurewicz image $h(w) \in H_*(\mathcal{Z}_{\partial\Delta_w})$. The cellular chain $h_c(w) \in \mathcal{C}_*(\mathcal{Z}_{\partial\Delta_w})$ corresponds to the simplicial chain $\partial\Delta_w^{\mathrm{sph}} \in C_*(\partial\Delta_w)$ via the isomorphism of Theorem 2.5. Now Proposition 4.5 implies that the simplicial homology class $[\partial\Delta_w^{\mathrm{sph}}] \in H_*(\partial\Delta_w)$ is nonzero. Thus, $h(w) \neq 0$ and the Whitehead product w is nontrivial. □

For a particular configuration of nested brackets, a more precise statement holds.

Theorem 5.2. *Let $w_j = [\mu_{j_1}, \dots, \mu_{j_{p_j}}]$, $j = 1, \dots, q$, be nontrivial single higher Whitehead products. Consider an iterated higher Whitehead product*

$$w = [w_1, \dots, w_q, \mu_{i_1}, \dots, \mu_{i_p}].$$

Then the product w is

(*a*) *defined in $\pi_*(\mathcal{Z}_\mathcal{K})$ if and only if $\mathcal{K}$ contains $\partial\Delta_w = \partial\Delta\big(\partial\Delta_{w_1}, \dots, \partial\Delta_{w_q}, i_1, \dots, i_p\big)$ as a subcomplex, where $\partial\Delta_{w_j} = \partial\Delta(j_1, \dots, j_{p_j})$, $j = 1, \dots, q$;*

(*b*) *trivial in $\pi_*(\mathcal{Z}_\mathcal{K})$ if and only if $\mathcal{K}$ contains*

$$\Delta\big(\partial\Delta_{w_1}, \dots, \partial\Delta_{w_q}, i_1, \dots, i_p\big) = \partial\Delta_{w_1} * \dots * \partial\Delta_{w_q} * \Delta(i_1, \dots, i_p)$$

as a subcomplex.

Note that assertion (a) implies that $\partial\Delta_w$ is the smallest simplicial complex realising the Whitehead product w.

Proof. We may assume that $q > 0$; otherwise the theorem reduces to the Proposition 3.3. We consider three cases: $p = 0$; $p = 1$; $p > 1$.

The case $p = 0$. We have $w = [w_1, \dots, w_q]$.

We first prove assertion (b). Let $d_1, \dots, d_q$ and $d = d_1 + \dots + d_q - 1$ be the dimensions of the Whitehead products $w_1, \dots, w_q$ and $[w_1, \dots, w_q]$, respectively. The condition that w vanishes implies the existence of the dashed arrow in the diagram

$$\begin{array}{ccccc} S^d & \longrightarrow & \mathrm{FW}(S^{d_1}, \dots, S^{d_q}) & \longrightarrow & \mathcal{Z}_\mathcal{K} \\ \downarrow & & \downarrow & \nearrow & \\ D^{d+1} & \longrightarrow & S^{d_1} \times \dots \times S^{d_q} & & \end{array}$$

Here $\mathrm{FW}(S^{d_1}, \dots, S^{d_q})$ denotes the fat wedge of spheres $S^{d_1}, \dots, S^{d_q}$, and the top left arrow is the attaching map of the top cell.

Let $\sigma_j \in H^{d_j}(\mathcal{Z}_\mathcal{K})$ be the cohomology class dual to the sphere $S^{d_j} \subset \mathrm{FW}(S^{d_1}, \dots, S^{d_q})$, $j = 1, \dots, q$. By the assumption, the single Whitehead product w_j is nontrivial, which implies that $\sigma_j \neq 0$ (see Propostion 3.3). The class $\sigma_j \in H^{d_j}(\mathcal{Z}_\mathcal{K})$ corresponds to the simplicial cohomology class $\big[\partial\Delta_{w_j}\big]^* \in \widetilde{H}^*(\mathcal{K}_{\partial\Delta_{w_j}})$ via the cohomological version of the isomorphism of Theorem 2.5. Here $\mathcal{K}_{\partial\Delta_{w_j}}$ is the full subcomplex $\partial\Delta_{w_j}$ of $\mathcal{K}$. Since the Whitehead product $[w_1, \dots, w_q]$ is trivial, the cohomology product $\sigma_1 \cdots \sigma_q$ is nontrivial in $H^*(\mathcal{Z}_\mathcal{K})$ (see the diagram above). By the cohomology product description in Theorem 2.5, this implies that $\mathcal{K}$ contains $\partial\Delta_1 * \dots * \partial\Delta_{w_q}$ as a full subcomplex, and assertion (b) follows.

To prove assertion (a), note that the existence of the product $[w_1, \dots, w_q]$ implies that each product $[w_1, \dots, \widehat{w_j}, \dots, w_q]$, $j = 1, \dots, q$, is trivial. By assertion (b), complex $\mathcal{K}$

contains the union $\bigcup_{j=1}^{q} \partial\Delta_{w_1} * \cdots * \widehat{\partial\Delta_{w_j}} * \cdots * \partial\Delta_{w_q}$ which is precisely $\partial\Delta(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q})$. This finishes the proof for the case $p = 0$.

The case $p = 1$. We have $w = [w_1, \ldots, w_q, \mu_{i_1}]$.

We first prove (b), that is, assume $w = 0$. This implies that $[w_1, \ldots, w_q] = 0$. By the previous case, we know that $\mathcal{K}$ contains $\Delta(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q})$ as a full subcomplex. We need to prove that $\mathcal{K}$ contains $\Delta(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q}) * \Delta(i_1)$, which is a cone with apex i_1. The Hurewicz image $h(w) \in H_*(\mathcal{Z}_\mathcal{K})$ is zero, because w is trivial. Therefore, the canonical cellular chain $h_c(w) = h_c(w_1) \cdots h_c(w_q) S_{i_1}$ (see Lemma 3.2) is a boundary. By Theorem 2.5, this implies that the simplicial cycle $\partial\Delta_{w_1} * \cdots * \partial\Delta_{w_q}$ is a boundary in $\mathcal{K}_{\Delta(\partial\Delta_{w_1},\ldots,\partial\Delta_{w_q})\cup\{i_1\}}$. This can only be the case when $\mathcal{K}_{\Delta(\partial\Delta_{w_1},\ldots,\partial\Delta_{w_q})\cup\{i_1\}} = \Delta(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q}) * \Delta(i_1)$, proving (b).

Now we prove (a). By the previous cases, the existence of w implies that $\mathcal{K}$ contains $\Delta(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q})$ and $\Delta(\partial\Delta_{w_1}, \ldots, \widehat{\partial\Delta_{w_j}}, \ldots, \partial\Delta_{w_q}, i_1)$ for $j = 1, \ldots, q$. The union of these subcomplexes is precisely $\partial\Delta(\partial\Delta_{w_1}, \ldots, \partial\Delta_{w_q}, i_1)$.

The case $p > 1$.

We induct on $p + q$. We have $w = \big[w_1, \ldots, w_q, \mu_{i_1}, \ldots, \mu_{i_p}\big]$.

To prove (b), suppose that $w = 0$ but $\mathcal{K}$ does not contain $\partial\Delta_{w_1} * \cdots * \partial\Delta_{w_q} * \Delta(i_1, \ldots, i_p)$. Then the cellular chain corresponding to $\partial\Delta_{w_1} * \cdots * \partial\Delta_{w_q} * \partial\Delta(i_1, \ldots, i_p)$ via Theorem 2.5 gives a nontrivial homology class in $H_*(\mathcal{Z}_\mathcal{K})$. This class coincides with the Hurewicz image $h(w)$, by Lemma 3.2. Hence, the Whitehead product w is nontrivial. A contradiction.

