Twisted toric structures

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Contents

§1 Symplectic toric manifolds

(X^4, ω): a 4-dimensional symplectic manifold

$T^k \curvearrowright (X^4, \omega)$:a Hamiltonian action with a moment map $\mu : X \to \mathbb{R}^k$

Example 1.1. $T^2 \curvearrowright (\mathbb{C}^2, \omega_{\mathbb{C}^2})$: $\theta \cdot z := (e^{2\pi\sqrt{-1}\theta_i} z_i)$

$$\Rightarrow \begin{array}{cccc} \mu_{\mathbb{C}^2} : & \mathbb{C}^2 & \to & \mathbb{R}^2 \\ & \Psi & & \Psi \\ & z & \mapsto & \left(|z_1|^2, |z_2|^2\right) \end{array}$$

Assume X *is compact, connected.*

Fact 1.2. *If the action is effective, then* $k \leq 2$.

In the case of $k = 2$, (X^4, ω) with an effective Ham. T^2-action is called a 4-dimensional **symplectic toric manifold**.

General Hamiltonian case

$T^k \curvearrowright (X^4, \omega)$:a Hamiltonian action with a moment map $\mu : X \to \mathbb{R}^k$

Theorem 1.3 (Atiyah, Guillemin-Sternberg). *$\mu(X)$ is a convex hull of the image of the fixed points.*

Symplectic toric case

$T^k \curvearrowright (X^4, \omega)$:a symplectic toric manifold with a moment map $\mu : X \to \mathbb{R}^k$

Theorem 1.4 (Delzant). *$\mu(X)$ is a Delzant polytope.*

Theorem 1.5 (Delzant).

$$\frac{\left\{ \begin{array}{c} T^2 \curvearrowright (X^4, \omega) \\ \textit{: sympl. toric mfd} \end{array} \right\}}{\textit{equiv.sympl.diffeo}} \overset{1:1}{\longleftrightarrow} \frac{\left\{ \begin{array}{c} \Delta \subset \mathbb{R}^2 \\ \textit{: Delzant polytope} \end{array} \right\}}{\textit{parallel transports in } \mathbb{R}^2}$$

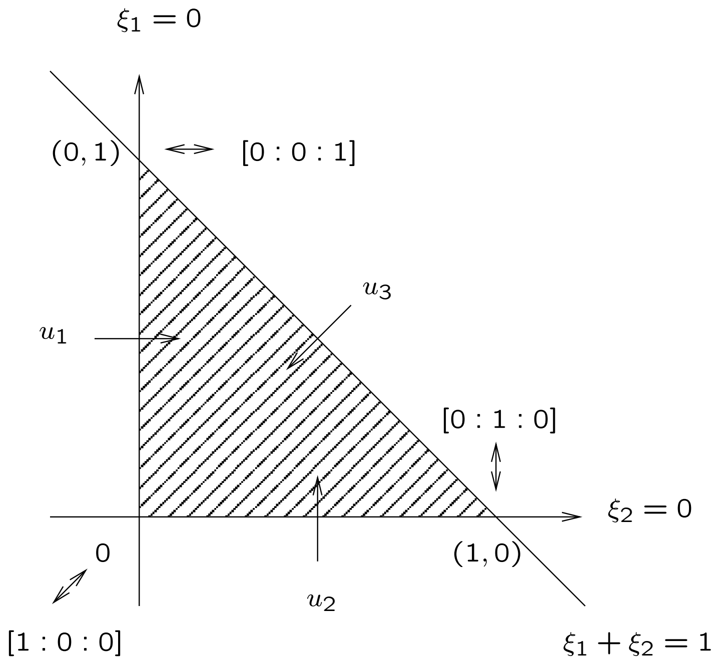
Example 1.6. $T^2 \curvearrowright (\mathbb{C}P^2, \omega_{FS})$:

$$\theta \cdot [z_0 : z_1 : z_2] = [z_0 : e^{2\pi\sqrt{-1}\theta_1} z_1 : e^{2\pi\sqrt{-1}\theta_2} z_2]$$

