TORIC TOPOLOGY AND COMPLEX COBORDISM

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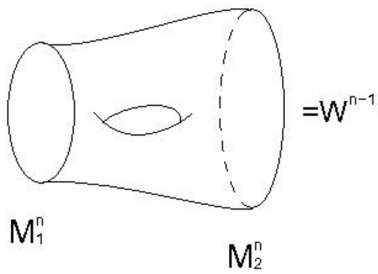
Abstract. We plan to discuss how the ideas and methodology of Toric Topology can be applied to one of the classical subjects of algebraic topology: finding nice representatives in complex cobordism classes. Toric and quasitoric manifolds are the key players in the emerging field of Toric Topology, and they constitute a sufficiently wide class of stably complex manifolds to additively generate the whole complex cobordism ring. In other words, every stably complex manifold is cobordant to a manifold with a nicely behaving torus action.

An informative setting for applications of toric topology to complex cobordism is provided by the combinatorial and convex-geometrical study of analogous polytopes. By way of application, we give an explicit construction of a quasitoric representative for every complex cobordism class as the quotient of a free torus action on a real quadratic complete intersection. The latter is a yet another disguise of the moment-angle manifold, another familiar object of toric topology. We suggest a systematic description for omnioriented quasitoric manifolds in terms of combinatorial data, and explain the relationship with non-singular projective toric varieties (otherwise known as toric manifolds).

Contents

Main Theorem . *Every complex cobordism class in* $\dim > 2$ *contains quasitoric representative.*



In cobordism theory, all manifolds are smooth and closed.

Complex cobordism.

complex manifolds $\subset$ almost complex $\subset$ stably (almost) complex manifolds

$$\tau M^n \oplus \mathbb{R}^N \xrightarrow{\text{complex bundle}} M$$

Quasitoric manifolds. manifold M^{2n} with "nice" T^n-action

- locally standard action
- The orbit space M^{2n}/T^n is a simple polytope.

Examples include projective smooth toric varieties and symplectic manifolds M^{2n} with Hamiltonian action of T^n.

1. Polytopes

$\mathbb{R}^n$: Euclidean vector space.
$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geq 0 \text{ for } 1 \leq i \leq m\}$, $\boldsymbol{a}_i \in \mathbb{R}^n, b_i \in \mathbb{R}$.
$H_i = \{\langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i = 0\}$, the ith bounding hyperplane.

Assume:

(1) $\dim P = n$;
(2) P is bounded.

Then P is called a (convex) n-dimensional *polytope*.

A *supporting hyperplane* H is characterised by the condition that P lies within one of the halfspaces determined by H.

A proper *face* of P is the intersection with a supporting hyperplane.
0-dim faces are *vertices*.
1-dim faces are *edges*.
$(n-1)$-dim faces are *facets*.
n-dim face is P.

Also assume:

(3) there are no redundant inequalities (cannot remove any inequality without changing P); then P has exactly m facets;
(4) bounding hyperplanes of P intersect in general position at every vertex; then there are exactly n facets of P meeting at each vertex.

Then P is a *simple* n-dim polytope with m facets.

The faces form a poset $\mathcal{L}(P)$ with respect to the inclusion. Two polytopes are said to be *combinatorially equivalent* if their face posets are isomorphic. The corresponding equivalence classes are called *combinatorial polytopes*.

Assume $|\boldsymbol{a}_i| = 1$. Then $\langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i$ is the distance from $\boldsymbol{x} \in \mathbb{R}^n$ to the ith hyperplane H_i.

2. Moment angle manifolds

P a simple polytope given as above, $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{in}), 1 \le i \le m$.

Set $A_P = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix} = (a_{ij})$ ($m \times n$-matrix), $\boldsymbol{b}_P = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. Then can write P as

$$P = \{\boldsymbol{x} \colon \ A_P\boldsymbol{x} + \boldsymbol{b}_P \ge 0\}.$$

Define $i_P(\boldsymbol{x}) = A_P\boldsymbol{x} + \boldsymbol{b}_P$, $\ i_P \colon \mathbb{R}^n \to \mathbb{R}^m$, so we have

$$\begin{array}{cccll} i_P: & \mathbb{R}^n & \to & \mathbb{R}^m & \\ & \cup & & \cup & \\ & P & \to & \mathbb{R}^m_{\ge} & = \{(y_1, \ldots, y_m) \colon \ y_i \ge 0\} \end{array}$$

$i_P(P)$ is the intersection of an n-dim affine plane in $\mathbb{R}^m$ with $\mathbb{R}^m_{\ge}$.
Consider the m-torus

$$T^m = \{(t_1, \ldots, t_m) = (e^{2\pi i\varphi_1}, \ldots, e^{2\pi i\varphi_m}) \in \mathbb{C}^m; \ \varphi_i \in \mathbb{R}\}.$$

Then $\mathbb{R}^m_{\ge}$ is the orbit space of the standard T^m-action on $\mathbb{C}^m$:

$$(t_1, \ldots, t_m) \cdot (z_1, \ldots, z_m) = (t_1z_1, \ldots, t_mz_m).$$

The orbit projection is

$$\begin{array}{ccc} \mathbb{C}^m & \to & \mathbb{R}^m_{\ge}, \\ (z_1, \ldots, z_m) & \mapsto & (|z_1|^2, \ldots, |z_m|^2). \end{array}$$

Now define the space $\mathcal{Z}_P$ from the pullback diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_{\mathcal{Z}}} & \mathbb{C}^m \\ \downarrow & & \downarrow \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\ge} \end{array}.$$

So $\mathcal{Z}_P$ is a T^m-space and $i_{\mathcal{Z}} : \mathcal{Z}_P \to \mathbb{C}^m$ is a T^m-equivariant embedding.

Example 2.1. $P^2 = \{x_1 \ge 0, \ x_2 \ge 0, \ -x_1 - x_2 + 1 \ge 0\}$ a triangle,

$$A_P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$i_P(\mathbb{R}^2) = \{A_P\boldsymbol{x} + \boldsymbol{b}_P\} = \{y_1 + y_2 + y_3 = 1\} \subset \mathbb{R}^3,$$

$$\begin{array}{ccc} \mathcal{Z}_P & \to & \mathbb{C}^3 \\ \downarrow & & \downarrow \\ P^2 & \to & \mathbb{R}^3 \end{array}, \qquad \mathcal{Z}_P = \{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \cong S^5.$$

Proposition 2.2. *$\mathcal{Z}_P$ is a smooth T^m-manifold with the canonical trivialisation of the normal bundle of $i_{\mathcal{Z}} : \mathcal{Z}_P \to \mathbb{C}^m$.*

Idea of proof.
1. Write the image $i_P(\mathbb{R}^n) \subset \mathbb{R}^m$ as the set of common solutions of $(m - n)$ linear equations in y_i, $\ 1 \le i \le m$.
2. Replace y_i's by $|z_i|^2$'s to get a representation of $\mathcal{Z}_P$ as an intersection of $(m - n)$ real quadratic hypersurfaces.
3. Check that (2) is a "complete" intersection, i.e. the gradients are linearly independent at each point of $\mathcal{Z}_P$.

□

In the presentation of P, let us fix $\boldsymbol{a}_i$, $1 \le i \le m$, but allow for b_i's to change. Let us consider "virtual polytopes" analogous to P ("analogous" here means "keep $\boldsymbol{a}_i$'s, change b_i's"), so

virtual polytope = arrangement of half-spaces.