Assertion (a) is proved similarly to the case $p = 1$. □

Remark 5.3. In our approach, the nontriviality of a higher Whitehead product w is understood as the nontriviality of its canonical representative constructed in § 2. Nevertheless, arguments similar to those given in the proof of the case $p = 0$ show that the nontriviality assertion in Theorem 5.2 remains valid if the nontriviality is understood in the classical sense, that is, as the absence of a trivial homotopy class in the set of all possible extensions.

Example 5.4. Consider the Whitehead product $w = \big[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5\big]$ in the moment-angle complex $\mathcal{Z}_\mathcal{K}$ corresponding to a simplicial complex $\mathcal{K}$ on 5 vertices. For the existence of w it is necessary that the brackets $\big[[\mu_1, \mu_2, \mu_3], \mu_4\big]$, $\big[[\mu_1, \mu_2, \mu_3], \mu_5\big]$ and $[\mu_4, \mu_5]$ vanish. By Theorem 5.2 (b), this implies that $\mathcal{K}$ contains subcomplexes $\partial\Delta(1, 2, 3) * \Delta(4)$, $\partial\Delta(1, 2, 3) * \Delta(5)$ and $\Delta(4, 5)$. In other words, $\mathcal{K}$ contains the complex $\partial\Delta\big(\partial\Delta(1, 2, 3), 4, 5\big)$ shown in Figure 1. Therefore, the latter is the smallest complex realising the Whitehead bracket $w = \big[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5\big]$.

The moment-angle complex $\mathcal{Z}_\mathcal{K}$ corresponding to $\mathcal{K} = \partial\Delta\big(\partial\Delta(1, 2, 3), 4, 5\big)$ is homotopy equivalent to the wedge of spheres $(S^5)^{\vee 4} \vee (S^6)^{\vee 3} \vee S^7 \vee S^8$, and each sphere is a Whitehead product, see [Ab, Example 5.4]. For example, S^7 corresponds to $w = \big[\big[[\mu_3, \mu_4, \mu_5], \mu_1\big], \mu_2\big]$, and S^8 corresponds to $w = \big[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5\big]$.

We expect that Theorem 5.2 holds for all iterated higher Whitehead products:

Problem 5.5. *Is it true that for any iterated higher Whitehead product w the substitution complex $\partial\Delta_w$ is the smallest complex realising w?*

6. Resolutions of the face coalgebra

Originally, cohomology of $\mathcal{Z}_\mathcal{K}$ was described in [BP1] as the Tor-algebra of the face algebra of $\mathcal{K}$. As observed in [BBP], the Koszul complex calculating the Tor-algebra can be identified with the cellular cochain complex of $\mathcal{Z}_\mathcal{K}$ with respect to the standard cell decomposition. On the other hand, the Tor-algebra, and therefore cohomology of $\mathcal{Z}_\mathcal{K}$, can be calculated via the Taylor resolution of the face algebra as a module over the polynomial ring, see [WZ], [Ay2, §4]. We dualise both approaches by identifying homology of $\mathcal{Z}_\mathcal{K}$ with the Cotor of the face coalgebra of $\mathcal{K}$, and use both co-Koszul and co-Taylor resolutions to describe cycles corresponding to iterated higher Whitehead products.

Let $\Bbbk$ be a commutative ring with unit. The *face algebra* $\Bbbk[\mathcal{K}]$ of a simplicial complex $\mathcal{K}$ is the quotient of the polynomial algebra $\Bbbk[v_1,\ldots,v_m]$ by the square-free monomial ideal generated by non-simplices of $\mathcal{K}$:

$$\Bbbk[\mathcal{K}]=\Bbbk[v_1,\ldots,v_m]\big/\big(v_{j_1}\cdots v_{j_k}\mid\{j_1,\ldots,j_k\}\notin\mathcal{K}\big).$$

The grading is given by $\deg v_j=2$. Given a subset $J\subset[m]$, we denote by v_J the square-free monomial $\prod_{j\in J}v_j$. Observe that

$$\Bbbk[\mathcal{K}]=\Bbbk[v_1,\ldots,v_m]\big/\big(v_J\mid J\in\mathrm{MF}(\mathcal{K})\big),$$

where $\mathrm{MF}(\mathcal{K})$ denotes the set of missing faces (minimal non-faces) of $\mathcal{K}$. The face algebra $\mathbb{Z}[\mathcal{K}]$ is also known as the *face ring*, or the *Stanley–Reisner ring* of $\mathcal{K}$.

We shall use the shorter notation $\Bbbk[m]$ for the polynomial algebra $\Bbbk[v_1,\ldots,v_m]$. Let M and N be two $\Bbbk[m]$-modules. The n-th derived functor of $\cdot\otimes_{\Bbbk[m]}N$ is denoted by $\mathrm{Tor}_n^{\Bbbk[m]}(M,N)$ or $\mathrm{Tor}^{-n}_{\Bbbk[m]}(M,N)$. (The latter notation is better suited for topological application of the Eilenberg–Moore spectral sequence, where the Tor appears naturally as cohomology of certain spaces.) Namely, given a projective resolution $R^\bullet\to M$ with the resolvents indexed by nonpositive integers, we have

$$\mathrm{Tor}^{-n}_{\Bbbk[m]}(M,N)=H^{-n}(R^\bullet\otimes_{\Bbbk[m]}N).$$

The standard argument using bicomplexes and commutativity of the tensor product gives a natural isomorphism

$$\mathrm{Tor}^{-n}_{\Bbbk[m]}(M,N)\cong\mathrm{Tor}^{-n}_{\Bbbk[m]}(N,M).$$

When M and N are graded $\Bbbk[m]$-modules, $\mathrm{Tor}^{-i}_{\Bbbk[m]}(M,N)$ inherits the intrinsic grading and we denote by $\mathrm{Tor}^{-i,2j}_{\Bbbk[m]}(M,N)$ the corresponding bigraded components.

Theorem 6.1 ([BP1, Theorem 4.2.1])**.** *There is an isomorphism of* $\Bbbk$*-algebras*

$$H^*(\mathcal{Z}_\mathcal{K};\Bbbk)\cong\mathrm{Tor}_{\Bbbk[v_1,\ldots,v_m]}\big(\Bbbk[\mathcal{K}],\Bbbk\big)$$

where the Tor *is viewed as a single-graded algebra with respect to the total degree.*

The Tor-algebra $\mathrm{Tor}_{\Bbbk[m]}(\Bbbk[\mathcal{K}],\Bbbk)$ can be computed either by resolving the $\Bbbk[m]$-module $\Bbbk$ and tensoring with $\Bbbk[\mathcal{K}]$, or by resolving the $\Bbbk[m]$-module $\Bbbk[\mathcal{K}]$ and tensoring with $\Bbbk$.

For the first approach, there is a standard resolution of the $\Bbbk[m]$-module $\Bbbk$, the *Koszul resolution.* It is defined as the acyclic differential graded algebra

$$\big(\Lambda[u_1,\ldots,u_m]\otimes\Bbbk[v_1,\ldots,v_m],d_\Bbbk\big),\quad d_\Bbbk=\sum_i\frac{\partial}{\partial u_i}\otimes v_i.$$

Here $\Lambda[u_1,\ldots,u_m]$ denotes the exterior algebra on the generators u_i of cohomological degree 1, or bidegree $(-1,2)$. After tensoring with $\Bbbk[\mathcal{K}]$ we obtain the *Koszul complex* $\big(\Lambda[u_1,\ldots,u_m]\otimes\Bbbk[\mathcal{K}],d_\Bbbk\big)$, whose cohomology is $\mathrm{Tor}_{\Bbbk[m]}(\Bbbk[\mathcal{K}],\Bbbk)$.