$$\Rightarrow \quad \begin{array}{cccc} \mu : & \mathbb{C}P^2 & \to & \mathbb{R}^2 \\ & \Uplus & & \Uplus \\ & [z_i] & \mapsto & \left(\frac{|z_1|^2}{\|z\|^2}, \frac{|z_2|^2}{\|z\|^2}\right) \end{array}$$

$$\mu(\mathbb{C}P^2) = \left\{\xi \in \mathbb{R}^2 \colon \langle u_i, \xi\rangle \geq \lambda_i, i = 1, 2, 3\right\}$$

$$\left(\begin{array}{l} \quad u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ u_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \\ \lambda_1 = \lambda_2 = 0, \ \lambda_3 = -1 \end{array} \right)$$



Remark 1.7. (1) A symplectic manifold is obtained from the trivial torus bundle on a Delzant polytope Δ by corrapsing each fiber on an edge by some circle subgroup determined by the data of Δ.

(2) For $\forall\ \xi \in \Delta$, there exist

(i) U :a neighborhood of $\xi \in \Delta$

(ii) $\rho \in SL_2(\mathbb{Z})$

such that

$$\begin{tikzcd}
\mu^{-1}(U) \arrow[rr, "\rho-\text{equiv.sympl.}", "\cong"'] \arrow[d, "\mu"'] \arrow[drr, phantom, "\circlearrowright"] & & (\mu_{\mathbb{C}^2})^{-1}({}^t\rho^{-1}(U)) \arrow[d, "\mu_{\mathbb{C}^2}"] \\
U \arrow[rr, "{}^t\rho^{-1}", "\cong"'] & & {}^t\rho^{-1}(U).
\end{tikzcd}$$

A symplectic toric manifold can be thought of as a trivial torus bundle with singular fibers which look like those of $\mu_{\mathbb{C}^2}$.

generalize to a possibly non-trivial bundle case $\Longrightarrow$ **"twisted toric manifolds"**

§2 Twisted toric manifold

$\pi_P : P \to B$:a principal $SL_2(\mathbb{Z})$-bundle on a surface B with corners

$$\left(\Longrightarrow \begin{array}{l} \pi_T : T^2_P := P \times_{SL_2(\mathbb{Z})} T^2 \to B, \\ \pi_{\mathbb{Z}} : \mathbb{Z}^2_P := P \times_{SL_2(\mathbb{Z})} \mathbb{Z}^2 \to B \end{array} \right)$$

X :a closed connected 4-manifold

$$\begin{tikzcd} T^2_P \arrow[rr, "\nu"] \arrow[dr, "\pi_T"'] & \circlearrowright & X \arrow[dl, "\mu"] \\ & B & \end{tikzcd}$$

: a commutative diagram of surjective maps

Definition 2.1. $\{X, \nu, \mu\}$ is a **twisted toric manifold** associated with $\pi_P : P \to B$, if for $\forall\ b \in B$, there exist

(i) (U, φ^B) :a coordinate neighborhood of b in B

$\left(\varphi^B : U \cong \mathbb{R}^2_{\geq 0} \cap D^2_\varepsilon(\xi_0)\right)$

(ii) $\varphi^P : \pi_P^{-1}(U) \cong U \times SL_2(\mathbb{Z})$:a local trivialization of P

$$\left(\Longrightarrow \begin{array}{l} \varphi^T : \pi_T^{-1}(U) \cong U \times T^2, \\ \varphi^{\mathbb{Z}} : \pi_{\mathbb{Z}}^{-1}(U) \cong U \times \mathbb{Z}^2 \end{array} \right)$$