Let $\mathbb{R}(P)$ be the space of virtual polytopes analogous P.

$$\begin{array}{rccll} \kappa\colon & \mathbb{R}^m & \to & \mathbb{R}(P) & \text{an isomorphism,} \\ & \boldsymbol{b}_P + \boldsymbol{h} & \mapsto & P(\boldsymbol{h}) & := \{\boldsymbol{x}\colon\ A_P\boldsymbol{x} + \boldsymbol{b}_P + \boldsymbol{h} \ge 0\} \end{array}$$

Remark 2.3. Sum in $\mathbb{R}^m$ corresponds to Minkowski sum of polytopes in $\mathbb{R}(P)$.

Now define

$$\chi_P = \kappa \circ i_P : \mathbb{R}^n \to \mathbb{R}(P).$$

So $\chi_P(\boldsymbol{y})$ is the polytope congruent to P obtained by translating the origin to $\boldsymbol{y} \in \mathbb{R}^n$. Indeed, $i_P(\boldsymbol{y}) = A_P\boldsymbol{y} + \boldsymbol{b}_P$ and $\chi_P(\boldsymbol{y}) = P(A_P\boldsymbol{y}) = \{\boldsymbol{x}\colon A_P\boldsymbol{x} + \boldsymbol{b}_P + A_P\boldsymbol{y} \ge 0\} = P - \boldsymbol{y}$.

Assume that the first n facets of P meet at a vertex v_1, called the *initial vertex*. So $H_1 \cap \dots \cap H_n = v_1$ in P, and therefore $(H_1 - \boldsymbol{h}) \cap \dots \cap (H_n - \boldsymbol{h}) = v_1(\boldsymbol{h})$ is the initial vertex of $P(\boldsymbol{h})$. Denote

$$d_i(\boldsymbol{h}) = \text{distance between } v_1(\boldsymbol{h}) \text{ and } H_i + \boldsymbol{h},$$

so $d_i(\boldsymbol{h}) = 0$ for $1 \le i \le n$. Define $C\colon \mathbb{R}^m \to \mathbb{R}^{m-n}$ by

$$C(\boldsymbol{b}_P + \boldsymbol{h}) = (d_{n+1}(\boldsymbol{h}), \dots, d_m(\boldsymbol{h})).$$

In other words,

$$\begin{array}{rccc} C: & \mathbb{R}(P) & \to & \mathbb{R}^{m-n}, \\ & P(\boldsymbol{h}) & \mapsto & (d_{n+1}(\boldsymbol{h}), \dots, d_m(\boldsymbol{h})) \end{array}$$

Claim 1. *The sequence* $0 \to \mathbb{R}^n \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \to 0$ *is exact.*

Proof. Use the fact that d_i are metric invariants, so they take the same values on congruent polytopes. ☐

In what follows assume $\boldsymbol{a}_i = \boldsymbol{e}_i$ for $1 \le i \le n$; so we have

$$A_P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ a_{n+1,1} & & \dots\dots & a_{n+1,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & & \dots\dots & a_{m,n} \end{pmatrix} = (a_{ij}).$$

Example 2.4. $\kappa\colon \mathbb{R}^m \to \mathbb{R}(P)$ maps the basis vector $\boldsymbol{e}_j$ to the virtual polytope $P(-\boldsymbol{b}_P + \boldsymbol{e}_j) =: P_j$; then

$$d_i(P_j) = \begin{cases} -a_{i,j} & \text{if } 1 \le j \le n, \\ \delta_{ij} & \text{if } n+1 \le j \le m, \end{cases} \qquad \text{for } n+1 \le i \le m,$$

and C is given by the $(m-n) \times m$ matrix

$$C = (c_{ij}) = \begin{pmatrix} -a_{n+1,1} & \dots & -a_{n+1,n} & 1 & 0 & \dots & 0 \\ -a_{n+2,1} & \dots & -a_{n+2,n} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -a_{m,1} & \dots & -a_{m,n} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Proof of Proposition 2.2. Step (1). We can write

$$\begin{aligned} i_P(\mathbb{R}^n) &= \{\boldsymbol{y} \in \mathbb{R}^m \colon \boldsymbol{y} = A_P\boldsymbol{x} + \boldsymbol{b}_P \text{ for some } \boldsymbol{x} \in \mathbb{R}^n\} \\ &= \{\boldsymbol{y} \colon C\boldsymbol{y} - C\boldsymbol{b}_P = 0\} \\ &(m-n \text{ linear equations in } \boldsymbol{y} \in \mathbb{R}^m). \end{aligned}$$

Step (2). Then

$$\mathcal{Z}_P = \{\boldsymbol{z} \in \mathbb{C}^m \colon \sum_{k=1}^m c_{jk}(|z_k|^2 - b_k) = 0, \quad 1 \le j \le m-n\}$$

Step (3). Now we want to check that the gradients in the presentation of $\mathcal{Z}_P$ in Step (2) are linearly independent at each point. Write $z_k = q_k + \sqrt{-1}r_k$; then the gradients are given by

$$2(c_{j1}q_1, c_{j1}r_1, \ldots, c_{jm}q_m, c_{jm}r_m), \quad 1 \le j \le m-n.$$

So the gradients form the rows of the $(m-n) \times 2m$ matrix $2CR$, where

$$R = \begin{pmatrix} q_1 & r_1 & 0 & \ldots\ldots\ldots\ldots & & & 0 \\ 0 & 0 & q_2 & r_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots\ldots\ldots\ldots & & & 0 & q_m & r_m \end{pmatrix} \quad m \times 2m \text{ matrix}$$

Assume that $q_{j_1} = r_{j_1} = \cdots = q_{j_k} = r_{j_k} = 0$ at $\boldsymbol{z} \in \mathcal{Z}_P$ so that $(z_{j_1} = \cdots = z_{j_k} = 0)$. Then the corresponding facets $F_{j_1}, \ldots, F_{j_k}$ of P intersect nontrivially. The condition $CA_P = 0$ guarantees that the submatrix obtained form C by deleting the columns $\boldsymbol{c}_{j_1}, \ldots, \boldsymbol{c}_{j_k}$ has rank $m-n$. Then rank of $2CR$ is also $m-n$. □

$\mathcal{Z}_P$ is called the *moment angle manifold* corresponding to P.

Remark 2.5. It can be proved that the equivariant smooth structure on $\mathcal{Z}_P$ depends only on the combinatorial type of P.

Summary (reminder). Given a simple polytope

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \ge 0 \text{ for } 1 \le i \le m\},\ \boldsymbol{a}_i \in \mathbb{R}^n,\ b_i \in \mathbb{R}$$

with m facets

$$F_i = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i = 0\} \cap P, \quad 1 \le i \le m.$$

The facets are *finely ordered*, i.e.

$$F_1 \cap \cdots \cap F_n = v_1 \quad \text{the } \textit{initial vertex}$$

May specify P by the matrix inequality $A_P\boldsymbol{x} + \boldsymbol{b}_P \ge 0$, where

$A_P \colon m \times n$ matrix of row vectors $\boldsymbol{a}_i$,

$\boldsymbol{b}_P \in \mathbb{R}^m \colon$ column vector of scalar b_i

The intersection of the affine subspace $A_P(\mathbb{R}^n) + \boldsymbol{b}_P$ with the positive cone $\mathbb{R}^m_\ge$ is a copy of P in $\mathbb{R}^m$:

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m,\ i_P(\boldsymbol{x}) = A_P\boldsymbol{x} + \boldsymbol{b}_P \quad \text{affine, injective}$$

$$\begin{array}{ccc} \text{moment angle manifold } \mathcal{Z}_P & \stackrel{i_Z}{\hookrightarrow} & \mathbb{C}^m \\ \downarrow & & \downarrow\rho \\ P & \stackrel{i_P}{\to} & \mathbb{R}^m_\ge \end{array}$$

$$\rho((z_1, \ldots, z_m)) = (|z_1|^2, \ldots, |z_m|^2).$$

We want to describe the isotropy subgroups of points of $\mathcal{Z}_P$ with respect to the T^m-action. We may write

$$T^m = \prod_{i=1}^{m} T_i,$$

where $T_i := \{(1,\dots,1,t,1,\dots,1)\} \subset T^m$ is the *i-th coordinate subcircle*. Given a multiindex $I = \{i_1,\dots,i_k\} \subset [m] = \{1,2,\dots,m\}$, define the corresponding *coordinate subgroup* of T^m as

$$T_I := \prod_{i\in I} T_i \subset T^m.$$

Now take $\boldsymbol{z} \in \mathbb{C}^m$. Its *isotropy subgroup* with respect to the coordinatewise T^m-action is

$$T^m_{\boldsymbol{z}} = \{\boldsymbol{t} \in T^m \colon \boldsymbol{t}\cdot\boldsymbol{z} = \boldsymbol{z}\} \subset T^m.$$

It is easy to see that

$$T^m_{\boldsymbol{z}} = T_{\omega(\boldsymbol{z})}$$

where $\omega(\boldsymbol{z}) = \{i \in [m] \colon z_i = 0\} \subset [m]$. Obviously, every coordinate subgroup of T^m arises as $T_{\omega(\boldsymbol{z})}$ for some $\boldsymbol{z} \in \mathbb{C}^m$. However not every coordinate subgroup of T^m arises as the isotropy subgroup for some $\boldsymbol{z} \in \mathcal{Z}_P$.

The isotropy subgroups of the T^m-action on $\mathcal{Z}_P$ are described as follows. Given $p \in P$, set

$$F(p) := \bigcap_{p\in F_i} F_i.$$

It is the unique face of P containing p in its relative interior. Note

- if p is a vertex, then $F(p) = p$;
- if $p \in intP$, then $F(p) = P$.

Now set

$$T(p) = \prod_{p\in F_i} T_i \subset T^m.$$

Note that $0 \le \dim T(p) \le n$ ($\because P^n$ is simple).

Now if $\boldsymbol{z} \in \mathcal{Z}_P$, then $\rho(\boldsymbol{z}) \in P$, and

$$T^m_{\boldsymbol{z}} = T(\rho(\boldsymbol{z})).$$

3. Quasitoric manifolds

Assume given P as above, and an $n \times m$ matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & \lambda_{n,m+1} & \dots & \lambda_{n,m} \end{pmatrix} = (I_n, \Lambda_*),$$

$I_n : n \times n$ unit matrix,

$\Lambda_* : n \times (m-n)$ matrix,

satisfying

(∗) the columns $\lambda_{j_1}, \dots, \lambda_{j_n}$ corresponding to any vertex $F_{j_1} \cap \dots \cap F_{j_n}$ of P form a basis for $\mathbb{Z}^n$.

Definition 3.1. A *combinatorial quasitoric pair* is (P, Λ) as above.

We may view Λ as a homomorphism $T^m \to T^n$. Now set

$$K(\Lambda) = \ker(T^m \xrightarrow{\Lambda} T^n) \cong T^{m-n}.$$

Proposition 3.2. *$K(\Lambda)$ acts freely on $\mathcal{Z}_P$.*

Proof. The map $\Lambda\colon T^m \to T^n$ is injective when restricted to $T(p)$, for all $p \in P$. Therefore, $K(\Lambda)$ meets every isotropy subgroup of the T^m-action on $\mathcal{Z}_P$ trivially. □

Definition 3.3. The quotient

$$M(P, \Lambda) := \mathcal{Z}_P / K(\Lambda)$$

is the *quasitoric manifold* corresponding to (P, Λ). The $2n$-dimensional manifold $M = M(P, \Lambda)$ has a $T^n \cong T^m/K(\Lambda)$-action which satisfies the two *Davis–Januszkiewicz conditions*:

(a) the T^n-action $\alpha\colon T^n \times M^{2n} \to M^{2n}$ is *locally standard*, or locally isomorphic to the standard coordinatewise representation of T^n in $\mathbb{C}^n$. More precisely, every $\boldsymbol{x} \in M$ is contained in a T^n-invariant neighborhood $U(\boldsymbol{x}) \subset M$ for which there is a T^n-invariant subset $W \subset \mathbb{C}^n$, an automorphism $\theta\colon T^n \to T^n$, and a homeomorphism $f : U(\boldsymbol{x}) \to W$ satisfying $f(\boldsymbol{ty}) = \theta(\boldsymbol{t}) f(\boldsymbol{y})$ for all $\boldsymbol{t} \in T^n$, $\boldsymbol{y} \in U(\boldsymbol{x})$.

(b) there is a projection $\pi : M \to P$ whose fibres are orbits of α.

It follows from the construction that M is canonically smooth.

Question 3.4 (open)**.** Unlike $\mathcal{Z}_P$, we don't know whether the equivariant smooth structure on M is unique.

Example 3.5. Assume that the initial vertex v_1 is the origin, and the first n normal vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ form the *standard* basis of $\mathbb{R}^n$. (We can always achieve this by applying an affine transformation). Then

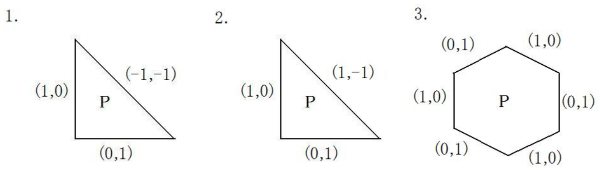
$$A_P^t = \begin{pmatrix} 1 & 0 & \dots & 0 & a_{n+1,1} & \dots & a_{m,1} \\ 0 & 1 & \dots & 0 & a_{n+1,2} & \dots & a_{m,2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & a_{n+1,n} & \dots & a_{m,n} \end{pmatrix}$$

has the same form as Λ, although with real (rather than integer) matrix elements. We can always achieve that P has integral coordinates of vertices without changing its combinatorial type. So we may assume $a_{ij} \in \mathbb{Z}$. However, condition (∗) on the minors of Λ is more severe: there are combinatorial polytopes with no integral realisation satisfying $(*)$. But if you can realise P so that A_P^t satisfies $(*)$, then

$$M(P) = \mathcal{Z}_P / K(A_P^t)$$

is the *projective toric variety* corresponding to P.

Example 3.6.



1. $\Lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = A_P^t$, and $K(\Lambda) = \langle (t,t,t) \rangle \subset T^3$, the diagonal subcircle. Then

$$M(P) = \mathcal{Z}_P / K(\Lambda) = S^5/S^1 \cong \mathbb{C}P^2.$$

The T^2-action is given by

$$(t_1, t_2) \cdot (z_0 : z_1 : z_2) = (z_0 : t_1 z_1 : t_2 z_2)$$

2. $\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, and $M(P) = \overline{\mathbb{C}P^2}$ (the standard orientation is reversed). The action is

$$(t_1, t_2) \cdot (z_0 : z_1 : z_2) = (z_0 : t_1 z_1 : t_2^{-1} z_2)$$

3. $\Lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$, $M \cong (S^2 \times S^2) \# (S^2 \times S^2)$.