Furthermore, by [BP1, Lemma 4.2.5], the monomials v_i^2 and u_iv_i generate an acyclic ideal in the Koszul complex. The quotient algebra

$$R^*(\mathcal{K})=\Lambda[u_1,\ldots,u_m]\otimes\Bbbk[\mathcal{K}]\big/\big(v_i^2=u_iv_i=0,\ 1\leqslant i\leqslant m\big)\tag{6.1}$$

has a finite $\Bbbk$-basis of monomials $u_J\otimes v_I$ with $J\subset[m]$, $I\in\mathcal{K}$ and $J\cap I=\varnothing$. The algebra $R^*(\mathcal{K})$ is nothing but the cellular cochain complex of $\mathcal{Z}_\mathcal{K}$ (see Construction 2.4):

Theorem 6.2 ([BBP])**.** *There is an isomorphism of cochain complexes*

$$R^*(\mathcal{K}) \xrightarrow{\cong} \mathcal{C}^*(\mathcal{Z}_\mathcal{K}), \quad u_J \otimes v_I \mapsto \varkappa(J,I)^*$$

inducing the cohomology algebra isomorphism of Theorem 6.1*.*

Remark 6.3. The isomorphism of cochain complexes in the theorem above is by inspection. The result of [BBP] is that it induces an algebra isomorphism in cohomology. Also, the Koszul complex $\big(\Lambda[u_1,\dots,u_m]\otimes\Bbbk[\mathcal{K}],d_\Bbbk\big)$ itself can be identified with the cellular cochains of the polyhedral product $(S^\infty,S^1)^\mathcal{K}$; then taking the quotient by the acyclic ideal in (6.1) corresponds to the homotopy equivalence $\mathcal{Z}_\mathcal{K}=(D^2,S^1)^\mathcal{K}\xrightarrow{\simeq}(S^\infty,S^1)^\mathcal{K}$. See the details in [BP2, §4.5].

In the second approach, $\mathrm{Tor}_{\Bbbk[m]}(\Bbbk[\mathcal{K}],\Bbbk)$ is computed by resolving the $\Bbbk[m]$-module $\Bbbk[\mathcal{K}]$ and tensoring with $\Bbbk$. The *minimal resolution* has a disadvantage of not supporting a multiplicative structure. There is a nice non-minimal resolution, constructed in the 1966 PhD thesis of Diana Taylor. It has a natural multiplicative structure inducing the algebra isomorphism of Theorem 6.1. This *Taylor resolution* of $\Bbbk[\mathcal{K}]$ is defined in terms of the missing faces of $\mathcal{K}$ and is therefore convenient for calculations with higher Whitehead products. We describe the resolution and its coalgebraic version next.

Construction 6.4 (Taylor resolution)**.** Given a monomial ideal $(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$ in the polynomial algebra $\Bbbk[m]$, we define a free resolution of the $\Bbbk[m]$-module $\Bbbk[m]/(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$.

For each $s=0,\dots,t$, let F_s be a free $\Bbbk[m]$-module of rank $\binom{m}{s}$ with basis $\{e_J\}$ indexed by subsets $J\subset\{1,\dots,t\}$ of cardinality s. Define a morphism $d\colon F_s\to F_{s-1}$ by

$$d(e_J)=\sum_{j\in J}\mathrm{sign}(j,J)\frac{\mathfrak{m}_J}{\mathfrak{m}_{J\setminus j}}e_{J\setminus j},$$

where $\mathfrak{m}_J=\mathrm{lcm}_{j\in J}(\mathfrak{m}_j)$ and $\mathrm{sign}(j,J)=(-1)^{n-1}$ if j is the n-th element in the ordered set J. It can be verified that $d^2=0$. We therefore obtain a complex

$$T(\mathfrak{m}_1,\dots,\mathfrak{m}_t):\ 0\to F_t\to F_{t-1}\to\cdots\to F_1\to F_0\to 0.$$

By the theorem of D. Taylor, $T(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$ is a free resolution of the $\Bbbk[m]$-module $\Bbbk[m]/(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$. For the convenience of the reader, we include the proof of this result in the Appendix as Theorem A.1.

Next we describe the dualisation of the constructions above in the coalgebraic setting. The dual of $\Bbbk[v_1,\dots,v_m]$ is the symmetric coalgebra, which we denote by $\Bbbk\langle x_1,\dots,x_m\rangle$ or $\Bbbk\langle m\rangle$. It has a $\Bbbk$-basis consisting of monomials $\mathfrak{m}$, with the comultiplication defined by the formula

$$\Delta\mathfrak{m}=\sum_{\mathfrak{m}'\cdot\mathfrak{m}''=\mathfrak{m}}\mathfrak{m}'\otimes\mathfrak{m}''. \tag{6.2}$$

Given a set of monomials $\mathfrak{m}_1,\dots,\mathfrak{m}_t$ in the variables $x_1,\dots,x_m$, we define a subcoalgebra $C(\mathfrak{m}_1,\dots,\mathfrak{m}_t)\subset\Bbbk\langle x_1,\dots,x_m\rangle$ with a $\Bbbk$-basis of monomials $\mathfrak{m}$ that are not divisible by any of the $\mathfrak{m}_i$, $i=1,\dots,t$. The *face coalgebra* of a simplicial complex $\mathcal{K}$ is defined as

$$\Bbbk\langle\mathcal{K}\rangle=C\big(x_J\mid J\in\mathrm{MF}(\mathcal{K})\big).$$

The coalgebra $\Bbbk\langle\mathcal{K}\rangle$ has a $\Bbbk$-basis of monomials $\mathfrak{m}$ whose support is a face of $\mathcal{K}$, with the comultiplication given by (6.2).

Let Λ be a coalgebra, let A be a right Λ-comodule with the structure morphism $\nabla_A\colon A\to A\otimes\Lambda$, and let B be a left Λ-comodule with the structure morphism $\nabla_B\colon B\to\Lambda\otimes B$. The *cotensor product* of A and B is defined as the $\Bbbk$-comodule

$$A\boxtimes_\Lambda B=\ker\big(\nabla_A\otimes\mathbb{1}_B-\mathbb{1}_A\otimes\nabla_B\colon A\otimes B\to A\otimes\Lambda\otimes B\big).$$

When Λ is cocommutative, $A \boxtimes_\Lambda B$ is a Λ-comodule.

The n-th derived functor of $\cdot \boxtimes_\Lambda B$ is denoted by $\mathrm{Cotor}^n_\Lambda(A, B)$ or $\mathrm{Cotor}^\Lambda_{-n}(A, B)$. Namely, given an injective resolution $A \to I^\bullet$ with the resolvents indexed by nonnegative integers, we have

$$\mathrm{Cotor}^\Lambda_{-n}(A, B) = \mathrm{Cotor}^n_\Lambda(A, B) = H^n(I^\bullet \boxtimes_\Lambda B).$$

If $B \to J^\bullet$ is an injective resolution of B, then the standard argument using a bicomplex gives isomorphisms

$$\mathrm{Cotor}^n_\Lambda(A, B) = H^n(I^\bullet \boxtimes_\Lambda B) \cong H^n(I^\bullet \boxtimes_\Lambda J^\bullet) \cong H^n(A \boxtimes_\Lambda J^\bullet). \tag{6.3}$$

The isomorphism $H^n(I^\bullet \boxtimes_\Lambda B) \xrightarrow{\cong} H^n(A \boxtimes_\Lambda J^\bullet)$ can be described explicitly as follows.