(iii) $\varphi^X : \mu^{-1}(U) \cong \mu_{\mathbb{C}^2}^{-1}(D^2_\varepsilon(\xi_0))$:a diffeomorphism

such that the following diagram commutes

$$
\begin{tikzcd}
\pi_T^{-1}(U) \arrow[rrrr, "\nu"] \arrow[drr, "\pi_T"'] \arrow[ddr, "(\varphi^B \times \mathrm{id}_{T^2}) \circ \varphi^T"'] & & & & \mu^{-1}(U) \arrow[dll, "\mu"] \arrow[ddr, "\varphi^X"] & \\
& & U \arrow[ddr, "\varphi^B"] & & & \\
& \varphi^B(U) \times T^2 \arrow[rrrr, "\nu_{\mathbb{C}^2}"] \arrow[drr, "\mathrm{pr}_1"'] & & & & \mu_{\mathbb{C}^2}^{-1}(D_\varepsilon^2(\xi_0)) \arrow[dll, "\mu_{\mathbb{C}^2}"] \\
& & & \varphi^B(U) & &
\end{tikzcd}
$$

Example 2.2. symplectic toric manifolds

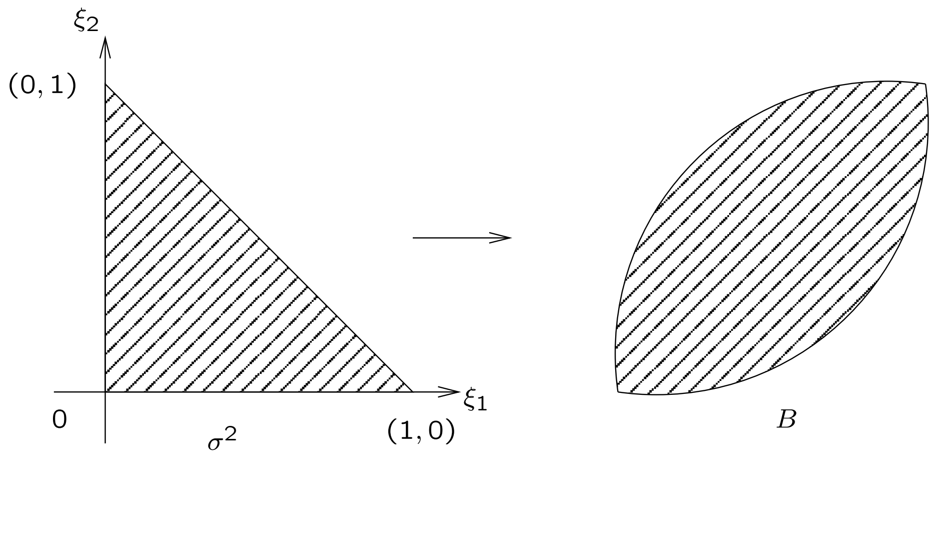
Example 2.3. If B is closed, T^2_P is a twisted toric mfd associated with $\pi_P : P \to B$.

Example 2.4. $S^3 \times S^1$

$$\begin{tikzcd} \overline{D}^2 \times T^2 \arrow[rr, "\nu"] \arrow[dr, "\mathrm{pr}_1"'] & \arrow[d, phantom, "\circlearrowright"] & S^3 \times S^1 \arrow[dl, "\mu"] \\ & \overline{D}^2 & \end{tikzcd}$$

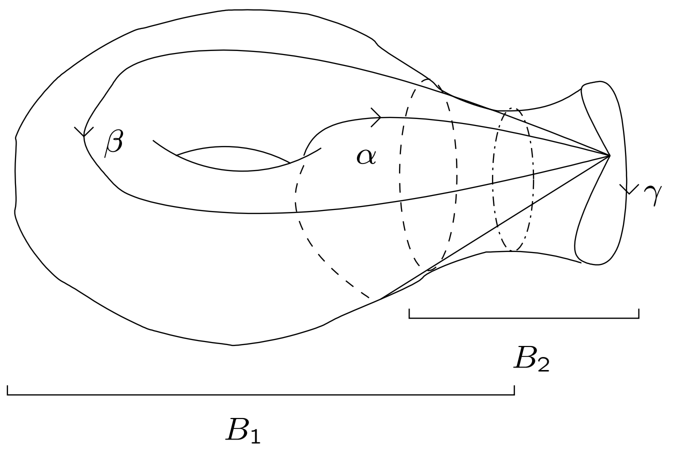
Example 2.5. S^4

$$S^4 = \overline{D}^4 / \partial \overline{D}^4, \ B = \sigma^2 / \{\xi_1 + \xi_2 = 1\}$$