The T^n-action on M is free over the interior $intP = P^\circ$.

$$\boldsymbol{p} \in P^\circ, \quad \pi^{-1}(\boldsymbol{p}) = (\boldsymbol{p}, \boldsymbol{t}), \quad \pi\colon M \to P.$$

We orient M using the decomposition

$$\tau_{(\boldsymbol{p},\boldsymbol{t})} M \cong \tau_{\boldsymbol{p}} P \oplus \tau_{\boldsymbol{t}} T^n$$

by insisting that $(\xi_1, \eta_1, \dots, \xi_n, \eta_n)$ is a positive basis of $\tau_{(\boldsymbol{p},\boldsymbol{t})} M$ whenever

$$(\xi_1, \dots, \xi_n) > 0 \text{ in } \tau_{\boldsymbol{p}} P = \mathbb{R}^n \text{ and } (\eta_1, \dots, \eta_n) > 0 \text{ in } \tau_{\boldsymbol{t}} T^n.$$

This is similar to orienting $\mathbb{C}^n$ by the basis $(\boldsymbol{e}_1, i\boldsymbol{e}_1, \dots, \boldsymbol{e}_n, i\boldsymbol{e}_n)$.

Corollary 3.7. *M is canonically oriented by the orientations of P and T^n.*

The *facial* (or *characteristic*) submanifolds of M are defined as

$$M_i := \pi^{-1}(F_i) = \mathcal{Z}_{F_i}/K \quad \text{for } 1 \le i \le m.$$

$\mathcal{Z}_{F_i}$ is the fixed point set of $\mathcal{Z}_P$ with respect to the action of $T_i \subset T^m$. So $M_i \subset M$ is fixed by the circle subgroup $\Lambda(T_i) \subset T^n$ determined by the ith column of $\Lambda\colon T^m \to T^n$.

Let $\mathbb{C}_i$ denote the 1-dim complex T^m-representation defined via the quotient projection $\mathbb{C}^m \to \mathbb{C}_i$ onto the ith factor. Define

$$\mathcal{Z}_P \times_K \mathbb{C}_i = \{(\boldsymbol{z}, w)\colon \boldsymbol{z} \in \mathcal{Z}_P,\ w \in \mathbb{C}_i\} /\!\sim,$$
$$(\boldsymbol{z}, w) \sim (\boldsymbol{z}t^{-1}, tw) \text{ for every } t \in K.$$

Then we have a complex line bundle

$$\rho_i\colon \mathcal{Z}_P \times_K \mathbb{C}_i \to M$$

over M whose restriction to M_i is the normal bundle of the inclusion $M_i \hookrightarrow M$.

Definition 3.8. The *ominiorientation* of M is a choice of orientation for M and for every M_i, $1 \le i \le m$.

By the above considerations, (P, Λ) determines a canonical omniorientation for $M(P, \Lambda)$.

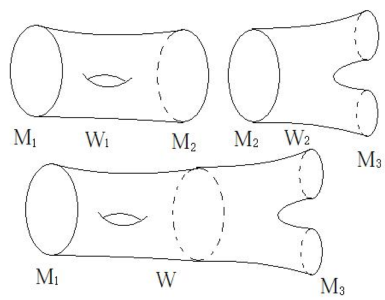
4. Cobordism theories

4.1. **General notion of cobordism.** All manifolds are closed, smooth.

Definition 4.1. M_1^n and M_2^n are *(co)bordant* (notation: $M_1^n \sim M_2^n$) if there exists a manifold W^{n+1} with boundary such that $\partial W^{n+1} = M_1 \sqcup M_2$.

Proposition 4.2. $\sim$ *is an equivalence relation.*

Proof.
(1) $M \sim M$. Indeed, $W = M \times [0, 1]$;
(2) $M_1 \sim M_2 \Rightarrow M_2 \sim M_1$ obvious;
(3) $M_1 \sim M_2$ & $M_2 \sim M_3 \Longrightarrow M_1 \sim M_3$.



□

Denote by $[M]$ the cobordism equivalence class of M.
$\Omega_n^O = \{[M^n]\}$ the set of cobordism classes of n-dimensional manifolds.

Proposition 4.3. Ω_n^O *an abelian group with respect to* $[M_1^n] + [M_2^n] = [M_1^n \sqcup M_2^n]$.

Proof. Zero is the cobordism class of an empty set, $-[M] = [M]$. □

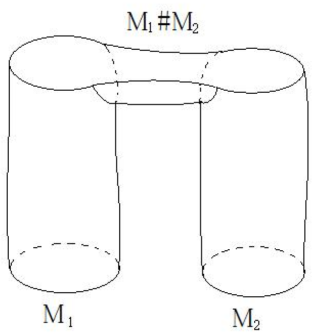
In particular, Ω_n^O is a 2-torsion.
Set $\Omega_*^O := \bigoplus_{n \geq 0} \Omega_n^O$.

Proposition 4.4. Ω_*^O *is a ring with respect to* $[M_1] \times [M_2] = [M_1 \times M_2]$.

Ω_*^O is called the *unoriented (co)bordism ring* (in fact, it is a $\mathbb{Z}/2$-algebra).

4.2. **Oriented cobordism.** Now all manifolds are oriented.
$M_1^n \sim M_2^n$ if there is an oriented W^{n+1} such that $\partial W = M_1 \sqcup \overline{M_2}$ where $\overline{M_2}$ denotes M_2 with orientation reversed.
Ω_*^{SO} is defined in the same way as Ω_*^O except $-[M] = [\overline{M}]$. So Ω_*^{SO} is no longer a 2-torsion! It is a $\mathbb{Z}$-algebra.

Remark 4.5. $[M_1] + [M_2] = [M_1 \# M_2]$. In other words, $M_1 \sqcup M_2 \sim M_1 \# M_2$.



Example 4.6.

1. $\Omega_0^O \cong \mathbb{Z}/2$ (with two cobordism classes $\emptyset$ and $\cdot = pt$).
2. $\Omega_1^O = 0$ (every 1-manifold bounds).
3. $\Omega_2^O \cong \mathbb{Z}/2$ with generator $[\mathbb{R}P^2]$;

   $2[\mathbb{R}P^2] = [\mathbb{R}P^2 \# \mathbb{R}P^2] = [K^2] = 0.$

   Here K^2 is the Klein bottle (it bounds).
4. $\Omega_3^O \cong 0$ elementary, but hard. Established by Rohlin in 1951.
5. Ω_*^O was completely calculated by Thom in 1954 using algebraic and homotopy methods.

Example 4.7.

1. $\Omega_0^{SO} \cong \mathbb{Z}$. The generator is $[pt]$.
2. $\Omega_1^{SO} = 0$.
3. $\Omega_2^{SO} = 0$ (every oriented 2-manifold bounds).
4. $\Omega_3^{SO} = 0$ by Rohlin.
5. $\Omega_4^{SO} \cong \mathbb{Z}$ with generator $[\mathbb{C}P^2]$; hard.
6. Ω_*^{SO} was completely calculated by the efforts of several people by 1960.

Exercise 4.8. $\mathbb{R}P^{2n+1}$, $\mathbb{C}P^{2n+1}$ *bound.*

4.3. **Complex cobordism.** Idea: try to work with complex manifolds. This runs into a complication as W cannot be complex. The remedy is to consider complex structures on M up to "stabilisation", i.e. assume chosen a real bundle isomorphism

$$c_\tau\colon \tau(M) \oplus \mathbb{R}^k \to \xi$$

where $\tau(M)$ denotes the tangent bundle, $\mathbb{R}^k$ a trivial real k-plane bundle over M, and ξ a complex bundle over M.