Construction 6.5. Let $\eta \in H^n(I^\bullet \boxtimes_\Lambda B)$ be a homology class represented by a cycle $\eta^{(0)} \in I^n \boxtimes_\Lambda B$. We describe how to construct a cycle $\eta^{(n+1)} \in A \boxtimes_\Lambda J^n$ representing the same homology class in $\mathrm{Cotor}^n_\Lambda(A, B)$. Consider the bicomplex

$$\begin{array}{ccccccc}
A \boxtimes_\Lambda B & \longrightarrow & I^0 \boxtimes_\Lambda B & \longrightarrow & \dots & \longrightarrow & I^n \boxtimes_\Lambda B \\
\downarrow & & \downarrow & & & & \downarrow{\scriptstyle \eta^{(0)} \mapsto \partial_B(\eta^{(0)})} \\
A \boxtimes_\Lambda J^0 & \longrightarrow & I^0 \boxtimes_\Lambda J^0 & \longrightarrow & \dots & \xrightarrow{\eta^{(1)} \mapsto \partial_A(\eta^{(1)}) = \partial_B(\eta^{(0)})} & I^n \boxtimes_\Lambda J^0 \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \ddots & & \vdots \\
\downarrow & & \downarrow{\scriptstyle \eta^{(n)} \mapsto \partial_B(\eta^{(n)})} & & & & \\
A \boxtimes_\Lambda J^n & \xrightarrow{\eta^{(n+1)} \mapsto \partial_A(\eta^{(n+1)}) = \partial_B(\eta^{(n)})} & I^0 \boxtimes_\Lambda J^n & \longrightarrow & \dots & &
\end{array}$$

The rows and columns are exact by the injectivity of the comodules I^m and J^l. We have $\partial_A\bigl(\partial_B \eta^{(0)}\bigr) = -\partial_B\bigl(\partial_A \eta^{(0)}\bigr) = 0$. Hence, there exists $\eta^{(1)} \in I^{n-1} \boxtimes_\Lambda J^0$ such that $\partial_A \eta^{(1)} = \partial_B \eta^{(0)}$. Similarly, there exists $\eta^{(2)} \in I^{n-2} \boxtimes_\Lambda J^1$ such that $\partial_A \eta^{(2)} = \partial_B \eta^{(1)}$. Proceeding in this fashion, we arrive at an element $\eta^{(n+1)} \in A \boxtimes_\Lambda J^n$, which represents η by construction.

We apply this construction in the following setting. Here is the dual version of Theorem 6.1:

Theorem 6.6. *There is an isomorphism of* $\Bbbk$*-coalgebras*

$$H_*(\mathcal{Z}_\mathcal{K}; \Bbbk) \cong \mathrm{Cotor}^{\Bbbk\langle x_1, \dots, x_m \rangle}\bigl(\Bbbk\langle\mathcal{K}\rangle, \Bbbk\bigr).$$

The coalgebra $\mathrm{Cotor}^{\Bbbk\langle m \rangle}(\Bbbk\langle\mathcal{K}\rangle, \Bbbk)$ can be computed using the dual version of the Koszul resolution.

Construction 6.7 (Koszul complex of the face coalgebra)**.** The *Koszul resolution* for the $\Bbbk\langle m \rangle$-comodule $\Bbbk$ is defined as the acyclic differential graded coalgebra

$$\bigl(\Bbbk\langle x_1, \dots, x_m \rangle \otimes \Lambda\langle y_1, \dots, y_m \rangle, \partial_\Bbbk\bigr), \quad \partial_\Bbbk = \sum_i \frac{\partial}{\partial x_i} \otimes y_i.$$

After cotensoring with $\Bbbk\langle\mathcal{K}\rangle$ we obtain the *Koszul complex* $\bigl(\Bbbk\langle\mathcal{K}\rangle \otimes \Lambda\langle y_1, \dots, y_m \rangle, \partial_\Bbbk\bigr)$, whose homology is $\mathrm{Cotor}^{\Bbbk\langle m \rangle}(\Bbbk\langle\mathcal{K}\rangle, \Bbbk)$.

The relationship between the cellular chain complex of $\mathcal{Z}_\mathcal{K}$ and the Koszul complex of $\Bbbk\langle\mathcal{K}\rangle$ is described by the following dualisation of Theorem 6.2.

Theorem 6.8. *There is an inclusion of chain complexes*

$$\mathcal{C}_*(\mathcal{Z}_\mathcal{K}) \to \big(\Bbbk\langle\mathcal{K}\rangle \otimes \Lambda\langle y_1,\dots,y_m\rangle, \partial_\Bbbk\big), \quad \varkappa(J,I) \mapsto x_I \otimes y_J$$

inducing an isomorphism in homology:

$$H_*(\mathcal{Z}_\mathcal{K};\Bbbk) \cong H\big(\Bbbk\langle\mathcal{K}\rangle \otimes \Lambda\langle y_1,\dots,y_m\rangle, \partial_\Bbbk\big) = \mathrm{Cotor}^{\Bbbk\langle x_1,\dots,x_m\rangle}\big(\Bbbk\langle\mathcal{K}\rangle, \Bbbk\big).$$

On the other hand, $\mathrm{Cotor}^{\Bbbk\langle m\rangle}(\Bbbk\langle\mathcal{K}\rangle,\Bbbk)$ can be computed using the dual version of the Taylor resolution for the $\Bbbk\langle m\rangle$-comodule $\Bbbk\langle\mathcal{K}\rangle$.

Construction 6.9 (Taylor resolution for comodules)**.** Given a set of monomials $\mathfrak{m}_1,\dots,\mathfrak{m}_t$, we describe a cofree resolution of the $\Bbbk\langle m\rangle$-comodule $C(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$.

For each $s = 0,\dots,t$, let I^s be a cofree $\Bbbk\langle m\rangle$-comodule of rank $\binom{m}{s}$ with basis $\{e^J\}$ indexed by subsets $J \subset \{1,\dots,t\}$ of cardinality s. The differential $\partial\colon I^s \to I^{s+1}$ is defined by

$$\partial(x_1^{\alpha_1}\cdots x_m^{\alpha_m}e^J) = \sum_{j\notin J}\mathrm{sign}(j,J)\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{j\}}}e^{J\cup\{j\}}.$$

Here we assume that $\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{j\}}}$ is zero if it is not a monomial. The resulting complex

$$T'(\mathfrak{m}_1,\dots,\mathfrak{m}_t):\ 0 \to I^0 \to I^1 \to \cdots \to I^t \to 0$$

is called the *Taylor resolution* of the $\Bbbk\langle m\rangle$-comodule $C(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$. The proof that it is indeed a resolution is given in Theorem A.1.

Construction 6.10 (Taylor complex of the face coalgebra)**.** Let $\Bbbk\langle\mathcal{K}\rangle = C\big(x_J \mid J \in \mathrm{MF}(\mathcal{K})\big)$ be the face coalgebra of a simplicial complex $\mathcal{K}$. In this case it is convenient to view the s-th term I^s in the Taylor resolution as the cofree $\Bbbk\langle m\rangle$-comodule with basis consisting of exterior monomials $w_{J_1}\wedge\cdots\wedge w_{J_s}$, where $J_1,\dots,J_s$ are different missing faces of $\mathcal{K}$. The differential then takes the form

$$\partial_{\Bbbk\langle\mathcal{K}\rangle}(x_1^{\alpha_1}\cdots x_m^{\alpha_m}\cdot w_{J_1}\wedge\cdots\wedge w_{J_s}) = \sum_{J\ne J_1,\dots,J_s}\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}}{x_{(J_1\cup\cdots\cup J_s\cup J)\setminus(J_1\cup\cdots\cup J_s)}}\cdot w_J\wedge w_{J_1}\wedge\cdots\wedge w_{J_s}$$

(the sum is taken over missing faces $J\in\mathrm{MF}(\mathcal{K})$ different from $J_1,\dots,J_s$).

