Hamiltonian GKM Spaces
and
Their Moment Graphs

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Outline

- Hamiltonian GKM Spaces
- Moment Graphs
- Abstract GKM Graphs
- Abstract cohomology ring
- Open Questions

(Complex) Hamiltonian GKM Spaces

- M^{2d}: compact, connected, Kaehler manifold of (real) dimension $2d$

  ω: Kaehler form

- $T = (S^1)^n$: compact torus, with Lie algebra

  $\mathfrak{t} = Lie(T) \simeq \mathbb{R}^n$

- $T \times M \to M$: holomorphic action, such that

  $T \times (M, \omega) \to (M, \omega)$: Hamiltonian action

  $\Phi\colon M \to \mathfrak{t}^* \simeq \mathbb{R}^n$, moment map

- The action is a GKM action if
  - M^T is finite
  - For every p in M^T, the isotropy representation

$$T \times T_pM \to T_pM$$

  has mutually non-collinear (complex) weights.

- Examples:

  T^n action on $\mathbb{C}^n$ induces
  effective GKM action of T^{n-1} on

  $\mathcal{F}l_{[n]}(\mathbb{C}^n)$, $Gr(k, n, \mathbb{C})$, $\mathbb{C}P^{n-1}$

Moment Graph

- $\Gamma = (V, E)$: regular, $d-$valent graph

  $V \longleftrightarrow M^T$, fixed points

  $E \longleftrightarrow$ nontrivial connected components of X^K for codimension one subtori $K \subset T$.

- $\alpha\colon E \to \mathfrak{t}^*$ axial function

  $p \in M^T$, $\{\alpha_1, \ldots, \alpha_d\}$ weights of $T \times T_pM \to T_pM$

  $K_j = \exp(\ker(\alpha_j)) \subset T$

  X_j connected component of M^{K_j} with $p \in X_j$

  e_j edge of Γ corresponding to X_j

  $$\alpha(p, e_j) = \alpha_j$$

- $\Gamma(M, T) = (\Gamma, \alpha)$: moment graph

  1-skeleton of the moment polytope (+ internal edges)

- $H^*_\alpha(\Gamma) \subset \text{Maps}(V, \mathbb{S}(\mathfrak{t}^*))$, $f \in H^*_\alpha(\Gamma)$ iff

  $$f(p) \equiv f(q) \mod \alpha_{pq} \text{ in } \mathbb{S}(\mathfrak{t}^*)$$

  for every $e = (p, q) \in E$.

  $H^*_T(pt, \mathbb{R}) = \mathbb{S}(\mathfrak{t}^*)$, symmetric algebra of $\mathfrak{t}^*$

$$H^k_\alpha(\Gamma) = H_\alpha(\Gamma) \cap \text{Maps}(V, \mathbb{S}^k(\mathfrak{t}^*)) \quad , \quad H^*_\alpha(\Gamma) = \bigoplus_{k \geqslant 0} H^k_\alpha(\Gamma)$$

$$\boxed{H^k_\alpha(\Gamma) \simeq H^{2k}_T(M, \mathbb{R})}$$

The Complex Grassmannian $Gr(k,n)$

- $T^n \times \mathbb{C}^n \to \mathbb{C}^n$,

  $$e^{i\xi} \cdot (z_1, .., z_n) = (e^{i\xi_1} z_1, ..., e^{i\xi_n} z_n)$$

  induces an action on $Gr(k,n)$ ($k-$planes in $\mathbb{C}^n$).
  Diagonal S^1 acts trivially

  $$T^n = S^1 \times T^{n-1}$$

  $\Longrightarrow$ effective action of T^{n-1}

  $$T^{n-1} \times Gr(k,n) \to Gr(k,n) \ .$$

- $\Gamma = J(n,k)$, the Johnson graph.
  - Vertices: $S \subset \{1, .., n\}$, with $\#S = k$.