Definition 4.9. A *(tangentially) stably complex manifold* is an equivalence class of pairs (M, c_τ) as above, where $(M, c_\tau) \sim (M, c_{\tau'})$ if there are some m, m' and a complex bundle isomorphism $\xi \oplus \mathbb{C}^m \to \xi' \oplus \mathbb{C}^{m'}$ such that the composition

$$\begin{array}{ccc} \tau(M) \oplus \mathbb{R}^k \oplus \mathbb{C}^m & \xrightarrow{c_\tau \oplus id} & \xi \oplus \mathbb{C}^m \\ & & \downarrow\cong \\ \tau(M) \oplus \mathbb{R}^{k'} \oplus \mathbb{C}^{m'} & \xrightarrow{c_{\tau'} \oplus id} & \xi' \oplus \mathbb{C}_{m'} \end{array}$$

is an isomorphism of real bundles.

FACT 1. We can do cobordism with tangentially stably complex manifolds. The opposite element in the resulting cobordism group is given by

$$-[M, c_\tau] := [M, \overline{c}_\tau]$$

where $\overline{c}_\tau\colon \tau(M^n) \oplus \mathbb{R}^k \to \overline{\xi}$ (the *conjugate* stably complex structure).

If M is an (almost) complex manifold then it has the *canonical* tangentially stably complex structure $id = c_\tau\colon \tau(M) \to \tau(M)$.

Example 4.10. $M = \mathbb{C}P^1$. Then we have a complex bundle isomorphism

$$\alpha\colon \tau(\mathbb{C}P^1) \oplus \mathbb{C} \cong \overline{\eta} \oplus \overline{\eta}$$

where η is the Hopf line bundle. So $[\mathbb{C}P^1, \alpha]$ is the canonical stably complex structure. The opposite element $-[\mathbb{C}P^1, \alpha]$ is determined by the real bundle isomorphism

$$\tau(\mathbb{C}P^1) \oplus \mathbb{R}^2 \to \eta \oplus \eta.$$

Finally, the real bundle isomorphism

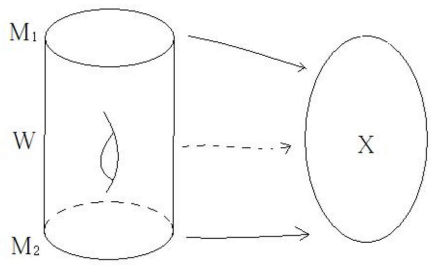
$$\beta : \tau(\mathbb{C}P^1) \oplus \mathbb{R}^2 \to \eta \oplus \overline{\eta} \cong \mathbb{C}^2$$

gives rise to the trivial stably complex structure on $\mathbb{C}P^1$.

FACT 2. $\Omega_2^U \cong \mathbb{Z}$, genetated by $[\mathbb{C}P^1]$.

4.4. **Generalised (co)homology theories.**

Definition 4.11. Let X be a "good" topological space. Define $O_n(X)$ as the set cobordism classes of maps $M^n \to X$, where $(M_1 \to X) \sim (M_2 \to X)$ if there is W such that $\partial W = M_1 \sqcup M_2$ and the map $M_1 \sqcup M_2 \to X$ extends to W:



$O_*(X)$ satisfied 3 of 4 Steenrod axioms for homology theory. It is

- homotopy invariant;
- has exact sequences of pairs;
- has the excision axiom.

But $O_*(pt) = \Omega_*^O$. The forth Steenrod axiom fails. So $O_*(X)$ gives rise to a *generalised homology theory.*
We can also define the "cohomology theory" $O^*(X)$, with

$$O^*(pt) = O_{-*}(pt).$$

In other words, $\Omega_O^* = \Omega_{-*}^O$.
Other (co)bordism theories $SO_*(X)$, $SO^*(X)$, $U_*(X)$, $U^*(X)$ are treated similarly.
Another common notation: use $MO^*(X)$, $MSO^*(X)$, etc. instead of $O^*(X)$, $SO^*(X)$, etc.

4.5. **Main results on cobordism.**

$$\begin{aligned} O : M,\quad & w(\tau M) = 1 + w_1(\tau M) + w_2(\tau M) + \dots && \text{total Stiefel–Whitney class} \\ SO : M,\quad & p(\tau M) = 1 + p_1(\tau M) + p_2(\tau M) + \dots && \text{total Pontrjagin class} \\ U : (M, c_\tau, \xi),\quad & c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \dots && \text{total Chern class of } \xi \end{aligned}$$

Given a sequence $\omega = (i_1, i_2, \dots, i_k)$ such that $i_1 + 2i_2 + \dots + ki_k = n$ (a *partition* of n), define the corresponding *characteristic numbers* as

$$\begin{aligned} w_\omega(M^n) &= w_1^{i_1} w_2^{i_2} \dots w_k^{i_k}(\tau M)\langle M\rangle \in \mathbb{Z}/2, && \dim M = n, \\ p_\omega(M^{4n}) &= p_1^{i_1} p_2^{i_2} \dots p_k^{i_k}(\tau M)\langle M\rangle \in \mathbb{Z}, && \dim M = 4n, \\ c_\omega(M^{2n}, \xi) &= c_1^{i_1} c_2^{i_2} \dots c_k^{i_k}(\xi)\langle M\rangle \in \mathbb{Z}, && \dim M = 2n, \end{aligned}$$

where $\langle M\rangle$ denotes the *fundamental homology class* of M (with $\mathbb{Z}/2$ or $\mathbb{Z}$ coefficients).

Example 4.12. $M^4 = \mathbb{C}P^2$, $\xi = \tau(M)$, $\tau(\mathbb{C}P^2) \oplus \mathbb{C} = \overline{\eta} \oplus \overline{\eta} \oplus \overline{\eta}$.

$$c(\tau(M)) = (1+u)^3 = 1 + \underbrace{3u}_{c_1} + \underbrace{3u^2}_{c_2}, \text{ where } u = c_1(\overline{\eta}) \in H^2(\mathbb{C}P^2),$$

$$c_2(\mathbb{C}P^2) = 3, \quad c_1^2(\mathbb{C}P^2) = 9, \quad u^2\langle \mathbb{C}P^2\rangle = 1.$$

Theorem 4.13 (Thom, Milnor)**.**

1. *$M_1 \sim M_2$ unorientedly cobordant $\Leftrightarrow \forall\omega, w_\omega(M_1) = w_\omega(M_2)$.*
2. *$[M_1] - [M_2]$ is a torsion element in $\Omega_*^{SO} \Leftrightarrow \forall\omega, p_\omega(M_1) = p_\omega(M_2)$.*
3. *$(M_1, \xi_1) \sim (M_2, \xi_2)$ complex cobordant $\Leftrightarrow \forall\omega, c_w(M_1, \xi_1) = c_w(M_2, \xi_2)$.*

Theorem 4.14 (Thom'1954)**.** *$\Omega_*^O \cong \mathbb{Z}/2[\{a_i, i \neq 2^k - 1\}]$ with $\deg a_i = i$. So in small dimensions, $\Omega_*^O \cong \mathbb{Z}/2[a_2, a_4, a_5, \dots]$.. Moreover, we can take $a_{2n} = [\mathbb{R}P^{2n}]$.*

Theorem 4.15 (Novikov, Milnor, Averbuh, Wall, Rohlin, Thom)**.**

$$\Omega_*^U \cong \mathbb{Z}[a_1, a_2, \dots], \qquad \deg a_i = 2i;$$
$$\Omega_*^{SO}/Tors \cong \mathbb{Z}[b_1, b_2, \dots], \qquad \deg b_i = 4i.$$

Moreover, Ω_*^{SO} has only 2-torsion, which is completely described.