After cotensoring with $\Bbbk$ over $\Bbbk\langle m\rangle$ we obtain the *Taylor complex* of $\Bbbk\langle\mathcal{K}\rangle$ calculating $\mathrm{Cotor}^{\Bbbk\langle x_1,\dots,x_m\rangle}\big(\Bbbk\langle\mathcal{K}\rangle,\Bbbk\big)$. Its $(-s)$th graded component is a free $\Bbbk$-module with basis of exterior monomials $w_{J_1}\wedge\cdots\wedge w_{J_s}$, where $J_1,\dots,J_s$ are different missing faces of $\mathcal{K}$. The differential is given by

$$\partial_{\Bbbk\langle\mathcal{K}\rangle}(w_{J_1}\wedge\cdots\wedge w_{J_s}) = \sum_{J\subset J_1\cup\cdots\cup J_s} w_J\wedge w_{J_1}\wedge\cdots\wedge w_{J_s}$$

(the sum is over missing faces $J\subset J_1\cup\cdots\cup J_s$ different from any of the $J_1,\dots,J_s$).

We therefore have two methods of calculating $H_*(\mathcal{Z}_\mathcal{K}) = \mathrm{Cotor}^{\Bbbk\langle x_1,\dots,x_m\rangle}\big(\Bbbk\langle\mathcal{K}\rangle,\Bbbk\big)$: by resolving $\Bbbk$ (Koszul resolution) or by resolving $\Bbbk\langle\mathcal{K}\rangle$ (Taylor resolution). The two resulting complexes are related by the chain of quasi-isomorphisms (6.3) and Construction 6.5.

Example 6.11. Let $\mathcal{K}$ be the substitution complex $\partial\Delta\big(\partial\Delta(1,2,3),4,5\big)$, see Figure 1. After tensoring the Taylor resolution for $\mathbb{Z}\langle\mathcal{K}\rangle$ with $\mathbb{Z}$ we obtain the following complex:

$$\mathbb{Z} \xrightarrow[1\mapsto 0]{} \mathbb{Z}^4 \longrightarrow \mathbb{Z}^6 \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z}$$

$$\begin{aligned}
w_{123} &\mapsto 0\\
w_{145} &\mapsto 0\\
w_{245} &\mapsto 0\\
w_{345} &\mapsto 0
\end{aligned}$$

$$\begin{aligned}
w_{123}\wedge w_{145} &\mapsto \quad w_{123}\wedge w_{145}\wedge w_{245} + w_{123}\wedge w_{145}\wedge w_{345}\\
w_{123}\wedge w_{245} &\mapsto -w_{123}\wedge w_{145}\wedge w_{245} + w_{123}\wedge w_{245}\wedge w_{345}\\
w_{123}\wedge w_{345} &\mapsto -w_{123}\wedge w_{145}\wedge w_{345} - w_{123}\wedge w_{245}\wedge w_{345}\\
w_{145}\wedge w_{245} &\mapsto 0\\
w_{145}\wedge w_{345} &\mapsto 0\\
w_{245}\wedge w_{345} &\mapsto 0
\end{aligned}$$

$$\begin{aligned}
-w_{123}\wedge w_{145}\wedge w_{245}\wedge w_{345} &\leftarrow\!\shortmid\, w_{123}\wedge w_{145}\wedge w_{245}\\
w_{123}\wedge w_{145}\wedge w_{245}\wedge w_{345} &\leftarrow\!\shortmid\, w_{123}\wedge w_{145}\wedge w_{345}\\
-w_{123} \leftarrow\!\shortmid\, w_{145}\wedge w_{245}\wedge w_{345} &\leftarrow\!\shortmid\, w_{123}\wedge w_{245}\wedge w_{345}\\
w_{123}\wedge w_{145}\wedge w_{245}\wedge w_{345} &\leftarrow\!\shortmid\, w_{145}\wedge w_{245}\wedge w_{345}
\end{aligned}$$

We see that homology of this complex agrees with homology of the wedge $(S^5)^{\vee 4}\vee(S^6)^{\vee 3}\vee S^7\vee S^8$, in accordance with Example 5.4.

7. Higher Whitehead products and Taylor resolution

Given an iterated higher Whitehead product w, Lemma 3.2 gives a canonical cellular cycle representing the Hurewicz image of w. By Theorem 6.8, this cellular cycle can be viewed as a cycle in the Koszul complex calculating $\operatorname{Cotor}^{\Bbbk\langle m\rangle}\big(\Bbbk\langle\mathcal{K}\rangle,\Bbbk\big)$. Here we use Construction 6.5 to describe a canonical cycle representing an iterated higher Whitehead product w in the coalgebraic Taylor resolution. This gives a new criterion for the realisability of w.

Theorem 7.1. *Let w be a nested iterated higher Whitehead product*

$$w = \Big[\big[\ldots\big[[\mu_{i_{11}},\ldots,\mu_{i_{1p_1}}],\mu_{i_{21}},\ldots,\mu_{i_{2p_2}}\big],\ldots\big],\mu_{i_{n1}},\ldots,\mu_{i_{np_n}}\Big]. \tag{7.1}$$

Then the Hurewicz image $h(w)\in H_*(\mathcal{Z}_{\mathcal{K}}) = \operatorname{Cotor}^{\mathbb{Z}\langle m\rangle}(\mathbb{Z}\langle\mathcal{K}\rangle,\mathbb{Z})$ is represented by the following cycle in the Taylor complex of $\mathbb{Z}\langle\mathcal{K}\rangle$

$$\bigwedge_{k=1}^{n}\Bigg(\sum_{\substack{J\in\mathrm{MF}(\mathcal{K})\\ J\setminus\left(\bigcup\limits_{j=1}^{n-k} I_j\right)=I_{n-k+1}}} w_J\Bigg), \tag{7.2}$$

where $I_k=\{i_{k1},\ldots,i_{kp_k}\}$.

Proof. Recall from Construction 2.4 that for a given pair of non-intersecting index sets $I=\{i_1,\ldots,i_s\}$ and $J=\{j_1,\ldots,j_t\}$ we have a cell

$$\varkappa(J,I) = D_{i_1}\cdots D_{i_s}S_{j_1}\cdots S_{j_t}.$$

It belongs to $\mathcal{Z}_{\mathcal{K}}$ whenever $I\in\mathcal{K}$. Using this notation we can rewrite the canonical cellular chain $h_c(w)$ from Lemma 3.1 as follows:

$$h_c(w) = \prod_{k=1}^{n}\Bigg(\sum_{I\in\partial\Delta(I_k)}\varkappa\big(I_k\setminus I, I\big)\Bigg). \tag{7.3}$$

Here and below the sum is over maximal simplicies $I\in\partial\Delta(I_k)$ only (otherwise the right hand side above is not a homogeneous element).