$$\begin{tikzcd} \sigma^2 \times T^2 \arrow[rr, "\nu_{\mathbb{C}^2}"] \arrow[dr, "\mathrm{pr}_1"'] & \arrow[d, phantom, "\circlearrowright"] & \overline{D}^4 \arrow[dl, "\mu_{\mathbb{C}^2}"] \\ & \sigma^2 & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} B \times T^2 \arrow[rr, "\nu"] \arrow[dr, "\mathrm{pr}_1"'] & \arrow[d, phantom, "\circlearrowright"] & S^4 \arrow[dl, "\mu"] \\ & B & \end{tikzcd}$$

Example 2.6. B :a surface with one corner point



$\rho : \pi_1(B) \to SL_2(\mathbb{Z})$

$$\rho(\alpha) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ \rho(\beta) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},\ \rho(\gamma) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow P := \tilde{B} \times_\rho SL_2(\mathbb{Z}),\ \ T^2_P := \tilde{B} \times_\rho T^2$$

<u>**On** B_1</u>

$$\begin{tikzcd} X_1 \arrow[dr, "\mu_1"'] & \overset{\nu_1}{:\cong} \arrow[d, phantom, "\circlearrowright"] & T^2_P\Big|_{B_1} . \arrow[dl, "\pi_T"] \\ & B_1 & \end{tikzcd}$$

On B_2

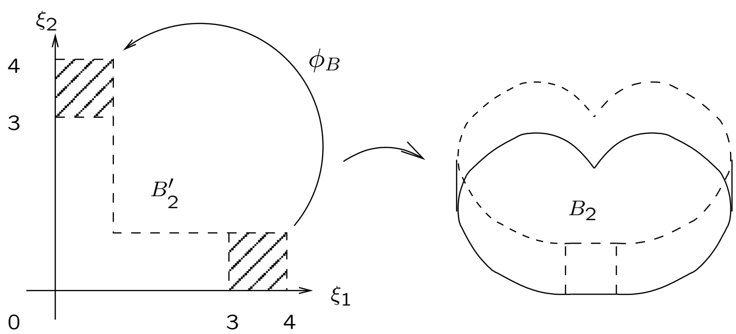
$$B'_2 := [0,4)\times[0,1)\cup[0,1)\times[0,4)(\subset\mathbb{R}^2),$$

$$B_2 := B'_2/\sim_B,\ T^2_P\Big|_{B_2} := B'_2\times T^2/\sim_T,$$

$$X_2 := \mu^{-1}_{\mathbb{C}^2}(B'_2)/\sim_X$$

$$\left(\begin{array}{l} \quad\ \xi\sim_B\xi' \\ \qquad \overset{\text{def}}{\Leftrightarrow}\ 3<\xi_1<4,\ 3<\xi'_2<4,\ \xi'=\phi_B(\xi) \\ (\xi,\theta)\sim_T(\xi',\theta') \\ \qquad \overset{\text{def}}{\Leftrightarrow}\ 3<\xi_1<4,\ 3<\xi'_2<4,\ \xi'=\phi_B(\xi),\ \theta'=\rho(\gamma)\theta \\ \quad\ z\sim_X z' \\ \qquad \overset{\text{def}}{\Leftrightarrow}\ \sqrt{3}<|z_1|<2,\ \sqrt{3}<|z_2|<2,\ z'=\phi_X(z) \end{array}\right)$$

$$\begin{tikzcd} B'_2\times T^2 \arrow[rr, "\nu_{\mathbb{C}^2}"] \arrow[dr, "\mathrm{pr}_1"'] & \arrow[d, phantom, "\circlearrowright"] & \mu^{-1}_{\mathbb{C}^2}(B'_2) \arrow[dl, "\mu_{\mathbb{C}^2}"] \\ & B'_2 & \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} T^2_P \arrow[rr, "\nu_2"] \arrow[dr, "\pi_T"'] & \arrow[d, phantom, "\circlearrowright"] & X_2 \arrow[dl, "\mu_2"] \\ & B_2 & \end{tikzcd}.$$