    $$v_S = \bigoplus_{j \in S} \mathbb{C}\varepsilon_j$$

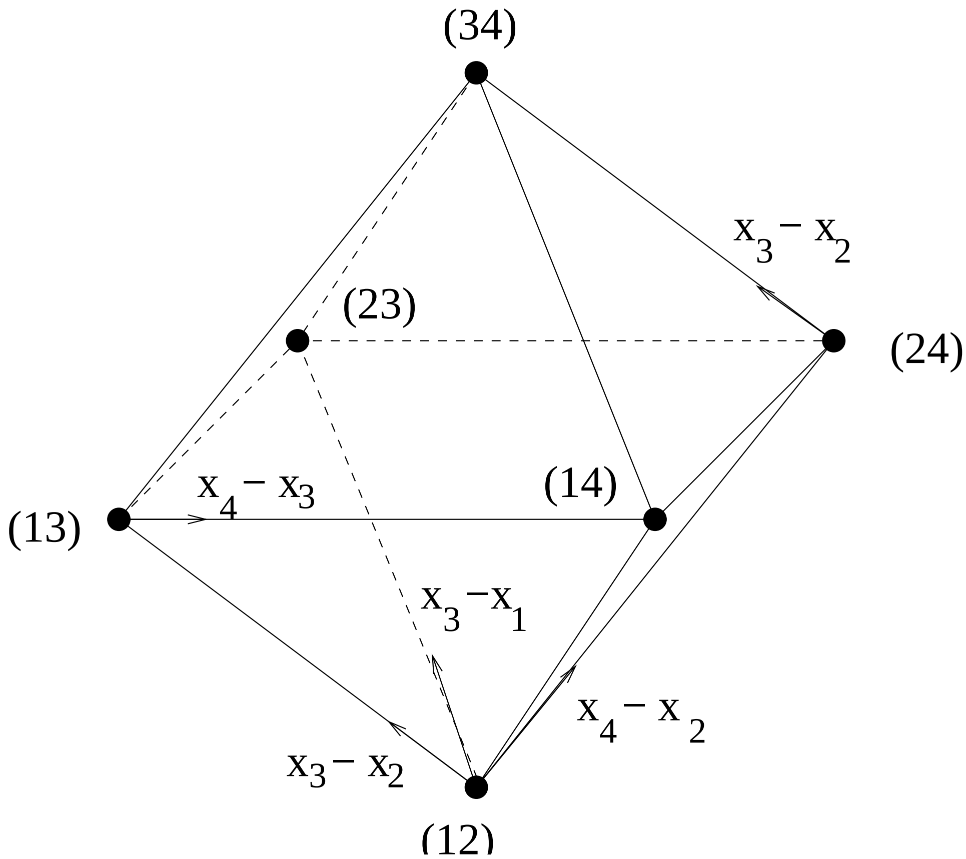
  - Edges: $S_1 S_2$, if $\#(S_1 \cap S_2) = k - 1$.

    $$\{W \mid v_{S_1} \cap v_{S_2} \subset W \subset v_{S_1} + v_{S_2}\} \simeq \mathbb{C}P^1$$

- Labels: If $S_2 = S_1 \setminus \{i\} \cup \{j\}$, then

  $$\alpha_{S_1 S_2} = x_j - x_i \ .$$

Example: $J(4,2)$



$\tau \in H^1_\alpha(\Gamma)$

$$\begin{array}{rlrl} & (12) \longrightarrow 0 & & (13) \longrightarrow x_2 - x_3 \\ (23) \longrightarrow x_1 - x_3 & & & (24) \longrightarrow x_1 - x_4 \\ (14) \longrightarrow x_2 - x_4 & & & (34) \longrightarrow x_1 + x_2 - x_3 - x_4 \end{array}$$

Generates a parallel redrawing (deformation) that fixes the vertex (12).

Abstract GKM graphs

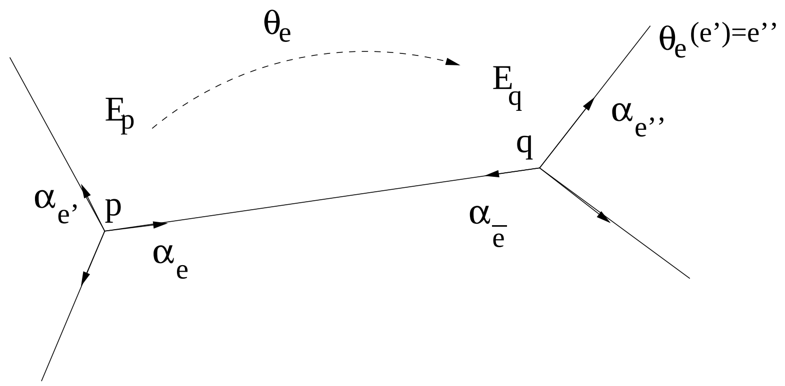
$\mathfrak{t}$: n-dimensional real vector space, $\mathfrak{t}^*$: dual space