Now we apply Construction 6.5 to (7.3). We obtain the following zigzag of elements in the bicomplex relating the Koszul complex with differential $\partial_{\mathbb{Z}}$ to the Taylor complex with differential $\partial_{\mathbb{Z}\langle\mathcal{K}\rangle}$:

$$\varkappa(\varnothing, I_1)\prod_{k=2}^{n}\Big(\sum_{I\in\partial\Delta(I_k)}\varkappa(I_k\setminus I, I)\Big) \xmapsto{\partial_{\mathbb{Z}}} \prod_{k=1}^{n}\Big(\sum_{I\in\partial\Delta(I_k)}\varkappa(I_k\setminus I, I)\Big)$$

$$\xmapsto{\partial_{\mathbb{Z}\langle\mathcal{K}\rangle}}$$

$$\varkappa(\varnothing, I_2)\prod_{k=3}^{n}\Big(\sum_{I\in\partial\Delta(I_k)}\varkappa(I_k\setminus I, I)\Big)w_{I_1} \xmapsto{\partial_{\mathbb{Z}}} \prod_{k=2}^{n}\Big(\sum_{I\in\partial\Delta(I_k)}\varkappa(I_k\setminus I, I)\Big)w_{I_1}$$

$$\xmapsto{\partial_{\mathbb{Z}\langle\mathcal{K}\rangle}}$$

$$\dots \xmapsto{\partial_{\mathbb{Z}}} \prod_{k=3}^{n}\Big(\sum_{I\in\partial\Delta(I_k)}\varkappa(I_k\setminus I, I)\Big)\Big(\sum_{(J\setminus I_1)=I_2} w_J\Big)\wedge w_{I_1}$$

It ends up precisely at element (7.2) in the Taylor complex. □

Example 7.2. Once again consider the complex $\mathcal{K} = \partial\Delta(\partial\Delta(1, 2, 3), 4, 5)$ shown in Figure 1. We have $\mathcal{Z}_{\mathcal{K}} \simeq (S^5)^{\vee 4} \vee (S^6)^{\vee 3} \vee S^7 \vee S^8$ by [Ab, Example 5.4], and each sphere is a Whitehead product. These Whitehead products together with the representing cycles in the Koszul and Taylor complexes are shown in Table 1 for each sphere.

Whitehead product	Koszul (cellular) cycle	Taylor cycle
$[\mu_1, \mu_2, \mu_3]$	$D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3$	w_{123}
$[\mu_1, \mu_4, \mu_5]$	$D_1D_4S_5 + D_1S_4D_5 + S_1D_4D_5$	w_{145}
$[\mu_2, \mu_4, \mu_5]$	$D_2D_4S_5 + D_2S_4D_5 + S_2D_4D_5$	w_{245}
$[\mu_3, \mu_4, \mu_5]$	$D_3D_4S_5 + D_3S_4D_5 + S_3D_4D_5$	w_{345}
$\big[[\mu_1, \mu_4, \mu_5], \mu_2\big]$	$(D_1D_4S_5 + D_1S_4D_5 + S_1D_4D_5)S_2$	$w_{245} \wedge w_{145}$
$\big[[\mu_1, \mu_4, \mu_5], \mu_3\big]$	$(D_1D_4S_5 + D_1S_4D_5 + S_1D_4D_5)S_3$	$w_{345} \wedge w_{145}$
$\big[[\mu_2, \mu_4, \mu_5], \mu_3\big]$	$(D_2D_4S_5 + D_2S_4D_5 + S_2D_4D_5)S_3$	$w_{345} \wedge w_{245}$
$\big[\big[[\mu_1, \mu_4, \mu_5], \mu_2\big]\mu_3\big]$	$(D_1D_4S_5 + D_1S_4D_5 + S_1D_4D_5)S_2S_3$	$(w_{123} + w_{345}) \wedge w_{245} \wedge w_{145}$
$\big[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5\big]$	$(D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3)(D_4S_5 + S_4D_5)$	$(w_{145} + w_{245} + w_{345}) \wedge w_{123}$

Table 1. Koszul and Taylor cycles representing Whitehead products

An important feature of the Taylor cycle (7.2) is that it has the form of a product of sums of generators w_J corresponding to missing faces, and the rightmost factor is a *single* generator w_{I_1}. This can be seen in the right column of Table 1. Below we give an example of a Taylor cycle which *does not* have this form. It corresponds to a sphere which *is not* a Whitehead product, although the corresponding $\mathcal{Z}_{\mathcal{K}}$ is a wedge of spheres. This example was discovered in [Ab, §7].

Example 7.3. Consider the simplicial complex

$$\begin{aligned}\mathcal{K} = \partial\Delta\bigl(\partial\Delta(1,2,3),4,5,6\bigr) \cup \Delta(1,2,3) \\ = \bigl(\partial\Delta(1,2,3) * \partial\Delta(4,5,6)\bigr) \cup \Delta(1,2,3) \cup \Delta(4,5,6).\end{aligned}$$

We have $\mathcal{Z}_\mathcal{K} \simeq (S^7)^{\vee 6} \vee (S^8)^{\vee 6} \vee (S^9)^{\vee 2} \vee S^{10}$, see [Ab, Proposition 7.1]. Here is the staircase diagram of Construction 6.5 relating the Koszul and Taylor cycles corresponding to S^{10}:

$$(D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3)(D_4D_5S_6 + D_4S_5D_6 + S_4D_5D_6)$$

$$D_1D_2D_3(D_4D_5S_6 + D_4S_5D_6 + S_4D_5D_6) \xmapsto{\partial_\mathbb{Z}} (D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3)(D_4D_5S_6 + D_4S_5D_6 + S_4D_5D_6)$$

$$D_1D_2D_3(D_4D_5S_6 + D_4S_5D_6 + S_4D_5D_6) \xmapsto{\partial_{\mathbb{Z}\langle\mathcal{K}\rangle}} (D_5S_6 + S_5D_6)w_{1234} + (D_4S_6 + S_4D_6)w_{1235} + (D_4S_5 + S_4D_5)w_{1236}$$

$$D_5D_6w_{1234} + D_4D_6w_{1235} + D_4D_5w_{1236} \xmapsto{\partial_\mathbb{Z}} (D_5S_6 + S_5D_6)w_{1234} + (D_4S_6 + S_4D_6)w_{1235} + (D_4S_5 + S_4D_5)w_{1236}$$

$$D_5D_6w_{1234} + D_4D_6w_{1235} + D_4D_5w_{1236} \xmapsto{\partial_{\mathbb{Z}\langle\mathcal{K}\rangle}} -(w_{1234} + w_{1235} + w_{1236}) \wedge (w_{1456} + w_{2456} + w_{3456})$$

We see that the Taylor cycle does not have a factor consisting of a single generator w_J. This reflects the fact that the sphere S^{10} in the wedge is not an iterated higher Whitehead product, see [Ab, Proposition 7.2].

Using the same argument as in the proof of Theorem 7.1, we can write down the Taylor cycle representing the Hurewicz image of an *arbitrary* iterated higher Whitehead product, not only a nested one. The general form of the answer is rather cumbersome though. Instead of writing a general formula, we illustrate it on an example.