Remark 4.16. Over rationals, the cobordism rings look much simpler:

$$\Omega_*^U \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \dots],$$
$$\Omega_*^{SO} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[[\mathbb{C}P^2], [\mathbb{C}P^4], \dots].$$

In what follows we consider only complex cobordism. Write formally the total Chern class of (M^{2n}, ξ) as

$$c(\xi) = 1 + c_1(\xi) + \dots + c_n(\xi) = (1 + x_1)\dots(1 + x_n),$$

so $c_i(\xi) = \sigma_i(x_1, \dots, x_n)$ is the ith elementary symmetric function. Consider $P_n(x_1, \dots x_n) = x_1^n + \dots + x_n^n$ and express it as a polynomial in elementary symmetric functions, $P_n(x_1, \dots, x_n) = s_n(\sigma_1, \dots, \sigma_n)$.

Definition 4.17. $s_n(M^{2n}, \xi) = s_n(c_1, \dots, c_n)\langle M\rangle$.

Theorem 4.18. *$[M^{2n}]$ can be taken as a multiplicative generator of Ω_*^U in degree $2n$ if and only if*

$$s_n(M^{2n}, \xi) = \pm\mu(n+1) \text{ where } \mu(k) = \begin{cases} p & \text{if there is a prime } p \text{ such that } k = p^s, \\ 1 & \text{else.} \end{cases}$$

in other words, $s_n(M^{2n}) = \pm 1$ except for $n = p^s - 1$ in which case $s_n(M^{2n}) = \pm p$.

Example 4.19. Can we take $[\mathbb{C}P^n]$ as a generator of Ω_{2n}^U?

1. $\mathbb{C}P^1$:
   $P_1(x_1) = x_1$, $s_1(\mathbb{C}P^1) = c_1\langle\mathbb{C}P^1\rangle = 2$. Since $n = 1 = 2^1 - 1$, $[\mathbb{C}P^1]$ is a generator or Ω_2^U.
2. $\mathbb{C}P^2$:
   $P_2(x_1, x_2) = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = c_1^2 - 2c_2$, so $s_2(\mathbb{C}P^2) = (c_1^2 - 2c_2)\langle\mathbb{C}P^2\rangle = 3$. Since $n = 2 = 3^1 - 1$, $[\mathbb{C}P^2]$ is a generator of Ω_4^U.
3. $\mathbb{C}P^3$:
   In general, $s_n(\mathbb{C}P^n) = n + 1$ (**Exercise**; use the fact $\tau(\mathbb{C}P^n) \oplus \mathbb{C} = \overline{\eta} \oplus \dots \oplus \overline{\eta}$). So for $n = 3$, $s_3(\mathbb{C}P^3) = 4$. Since $n = 3 = 2^2 - 1$, one should have $s_3(M) = \pm 2$ for a generator, and $[\mathbb{C}P^3]$ is *not* a generator!

Example 4.20 (Milnor hypersurfaces)**.** Given two integers $1 \leq i \leq j$, consider the following hypersurface in $\mathbb{C}P^i \times \mathbb{C}P^j$:

$$H_{i,j} = \{(z_0 : \dots : z_i) \times (w_0 : \dots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j : z_0w_0 + \dots + z_iw_i = 0\}$$

Consider $\mathbb{C}^{i+1} \subset \mathbb{C}^{j+1}$ embedded onto first $i + 1$ coordinates.

$$\mathbb{C}P^i = \{l \subset \mathbb{C}^{i+1}\},$$
$$E = \{(l, \alpha) : l \text{ a line in } \mathbb{C}^{i+1}, \ \alpha \text{ a hyperplane in } \mathbb{C}^{j+1} \text{ containing } l\}.$$

So we have a fibration $\mathbb{C}P^{j+1} \to E \to \mathbb{C}P^i$.

Proposition 4.21. $E = H_{i,j}$.

Also, set $H_{0,j} = \mathbb{C}P^{j-1}$.

Exercise 4.22. $s_{i+j-1}(H_{i,j}) = \binom{i+j}{i+1}$.

Corollary 4.23. *Ω^U_* is multiplicatively generated by the set of cobordism classes $\{[H_{i,j}],\ 0 \le i \le j\}$.*

Proof. Use the fact that

$$\gcd_{1\le j\le k-1}\left\{\binom{k}{j}\right\} = \begin{cases} p & \text{if } k = p^s, \\ 1 & \text{else.} \end{cases}$$

□

5. (Quasi)toric representatives in complex cobordism classes

Theorem 5.1. *In* dim > 2*, every complex cobordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is compatible with the action of the torus.*

Plan of proof.

1. Identify equivariant stably complex structures on quasitoric manifolds.
2. Observe that $H_{i,j}$ are not quasitoric manifolds.
3. Replace $H_{i,j}$ by a toric manifold, denoted $B_{i,j}$, with the same characteristic number s_{i+j-1}. This provides a set of toric multiplicative generators for Ω^U_*.
4. Replace disjoint unions by connected sums. This is tricky because we need to keep track of both the action and the stably complex structure.

The above theorem provides a solution to a toric version of the following famous problem:

Problem 5.2 (Hirzebruch)**.** Describe cobordism classes in Ω^U_* which have connected algebraic representatives.

Example 5.3. We have $\Omega^U_2 = \langle[\mathbb{C}P^1]\rangle$. For $k \le 1$, the class $k[\mathbb{C}P^1]$ contains a Riemanian surface of genus $1-k$. But $k[\mathbb{C}P^1]$ with $k > 1$ does not contain a connected algebraic representative. So the solution to the above problem in dim 2 is given by the inequality $c_1(M) \le 2$.
In dimension 4 (complex 2), some similar inequalities for c_1^2 and c_2 are known, but the complete answer is open.

5.1. Equivariant stably complex structure on quasitoric manifolds.

Recall: $i_{\mathcal{Z}}\colon \mathcal{Z}_P \to \mathbb{C}^m$ the framed T^m-equivariant embedding of the moment-angle manifold, (P,Λ) a combinatorial quasitoric pair,

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \hdotsfor{7} \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix},$$

$M(P,\Lambda) = \mathcal{Z}_P/K(\Lambda)$ the associated omnioriented quasitoric manifold,

$$\rho_i\colon \mathcal{Z}_P \times_K \mathbb{C}_i \to \mathcal{Z}_P/K = M$$

a $T^n = T^m/K$-equivariant $\mathbb{C}$-line bundle over M.

Theorem 5.4. *There is a real bundle isomorphism*

$$\tau(M) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \dots \oplus \rho_m.$$

Proof. There is a T^m-invariant decomposition

$$\tau(\mathcal{Z}_P) \oplus \nu(i_{\mathcal{Z}}) \cong \mathcal{Z}_P \times \mathbb{C}^m$$

obtained by restricting $\tau(\mathbb{C}^m)$ to $\mathcal{Z}_P$. Factoring out $K = \ker(\Lambda\colon T^m \to T^n)$ gives

$$\tau(M) \oplus (\xi/K) \oplus (\nu(i_{\mathcal{Z}})/K) \cong \mathcal{Z}_P \times_K \mathbb{C}^m,$$

where ξ denotes the $(m-n)$-plane bundle of tangents along the fibres of $\mathcal{Z}_P \to M$. Both ξ and $\nu(i_{\mathcal{Z}})$ are trivial real $(m-n)$-plane bundles. Moreover, the matrix A_P provides a canonical framing (trivialisation) of $\nu_{\mathcal{Z}}$, as described in Section 2. Similarly, the matrix Λ provides a canonical choice of basis in $K = \ker\Lambda$, and therefore a canonical framing of ξ. It remains to note that

$$\mathcal{Z}_P \times_K \mathbb{C}^m = \rho_1 \oplus \dots \oplus \rho_m.$$

□

Remark 5.5. Everything is T^m/K-invariant.