Definition: An *abstract GKM graph* is a pair (Γ, α), consisting of a regular $d-$valent graph Γ and a labeling of oriented edges $\alpha \colon E \to \mathfrak{t}^*$, such that:

1. For every vertex p, the vectors
$$\alpha_e \ ; \ e \in E_p = \{e \ : \ i(e) = p\}$$
are pairwise linearly independent;

2. For every oriented edge $e \in E$, $\alpha_{\bar{e}} = -\alpha_e$

3. For every oriented edge $e = (p, q)$, there exists a bijection $\theta_e \colon E_p \to E_q$ such that, for every $e' \in E_p$,
$$\alpha_{\theta_e(e')} - \alpha_{e'} = c_{e,e'}\alpha_e$$
for some $c_{e,e'} \in \mathbb{R}$.



4. (Effectiveness condition): There exists a vertex p such that $\{\alpha_e : e \in E_p\}$ spans $\mathfrak{t}^*$.

Betti Numbers

Let $\xi \in \mathfrak{t}$, generic: $\alpha_e(\xi) \neq 0$, for all edges e.

Definition. The $\xi-$*index* of $p \in V$ is

$$ind_\xi(p) = \#\{e \in E_p \ : \ \alpha_e(\xi) < 0\} \ .$$

Definition. The j^{th} *Betti number* of (Γ, α) is

$$\beta_j(\Gamma) = \#\{p \in V \ : \ ind_\xi(p) = j\} \ .$$

Theorem. $\beta_j(\Gamma)$ *doesn't* depend on ξ.

"Poincaré duality": Change ξ to $-\xi \implies \beta_{d-j} = \beta_j$

Theorem. If $(\Gamma, \alpha) = \Gamma(M, T)$, then

$$\beta_j(\Gamma) = \beta_{2j}(M) \ .$$

Definition. Morse function compatible with ξ:

$\Phi \colon V \to \mathbb{R}$ such that if $e = (p, q) \in E$, then

$\Phi(q) - \Phi(p)$ and $\alpha_{pq}(\xi)$ have the same sign.