Example 7.4. Consider the substitution complex $\mathcal{K} = \partial\Delta\bigl(\partial\Delta(1,2,3), \partial\Delta(4,5,6), 7, 8\bigr)$. By Theorem 5.1, it realises the Whitehead product $w = \bigl[[\mu_1,\mu_2,\mu_3],[\mu_4,\mu_5,\mu_6],\mu_7,\mu_8\bigr]$. From the description of the missing faces in Definition 4.1 we obtain

$$\begin{aligned}\mathrm{MF}(\mathcal{K}) = \bigl\{&\Delta(1,2,3), \Delta(4,5,6), \Delta(1,4,7,8), \Delta(1,5,7,8), \Delta(1,6,7,8), \\ &\Delta(2,4,7,8), \Delta(2,5,7,8), \Delta(2,6,7,8), \Delta(3,4,7,8), \Delta(3,5,7,8), \Delta(3,6,7,8)\bigr\}.\end{aligned}$$

Applying Construction 6.5 to the canonical cellular cycle

$$h_c(w) = (D_1D_2S_3 + D_1S_2D_3 + S_1D_2D_3)(D_4D_5S_6 + D_4S_5D_6 + S_4D_5D_6)(D_7S_8 + S_7D_8)$$

we obtain the corresponding cycle in the Taylor complex:

$$(w_{1478} + w_{1578} + w_{1678} + w_{2478} + w_{2578} + w_{2678} + w_{3478} + w_{3578} + w_{3678}) \wedge w_{456} \wedge w_{123}.$$

Appendix A. Proof of Taylor's theorem

Here we prove that the complex $T(\mathfrak{m}_1,\ldots,\mathfrak{m}_t)$ introduced in Construction 6.4 is a free resolution and the complex $T'(\mathfrak{m}_1,\ldots,\mathfrak{m}_t)$ from Construction 6.9 is a cofree resolution. In the case of modules, the argument was outlined in [Ei, Exercise 17.11] (see also [HH, Theorem 7.1.1]). The comodule case is obtained by dualisation.

Theorem A.1.

(*a*) *$T(\mathfrak{m}_1,\ldots,\mathfrak{m}_t)$ is a free resolution of the $\Bbbk[m]$-module $\Bbbk[m]/(\mathfrak{m}_1,\ldots,\mathfrak{m}_t)$.*

(*b*) *$T'(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$ is a cofree resolution of the $\Bbbk\langle m\rangle$-comodule $C(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$.*

Proof. Denote $\mathfrak{n}_i = \frac{\mathfrak{m}_i}{\gcd(\mathfrak{m}_i,\mathfrak{m}_t)}$. Then we have[1] $(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1} : \mathfrak{m}_t) = (\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1})$.

In the case of modules, there is a short exact sequence

$$0 \to \Bbbk[m]/(\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1}) \xrightarrow{\cdot\mathfrak{m}_t} \Bbbk[m]/(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1}) \to \Bbbk[m]/(\mathfrak{m}_1,\dots,\mathfrak{m}_t) \to 0.$$

Assume by induction that $T(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1})$ is a resolution. Consider the injective morphism

$$\varphi\colon \Bbbk[m]/(\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1}) \xrightarrow{\cdot\mathfrak{m}_t} \Bbbk[m]/(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1})$$

and the induced morphism of resolutions

$$\widetilde{\varphi}\colon T(\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1}) \to T(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1}).$$

The proof consists of three lemmata, proved separately below. By Lemma A.4, the complex $T(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$ can be identified with the cone of the morphism $\widetilde{\varphi}$. Then Lemma A.2 implies that $T(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$ is a resolution for $\Bbbk[m]/(\mathfrak{m}_1,\dots,\mathfrak{m}_t)$.

Similarly, in the comodule case we consider the short exact sequence of comodules

$$0 \to C(\mathfrak{m}_1,\dots,\mathfrak{m}_t) \to C(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1}) \xrightarrow{\cdot\frac{1}{\mathfrak{m}_t}} C(\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1}) \to 0,$$

use induction, and apply the lemmata below. □

Lemma A.2.

(*a*) *Let $\varphi\colon \bar V \to V$ be an injective morphism of modules. Let $\bar U_\bullet \to \bar V$ and $U_\bullet \to V$ be resolutions. Then the cone $C(\widetilde{\varphi})$ of the induced morphism of resolutions $\widetilde{\varphi}\colon \bar U_\bullet \to U_\bullet$ is a resolution for $V/\varphi(\bar V)$.*

(*b*) *Let $\varphi'\colon A \to \bar A$ be a surjective morphism of comodules. Let $A \to B^\bullet$ and $\bar A \to \bar B^\bullet$ be resolutions. Then the cocone $C'(\widetilde{\varphi}')$ of the induced morphism of resolutions $\widetilde{\varphi}'\colon B^\bullet \to \bar B^\bullet$ is a resolution for $\ker(\varphi'\colon A \to \bar A)$.*

Proof. Consider the homology long exact sequence associated with the cone $C(\widetilde{\varphi})$:

$$\begin{array}{ccccccccccc}
\cdots\longrightarrow & H_1(U_\bullet) & \longrightarrow & H_1(C(\widetilde{\varphi})) & \longrightarrow & H_0(\bar U_\bullet) & \longrightarrow & H_0(U_\bullet) & \longrightarrow & H_0(C(\widetilde{\varphi})) & \to 0\\
 & \| & & & & \| & & \| & & \| & \\
 & 0 & & & & \bar V & \xrightarrow{\varphi} & V & \longrightarrow & V/\varphi(\bar V) & \to 0
\end{array}$$

Injectivity of $\varphi\colon \bar V \to V$ implies that $H_1(C(\widetilde{\varphi})) = 0$. Vanishing of the higher homology groups $H_i\big(C(\widetilde{\varphi})\big)$, $i > 1$, follows from the exactness. Hence, $C(\widetilde{\varphi})$ is a resolution for $H_0(C(\widetilde{\varphi})) \cong V/\varphi(\bar V)$.

The comodule case is proved by straightforward dualisation. □

Lemma A.3.

(*a*) *The morphism $\widetilde{\varphi}\colon T(\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1}) \to T(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1})$ is given by*

$$\widetilde{\varphi}(\bar e_J) = \frac{\mathfrak{m}_{J\cup\{t\}}}{\mathfrak{m}_J} e_J, \qquad J \subset \{1,\dots,t-1\}.$$

(*b*) *The morphism $\widetilde{\varphi}'\colon T'(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1}) \to T'(\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1})$ is given by*

$$\widetilde{\varphi}'(x_1^{\alpha_1}\cdots x_m^{\alpha_m} e^J) = \frac{\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{t\}}} x_1^{\alpha_1}\cdots x_m^{\alpha_m} \bar e^J, \qquad J \subset \{1,\dots,t-1\}.$$

Proof. We need to show that the described maps commute with the differentials, as this property defines a morphism of resolutions uniquely.

For (a), denote $T(\mathfrak{n}_1,\dots,\mathfrak{n}_{t-1}) = \{\bar F_\bullet, \bar d\}$ and $T(\mathfrak{m}_1,\dots,\mathfrak{m}_{t-1}) = \{F_\bullet, d\}$. Recall that $F_\bullet$ has basis $\{e^J\}$ indexed by subsets $J \subset \{1,\dots,t-1\}$, and denote the corresponding basis elements of $\bar F_\bullet$ by $\bar e^J$. The required property follows by considering the diagram

[1]Given ideals $\mathcal{I},\mathcal{J}$ in a commutative ring R, the ideal quotient is defined as $(\mathcal{I} : \mathcal{J}) = \{f \in R \,|\, f\mathcal{J} \subset \mathcal{I}\}$.

$$\begin{array}{ccc}
\bar F_s & \xrightarrow{\quad\bar d\quad} & \bar F_{s-1}\\
& \bar e_J \stackrel{\bar d}{\longmapsto} \sum_{j\in J}\operatorname{sign}(j,J)\frac{\mathfrak n_J}{\mathfrak n_{J\setminus\{j\}}}\bar e_{J\setminus\{j\}} & \\
\Big\downarrow\widetilde\varphi & \widetilde\varphi\Big\downarrow \qquad\qquad \Big\downarrow\widetilde\varphi & \Big\downarrow\widetilde\varphi\\
& \sum_{j\in J}\operatorname{sign}(j,J)\frac{\mathfrak n_J}{\mathfrak n_{J\setminus\{j\}}}\frac{\mathfrak m_{(J\setminus\{j\})\cup\{t\}}}{\mathfrak m_{J\setminus\{j\}}}e_{J\setminus\{j\}} & \\
& \Big\| & \\
& \frac{\mathfrak m_{J\cup\{t\}}}{\mathfrak m_J}e_J \stackrel{d}{\longmapsto} \sum_{j\in J}\operatorname{sign}(j,J)\frac{\mathfrak m_J}{\mathfrak m_{J\setminus\{j\}}}\frac{\mathfrak m_{J\cup\{t\}}}{\mathfrak m_J}e_{J\setminus\{j\}} & \\
F_s & \xrightarrow{\quad d\quad} & F_{s-1}.
\end{array}$$

Here we used the identity

$$\frac{\mathfrak m_{J\cup\{t\}}}{\mathfrak m_{(J\setminus\{j\})\cup\{t\}}}=\frac{\mathfrak n_J}{\mathfrak n_{J\setminus\{j\}}},$$

which follows from the defintion of $\mathfrak n_i$.