Definition 5.6. Assume N is a G-manifold, $\alpha\colon G\times N\to N$ the action. A stably complex structure $c_\tau\colon \tau(N)\oplus\mathbb{R}^k\to\xi$ is said to be *G-equivariant* if

$$\xi \xrightarrow{c_\tau^{-1}} \tau(N)\oplus\mathbb{R}^k \xrightarrow{d\alpha(g,\cdot)\oplus id} \tau(N)\oplus\mathbb{R}^k \xrightarrow{c_\tau} \xi$$

is an isomorphism of complex bundles for every $g\in G$.

Corollary 5.7. *The quasitoric manifold $M(P,\Lambda)$ admits a canonical T^n-equivariant stably complex structure.*

Remark 5.8. Using the 1-1 correspondence

$$\left\{\begin{matrix}\text{combinatorial}\\ \text{quasitoric pairs } (P,\Lambda)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{omnioriented}\\ \text{quaritoric manifolds}\end{matrix}\right\}$$

we see that the T^n-equivariant stably complex structure is determined by the omniorientation. Changing the orientation of one M_i in the omniorientation data results in changing the corresponding ρ_i to its conjugate in the stably complex structure. This is equivalent to reversing the sign of the ith column in Λ.

5.2. **$H_{i,j}$ are not quasitoric.**
Recall:

$$H_{i,j}=\{(l,\alpha)\colon l\subset\mathbb{C}^{i+1}\text{ a line, }\alpha\subset\mathbb{C}^{j+1}\text{ a hyperplane containing } l\},\quad 0\le i\le j,$$

so $H_{i,j}=\mathbb{C}P(\zeta)$, where ζ is the complex j-plane bundle whose fibre over $l\in\mathbb{C}P^i$ is the j-plane $l^\perp$ in $\mathbb{C}^{j+1}$:

$$\mathbb{C}P^{j-1}\to\mathbb{C}P(\zeta)\to\mathbb{C}P^i.$$

Theorem 5.9 (exercise)**.**

$$H^*(H_{i,j})\cong\mathbb{Z}[u,w]/\big(u^{i+1},\ v^{j-i}(u^i+u^{i-1}w+\cdots+uw^{i-1}+w^i)\big).$$

Theorem 5.10 (Davis-Januszkiewicz)**.**

$$H^*(M(P,\Lambda))=\mathbb{Z}[u_1,\dots,u_m]/\mathcal{I}+\mathcal{J},$$

where $u_i=c_1(\rho_i)\in H^2(M(P,\Lambda))$,

$$\mathcal{I}=\{v_{i_1},\dots,v_{i_k}\colon F_{i_1}\cap\cdots\cap F_{i_k}=\varnothing\}\ \textit{the Stanley-Reisner ideal of } P,$$

$$\mathcal{J}=\{\lambda_{i,1}u_1+\cdots+\lambda_{i,m}u_m,\quad 1\le i\le n\}.$$

Corollary 5.11. *$H_{i,j}$ is not a quasitoric manifold for $2\le i\le j$.*

Proof. Assume the converse. Comparing H^2, we obtain $2=m-n$. Therefore,

$$H^*(H_{i,j})=(\mathbb{Z}[u_1,\dots,u_m]/\mathcal{J})/\mathcal{I}=\mathbb{Z}[u,w]/\mathcal{I}',\quad \deg u=\deg w=2$$

where the ideal $\mathcal{I}'$ has a basis consisting of elements of deg ≥ 4 decomposable into linear factors. This gives a contradiction. □

5.3. **Toric multiplicative generator set for Ω^U_*.**

Construction 5.12 (the bounded flag manifold B_n)**.** A *bounded flag* in $\mathbb{C}^{n+1}$ is a complete flag $U=\{U_1\subset\cdots\subset U_{n+1}=\mathbb{C}^{n+1}\}$ such that U_k contains the coordinate subspace $\mathbb{C}^{k-1}$ generated by the first $k-1$ standard basis vectors, for $2\le k\le n$.

$$B_n=\{\text{set of bounded flags in } \mathbb{C}^{n+1}\}.$$

There is a projection $B_n\to B_{n-1}$

$$\begin{aligned}U=&(U_1\subset U_2\subset\cdots\subset U_{n-1}\subset U_n\subset\mathbb{C}^{n+1})\\ &\mapsto\\ U'=&U/\mathbb{C}^1=(U'_1=U_2/\mathbb{C}^1\subset U'_2=U_3/\mathbb{C}^1\subset\cdots\subset U'_{n-1}=U_n/\mathbb{C}^1\subset\mathbb{C}^n)\end{aligned}$$

The fibre of $B_n \to B_{n-1}$ is $\mathbb{C}P^1$ (to recover U_1 we need to choose a line in $U_1' \oplus \mathbb{C}$). Get a tower of fibrations

$$B_n \to B_{n-1} \to \cdots \to B_2 \to B_1 = \mathbb{C}P^1.$$

This is an example of a *Bott tower* of height n.

Proposition 5.13. *The action*

$$T^n \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1},$$
$$(\boldsymbol{t}, \boldsymbol{z}) \mapsto (t_1 z_1, \dots, t_n z_n, z_{n+1})$$

induces a T^n-action on B_n making it a quasitoric manifold over I^n.

Idea of proof. $B_n = (P, \Lambda)$ where $P = I^n$ (an n-dimensional cube), and

$$\Lambda = \left(\begin{array}{c|cccc} & -1 & 0 & \dots & 0 \\ & 1 & -1 & \dots & 0 \\ I_n & \vdots & \ddots & \ddots & \vdots \\ & 0 & \dots & 1 & -1 \end{array} \right), \qquad m = 2n,$$

so $K(\Lambda) \to T^{2n}$ as

$$(t_1, \dots, t_n) \mapsto (t_1, t_1^{-1} t_2, t_2^{-1} t_3, \dots, t_{n-1}^{-1} t_n, t_1, t_2, \dots, t_n),$$
$$\mathcal{Z}_P = \{(z_1, \dots z_{2n}) \in \mathbb{C}^{2n} \colon |z_k|^2 + |z_{n+k}|^2 = 1, \quad 1 \le k \le n\} \cong (S^3)^n.$$

To identify $\mathcal{Z}_P / K(\Lambda)$ with B_n, we do the following. Given $(z_1, \dots, z_{2n}) \in \mathcal{Z}_P$, define $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n+1} \in \mathbb{C}^{n+1}$

$$\boldsymbol{v}_{n+1} = \boldsymbol{e}_{n+1}, \quad \boldsymbol{v}_k = z_k \boldsymbol{e}_k + z_{k+n} \boldsymbol{v}_{k+1}, \quad k = n, \dots, 1.$$

Then we get a projection

$$\mathcal{Z}_P \to B_n,$$
$$\boldsymbol{z} \mapsto U = (U_1 \subset U_2 \subset \cdots \subset U_n \subset \mathbb{C}^{n+1}),$$
$$U_k = \langle \boldsymbol{e}_1, \dots, \boldsymbol{e}_{k-1}, \boldsymbol{v}_k \rangle.$$

□

Now, define

$$f_i \colon B_i \to \mathbb{C}P^i,$$
$$U = \{U_1 \subset U_2 \subset \dots\} \mapsto U_1 \subset \mathbb{C}^{i+1},$$

and define $B_{i,j}$ from the pullback diagram

$$\begin{array}{ccc} B_{i,j} & \to & H_{i,j} = \mathbb{C}P(\zeta) \\ \downarrow & & \downarrow \\ B_i & \xrightarrow{f_i} & \mathbb{C}P^i \end{array}$$

So

$$B_{i,j} = \{(U, \alpha) \colon U \text{ a bounded flag in } \mathbb{C}^{i+1}, \ \alpha \text{ a hyperplane in } \mathbb{C}^{j+1} \text{ containing } U_1\}$$

and there is a fibration $\mathbb{C}P^{j-1} \to B_{i,j} \to B_i$.