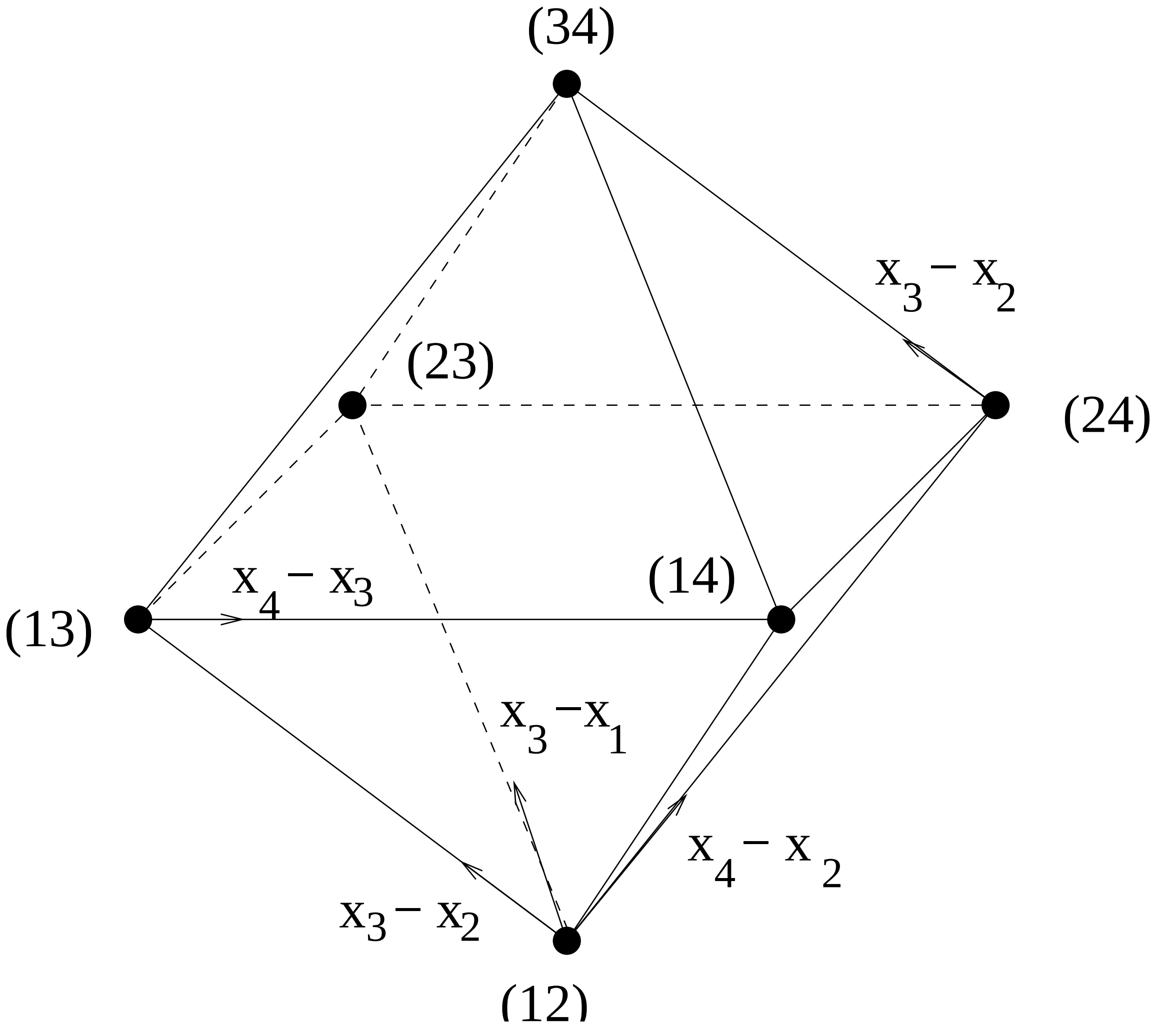
Morse order:

[If $e = (p, q)$, then $p \prec q$ if $\alpha_{pq}(\xi) > 0$] + transitive closure.

For $p \in V$, define the flow-up

$$F_p = \{q \in V \mid p \preccurlyeq q\}$$

F_p is not necessarily a regular subgraph!



Indices:

(12)	(13)	(23)	(14)	(24)	(34)
0	1	2	2	3	4

Betti Numbers:

$$\beta_0(\Gamma) = 1$$
$$\beta_1(\Gamma) = 2$$
$$\beta_2(\Gamma) = 2$$
$$\beta_3(\Gamma) = 1$$

The Ring $H^*_\alpha(\Gamma)$

(Γ, α) abstract GKM graph

$H^*_\alpha(\Gamma) \subset \mathrm{Maps}(V, \mathbb{S}(\mathfrak{t}^*)$, $f \in H^*_\alpha(\Gamma)$ iff

$$f(q) \equiv f(p) \mod \alpha_e$$

for every $e = (p, q) \in E$.

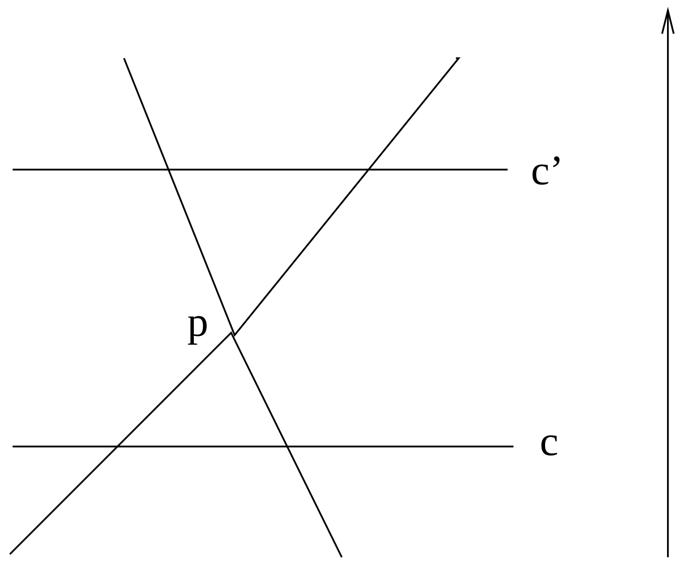
An Estimate

$\Phi \colon V \to \mathbb{R}$, Morse function.