Statement (b) is proved by dualisation. □

Lemma A.4. *Up to a sign in the differentials,*

(*a*) *the cone complex $C(\widetilde\varphi)$ is isomorphic to $T(\mathfrak m_1,\dots,\mathfrak m_t)$;*
(*b*) *the cocone complex $C'(\widetilde\varphi')$ is isomorphic to $T'(\mathfrak m_1,\dots,\mathfrak m_t)$.*

Proof. For (a), we denote $T(\mathfrak n_1,\dots,\mathfrak n_{t-1})=\{\bar F_\bullet,\bar d\}$, $T(\mathfrak m_1,\dots,\mathfrak m_{t-1})=\{F_\bullet,d\}$ and $T(\mathfrak m_1,\dots,\mathfrak m_t)=\{\widetilde F_\bullet,\widetilde d\}$.

We shall define a morphism $\psi\colon C(\widetilde\varphi)\to T(\mathfrak m_1,\dots,\mathfrak m_t)$, that is, $\psi\colon \bar F_s\oplus F_{s+1}\to\widetilde F_{s+1}$ commuting with the differentials. As $F_\bullet$ is a subcomplex of both $C(\widetilde\varphi)$ and $\widetilde F_\bullet$, we define ψ on $e_J\in F_{s+1}$ by $\psi(e_J)=\widetilde e_J$. Now we define ψ on $\bar e_J\in\bar F_s$ by the formula $\psi(\bar e_J)=\widetilde e_{J\cup\{t\}}$. The following diagram shows that the resulting map ψ indeed commutes with the differentials:

$$\begin{array}{ccc}
\bar F_s\oplus F_{s+1} & \xrightarrow{\quad d_{C(\widetilde\varphi)}\quad} & \bar F_{s-1}\oplus F_s\\
& \bar e_J \stackrel{\widetilde\varphi-\bar d}{\longmapsto} \frac{\mathfrak m_{J\cup\{t\}}}{\mathfrak m_J}e_J-\sum_{j\in J}\operatorname{sign}(j,J)\frac{\mathfrak n_J}{\mathfrak n_{J\setminus\{j\}}}\bar e_{J\setminus\{j\}} & \\
\Big\downarrow\psi & \psi\Big\downarrow \qquad\qquad \Big\downarrow\psi & \Big\downarrow\psi\\
& \pm\widetilde e_{J\cup\{t\}} \stackrel{\widetilde d}{\longmapsto} \frac{\mathfrak m_{J\cup\{t\}}}{\mathfrak m_J}\widetilde e_J\pm\sum_{j\in J}\operatorname{sign}(j,J)\frac{\mathfrak n_J}{\mathfrak n_{J\setminus\{j\}}}\widetilde e_{(J\setminus\{j\})\cup\{t\}} & \\
\widetilde F_{s+1} & \xrightarrow{\quad\widetilde d\quad} & \widetilde F_s.
\end{array}$$

Thus, ψ defines a morphism $C(\widetilde\varphi)\to T(\mathfrak m_1,\dots,\mathfrak m_t)$, which is clearly an isomorphism.

For (b), we use the notation $T'(\mathfrak n_1,\dots,\mathfrak n_{t-1})=\{\bar I^\bullet,\bar\partial\}$, $T'(\mathfrak m_1,\dots,\mathfrak m_{t-1})=\{I^\bullet,\partial\}$, and $T'(\mathfrak m_1,\dots,\mathfrak m_t)=\{\widetilde I^\bullet,\widetilde\partial\}$.

We define $\psi'\colon T'(\mathfrak{m}_1,\dots,\mathfrak{m}_t)\to C'(\widetilde{\varphi})$, that is, $\psi'\colon \widetilde{I}^s\to I^s\oplus\bar{I}^{s-1}$ by the formula

$$\psi'(x_1^{\alpha_1}\cdots x_m^{\alpha_m}\widetilde{e}^J)=\begin{cases}(-1^{|J|-1}x_1^{\alpha_1}\cdots x_m^{\alpha_m}\bar{e}^{J\setminus\{t\}}, & \text{for } t\in J,\\ (-1)^{|J|}x_1^{\alpha_1}\cdots x_m^{\alpha_m}e^J, & \text{for } t\notin J.\end{cases}$$

We need to check that ψ' commutes with the differentials. For $t\in J$ we have

$$\begin{array}{ccc} x_1^{\alpha_1}\cdots x_m^{\alpha_m}\widetilde{e}^J & \xmapsto{\ \widetilde{\partial}\ } & \sum\limits_{j\notin J}\operatorname{sign}(j,J)\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{j\}}}\widetilde{e}^{J\cup\{j\}} \\ \big\downarrow{\scriptstyle\psi'} & & \big\downarrow{\scriptstyle -\psi'} \\ (-1)^{|J|-1}x_1^{\alpha_1}\cdots x_m^{\alpha_m}\bar{e}^{J\setminus\{t\}} & \xmapsto{\ \bar{\partial}\ } & (-1)^{|J|-1}\sum\limits_{j\notin J}\operatorname{sign}(j,J)\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{n}_J}{\mathfrak{n}_{J\cup\{j\}}}\bar{e}^{J\cup\{j\}\setminus\{t\}}. \end{array}$$

For $t\notin J$ we have

$$\begin{array}{ccc} x_1^{\alpha_1}\cdots x_m^{\alpha_m}\widetilde{e}^J & \xmapsto{\ \widetilde{\partial}\ } & \sum\limits_{j\notin J,\, j\neq t}\operatorname{sign}(j,J)\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{j\}}}\widetilde{e}^{J\cup\{j\}}+(-1)^{|J|}\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{t\}}}\widetilde{e}^{J\cup\{t\}} \\ \big\downarrow{\scriptstyle\psi'} & & \big\downarrow{\scriptstyle\psi'} \\ x_1^{\alpha_1}\cdots x_m^{\alpha_m}e^J & \xmapsto{\ -\partial+\widetilde{\varphi}'\ } & -\sum\limits_{j\notin J,\, j\neq t}\operatorname{sign}(j,J)\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{j\}}}e^{J\cup\{j\}}+\frac{x_1^{\alpha_1}\cdots x_m^{\alpha_m}\mathfrak{m}_J}{\mathfrak{m}_{J\cup\{t\}}}\bar{e}^J; \end{array}$$

We therefore obtain the required isomorphism $\psi'\colon T'(\mathfrak{m}_1,\dots,\mathfrak{m}_t)\to C'(\widetilde{\varphi})$. □

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