Proposition 5.14. *$B_{i,j}$ has a T^{i+j-1}-action turning it into a quasitoric manifold over $I^i \times \Delta^{j-1}$.*

Idea of proof. Like always with "flag" manifolds, pulling back ζ along f_i splits it into a sum of line bundles. So $B_{i,j}$ is a projectivisation of a sum of line bundles over a toric manifold B_i. Under these circumstances, the torus action can be extended from the base to the total space. □

Remark 5.15. Both B_i and $B_{i,j}$ are toric manifolds, or Bott and generalised Bott towers respectively.

Lemma 5.16. *Assume $f\colon N_1^{2i} \to N_2^{2i}$ is a degree 1 map of stably complex manifolds, and $\zeta \to N_2^{2i}$ a j-plane complex bundle. Then*

$$s_{i+j-1}(\mathbb{C}P(f^*(\zeta))) = s_{i+j-1}(\mathbb{C}P(\zeta))$$

Theorem 5.17 (Buchstaber-Ray '98)**.** *$\{B_{i,j}\}$ is the set of multiplicative generators of Ω_*^U consisting of toric manifolds.*

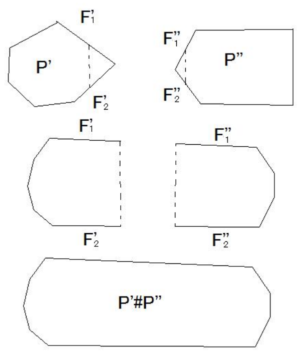
Proof. Indeed, $s_{i+j-1}(B_{i,j}) = s_{i+j-1}(H_{i,j})$ by the above Lemma. □

5.4. **Constructing connected representatives: replacing the disjoint union by the connected sum.**

Remark 5.18. We cannot find a toric representative in every cobordism class because e.g. $Td(M) = 1$ and $c_n(M) = \chi(M) > 0$ for every toric manifold M.

Construction 5.19 (connected sum of polytopes)**.**
P', P'' simple polytopes, finely ordered, of dim n:

$$v_0' = F_1' \cap \cdots \cap F_n', \quad v_0'' = F_1'' \cap \cdots \cap F_n'': \text{ initial vertices.}$$



Construction 5.20 (equivariant connected sum of quasitoric pairs and quasitoric manifolds)**.**

$$\Lambda' = \begin{pmatrix} 1 & 0 & \cdots\cdots & \lambda'_{1,n+1} & \cdots & \lambda'_{1,m'} \\ 0 & 1 & \cdots\cdots & \lambda'_{2,n+1} & \cdots & \lambda'_{2,m'} \\ \vdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \lambda'_{n,n+1} & \cdots & \lambda'_{n,m'} \end{pmatrix}$$

$$\Lambda'' = \begin{pmatrix} 1 & 0 & \cdots\cdots & \lambda''_{1,n+1} & \cdots & \lambda''_{1,m''} \\ 0 & 1 & \cdots\cdots & \lambda''_{2,n+1} & \cdots & \lambda''_{2,m''} \\ \vdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \lambda''_{n,n+1} & \cdots & \lambda''_{n,m''} \end{pmatrix}$$

$$\Lambda' \# \Lambda'' = \begin{pmatrix} 1 & 0 & \cdots\cdots & \lambda'_{1,n+1} & \cdots & \lambda'_{1,m'} & \lambda''_{1,n+1} & \cdots & \lambda''_{1,m''} \\ 0 & 1 & \cdots\cdots & \lambda'_{2,n+1} & \cdots & \lambda'_{2,m'} & \lambda''_{2,n+1} & \cdots & \lambda''_{2,m''} \\ \vdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \lambda'_{n,n+1} & \cdots & \lambda'_{n,m'} & \lambda''_{n,n+1} & \cdots & \lambda''_{n,m''} \end{pmatrix}$$

$$M' = M(P', \Lambda'), \quad M'' = M(P'', \Lambda''),$$

$$M := M(P'\#P'', \Lambda'\#\Lambda'').$$

Proposition 5.21. *M is the equivariant connected sum of M' and M'' at $\pi^{-1}(v_1')$ and $\pi^{-1}(v_1'')$.*

Difficulty: Both M' and M'' are oriented. The only possible obstruction to get the omniorientation of $M'\#M''$ right involves the associated orientations of M' and M'': the orientations must be preserved under the collapse maps

$$p' : M'\#M'' \to M' \quad \text{and} \quad p'' : M'\#M'' \to M''.$$

Definition 5.22. Let $w \in P$ be a vertex, $w = F_{i_1} \cap \cdots \cap F_{i_n}$. The *sign* $\sigma(w)$ is ± 1: it measures the difference between the orientations induced on $T_w M$ by $\rho_{i_1} \oplus \cdots \oplus \rho_{i_n}$ and by the orientation of M. It can be calculated by

$$\sigma(w) = u_{i_1}, \ldots, u_{i_n}\langle M\rangle$$

where $u_i = c_1(\rho_i) \in H^2(M)$, and $\langle M\rangle \in H_{2n}(M)$ the fundamental class.

Proposition 5.23. *$M'\#_{v_1',v_1''}M''$ admits an orientation compatible with those of M' and M'' if and only if $-\sigma(v_1') = \sigma(v_1'')$. In this case, $[M'\#M''] = [M'] + [M'']$ in Ω_*^U.*

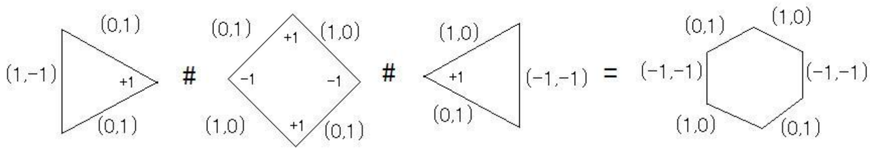
Lemma 5.24. *Let M be an omnioriented quasitoric manifold of dimension > 2 over P. Then there exists an ominioriented M' over P' such that $[M'] = [M]$ in Ω_*^U and P' has at least two vertices of opposite signs.*

Corollary 5.25. *The main theorem.*

Example 5.26. How to find a quasitoric representative in $2[\mathbb{C}P^2] \in \Omega_4^U$? We have

$$c_2([\mathbb{C}P^2]) = 3 = \text{ number of vertices in a triangle } \Delta,$$

and $c_2(2[\mathbb{C}P^2]) = 6$. So there is no quasitoric manifold over $\Delta\#\Delta = \square$ representing $2[\mathbb{C}P^2]$, because $\square$ has only 4 vertices. But it is possible to do over a hexagon:



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