For $c \in \mathbb{R}$, let

$$H_c(\Gamma) = \{f \in H^*_\alpha(\Gamma) \mid f(q) = 0 \text{ if } \Phi(q) < c\}$$

Let $c < c'$ such that $\Phi^{-1}([c, c']) = \{p\}$ and $c < \Phi(p) < c'$



Let

$$H^*_\alpha(\Gamma) \to \mathbb{S}(\mathfrak{t}^*) \quad , \quad f \to f(p)$$

$ind(p) = j$

$e_1, \ldots, e_j$ point down from p

$\alpha_s = \alpha(e_s) \in \mathfrak{t}^*$, for $s = 1, \ldots, j$

Then

$$0 \to H^k_{c'}(\Gamma) \to H^k_c(\Gamma) \xrightarrow{f \to f(p)} \alpha_1 \cdots \alpha_j \mathbb{S}^{k-j}(\mathfrak{t}^*)$$

is exact, so

$$\dim_{\mathbb{R}} H^k_c(\Gamma) - \dim_{\mathbb{R}} H^k_{c'}(\Gamma) \leqslant \dim_{\mathbb{R}} \mathbb{S}^{k-ind(p)}(\mathfrak{t}^*)$$

Add all:

$$\begin{aligned} \dim_{\mathbb{R}} H^k_\alpha(\Gamma) &\leqslant \sum_{p \in V} \dim_{\mathbb{R}} \mathbb{S}^{k-ind(p)}(\mathfrak{t}^*) \\ &= \sum_{j=0}^{d} \beta_j(\Gamma) \dim_{\mathbb{R}} \mathbb{S}^{k-j}(\mathfrak{t}^*) \end{aligned}$$

$$\boxed{\dim_{\mathbb{R}} H^k_\alpha(\Gamma) \leqslant \sum_{j=0}^{d} \beta_j(\Gamma) \dim_{\mathbb{R}} \mathbb{S}^{k-j}(\mathfrak{t}^*)}$$

Proof of GKM

$$i_T : M^T \to M \quad \Rightarrow \quad i_T^* : H_T^*(M) \to H_T^*(M^T)$$

$$H_T^*(M^T) \simeq \mathsf{Maps}(M^T, \mathbb{S}(\mathfrak{t}^*))$$

If $K \subset T$ is a codimension one subtorus,

$$\begin{array}{ccccccc} & M & & & & H_T^*(M) & \\ i_T \nearrow & & \nwarrow i_K & \Longrightarrow & i_T^* \swarrow & & \searrow i_K^* \\ M^T & \overset{i_{T,K}}{\hookrightarrow} & M^K & & H_T^*(M^T) & \overset{i_{T,K}^*}{\longleftarrow} & H_T^*(M^K) \end{array}$$

$$i_T^* H_T^*(M) \subseteq \bigcap_{\substack{K \subset T \\ codim=1}} i_{T,K}^* H_T^*(M^K)$$

$$H_T^*(M) \quad \hookrightarrow \quad H_\alpha^*(\Gamma)$$

$$H_T^*(M) \simeq H^*(M) \otimes \mathbb{S}(\mathfrak{t}^*) \qquad \text{as } \mathbb{S}(\mathfrak{t}^*) - \text{modules}$$

$$\dim_{\mathbb{R}} H_T^{2k}(M) = \sum_{j=0}^{k} \beta_{2j}(M) \dim_{\mathbb{R}} \mathbb{S}^{k-j}(\mathfrak{t}^*)$$

$$\dim_{\mathbb{R}} H_\alpha^k(\Gamma) \leqslant \sum_{j=0}^{k} \beta_j(\Gamma) \dim_{\mathbb{R}} \mathbb{S}^{k-j}(\mathfrak{t}^*)$$

Conclusion:

$$H_T^*(M) \simeq H_\alpha^*(\Gamma)$$

Generators for $H^*_\alpha(\Gamma)$

Definition. Let $p \in V$. A map $\tau_p : V \to \mathbb{S}(\mathfrak{t}^*)$ is a *special class* at p if

- $\tau_p \in H^*_\alpha(\Gamma)$ is homogeneous of degree $ind(p)$.
- τ_p is supported on F_p.
- τ_p is normalized by

$$\tau_p(p) = \prod_{\alpha_{p,e}(\xi)<0} (-\alpha_{p,e})$$

Theorem. For $p \in V$, there exists a special class at p.

Step 1:

$$H^k_c(\Gamma) \to \left[\alpha_1 \cdots \alpha_{ind(p)}\right] \mathbb{S}^{k-ind(p)}(\mathfrak{t}^*) \quad , \quad f \to f(p)$$

is surjective $\Longrightarrow$ first and last condition

Step 2: clean outside the flow-up $\Longrightarrow$ second condition.

Remark: τ_p may not be unique, but a canonical choice can be made using local indices.

Morse Interpolation

- Construct $\tau_p \Leftrightarrow$ Construct $\tau_p(q)$ for all $q \succcurlyeq p$

$$\Updownarrow$$

Interpolation problem

- Result:

$$\tau_p(q) = \sum_{\gamma} E_\gamma(x, \xi)$$

where

  - γ: ascending chain from p to q.
  - E_γ: rational expression in $x_1, x_2, .., x_n, \xi_1, .., \xi_n$.

- Limit trick:

$$\tilde{E}_\gamma(x) = \lim_{\xi_1 \to 0}(\lim_{\xi_2 \to 0}(...(\lim_{\xi_n \to 0} E_\gamma(x, \xi))...))$$

$$\tau_p(q) = \sum_{\gamma} \tilde{E}_\gamma(x)$$

where

  - γ: "relevant" ascending chain from p to q.
  - $\tilde{E}_\gamma$: rational expression in $x_1, x_2, .., x_n$.

Open Questions

- Realization: Which abstract GKM graphs are the GKM graphs of Hamiltonian GKM spaces? What additional conditions determine uniqueness?

- Multiplicative Structure: The multiplicative structure of $H_T^*(M)$ can be determined from the GKM graph. Combinatorial formulas for products of generators?

- Parallel Redrawings: For many GKM graphs, the dimension of the space of parallel redrawings is

$$\beta_1 + n\beta_0 \ .$$

  The formula is true not just for graphs coming from Hamiltonian GKM spaces, but for a larger class of abstract GKM graphs. How large?

- Extension: Let (Γ, α) be an abstract GKM graph, with axial function $\alpha \colon E \to \mathfrak{t}^*$. An extension of α is a map $\alpha' \colon E \to \mathfrak{k}^*$ such that $\alpha = \pi \circ \alpha'$, where $\pi \colon \mathfrak{k}^* \to \mathfrak{t}^*$ is a projection. Does there exist an extension of α to an effective axial function α'?

Necessary Conditions

- Integrality conditions:

  $\alpha\colon E \to \mathbb{Z}^n$ (modulo a linear isomorphism $\mathfrak{t}^* \to \mathbb{R}^n$)

  $c_{e,e'} \in \mathbb{Z}$

- Embedded as the one-skeleton of a convex polytope + internal edges

  Complexity: $k = d - n \geqslant 0$

Results

- $k = 0$: Complete classification of symplectic toric manifolds by their moment polytopes (no internal edges) – Delzant polytopes

- $k = 1$: Local/global classification of complexity one spaces

- $k > 1$: ??

Methods

- Glue.

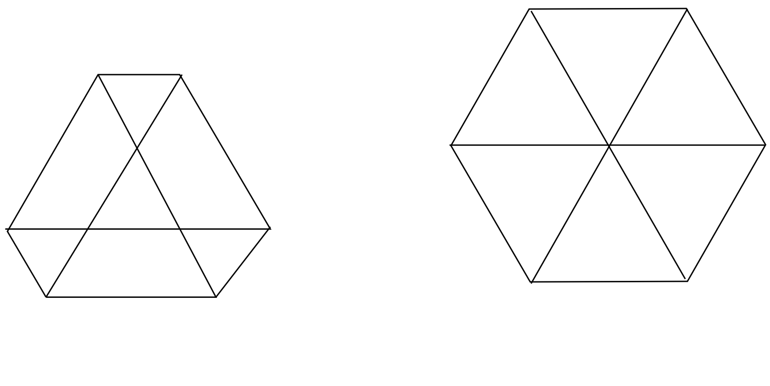
- Cut.

- Deform.

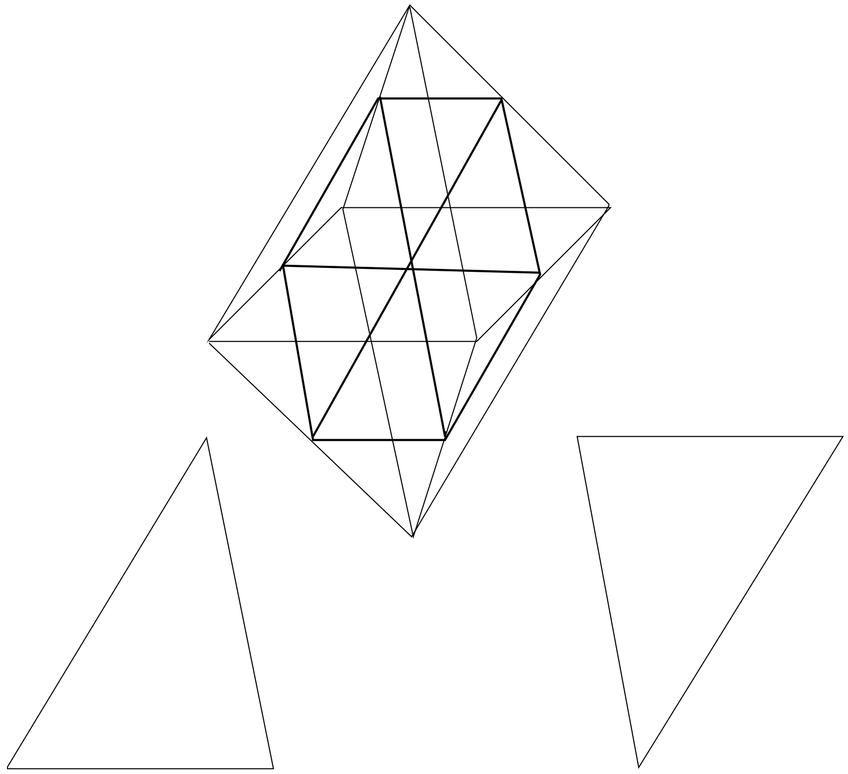
- Extend.

Deformations

Parallel redrawing:

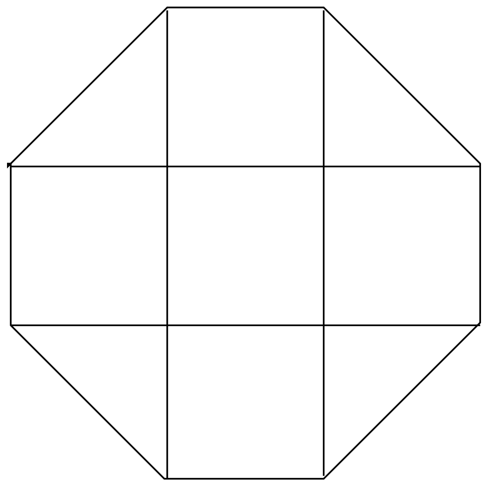


Cross-section:



More parallel redrawing $\Longrightarrow$ 2 types of collapsing
Front/back triangular face

Extensions:



(Γ, α): abstract one skeleton, $\alpha \colon E \to \mathfrak{t}^*$

$\mathfrak{k} \subset \mathfrak{t}$ subspace $\Longrightarrow$ $\mathfrak{t}^* \to \mathfrak{k}^*$ $\Longrightarrow$

(Γ, α'): abstract one skeleton, $\alpha' \colon E \to (\mathfrak{k})^*$

Question: Given (Γ, α'), with $\alpha' \colon E \to \mathfrak{k}^*$, does there exist an effective extension $\alpha \colon E \to \mathfrak{t}^*$ for $\mathfrak{k}^* \subsetneq \mathfrak{t}^*$?

Effect: Complexity decreases by $\dim_{\mathbb{R}} \mathfrak{t}^* - \dim_{\mathbb{R}} \mathfrak{k}^*$.