On B

$$
\begin{tikzcd}
 & T_P^2\big|_{B_1\cap B_2} \arrow[dl, "\nu_1"', "="] \arrow[dr, "\nu_2", "\cong"'] \arrow[d, phantom, "\circlearrowright"] & \\
X_1\big|_{B_1\cap B_2} \arrow[rr, "\psi", "\cong"'] \arrow[dr, "\mu_1"'] & \circlearrowright & X_2\big|_{B_1\cap B_2} \arrow[dl, "\mu_2"] \\
 & B_1 \cap B_2 &
\end{tikzcd}
$$

$$
\Rightarrow \quad
\begin{tikzcd}
T_P^2 \arrow[rr, "\nu"] \arrow[dr, "\pi_T"'] & \circlearrowright & X_1 \cup_\psi X_2 =: X. \arrow[dl, "\mu"] \\
 & B &
\end{tikzcd}
$$

§3 Classification

$\pi_P : P \to B$:a principal $SL_2(\mathbb{Z})$-bundle on a surface B with corners

$$\left(\Longrightarrow \begin{array}{l} \pi_T : T^2_P := P \times_{SL_2(\mathbb{Z})} T^2 \to B, \\ \pi_{\mathbb{Z}} : \mathbb{Z}^2_P := P \times_{SL_2(\mathbb{Z})} \mathbb{Z}^2 \to B \end{array} \right)$$

$\mathcal{S}^{(k)}B(\subset B)$:a k-dimensional strata

$\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$:a rank one sublattice bundle of $\pi_{\mathbb{Z}}\big|_{\mathcal{S}^{(1)}B} : \mathbb{Z}^2_P\big|_{\mathcal{S}^{(1)}B} \to \mathcal{S}^{(1)}B$

Definition 3.1. $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$ is **primitive**, if for $\forall b \in \mathcal{S}^{(k)}B$, there exist

(i) $U(\subset B)$: a neighborhood of b with $2-k = \#$ of comp.of $U \cap \mathcal{S}^{(1)}B$,

(ii) $\varphi^P : P\big|_U \cong U \times SL_2(\mathbb{Z})$: a local trivialization of P,

(iii) $\{L_1, \cdots, L_{2-k}\}$: a primitive tuple of rank one sublattices of $\mathbb{Z}^2$,

such that for $j = 1, \cdots, 2-k$, the following commutes

$$
\begin{array}{ccccc}
\mathbb{Z}^2_P\big|_U & \overset{\varphi^{\mathbb{Z}}}{\cong} & U \times \mathbb{Z}^2 & & . \\
\cup & \circlearrowright & \cup & & \\
\mathbb{Z}^2_P\big|_{(U\cap\mathcal{S}^{(1)}B)_j} & \cong & \big(U \cap \mathcal{S}^{(1)}B\big)_j \times \mathbb{Z}^2 & & \\
\cup & \circlearrowright & \cup & & \\
\mathcal{L}\big|_{(U\cap\mathcal{S}^{(1)}B)_j} & \cong & \big(U \cap \mathcal{S}^{(1)}B\big)_j \times L_j & &
\end{array}
$$

Remark 3.2. (1) Definition 3.1 does not depend on the choice of U.

(2) $\mathrm{Aut}(P) \curvearrowright \left\{ \begin{array}{c} \mathcal{L} \\ \pi_{\mathcal{L}} \downarrow \\ \mathcal{S}^{(1)}B \end{array} : \begin{array}{c} \text{primitive rank one} \\ \text{sublattice bundle} \end{array} \right\}$

$\{X_i, \nu_i, \mu_i\}$:twisted toric manifolds associated with $\pi_P : P \to B$

Definition 3.3. $\{X_1, \nu_1, \mu_1\}$ and $\{X_2, \nu_2, \mu_2\}$ are **topologically isomorphic**, if there exist

(i) $\psi^P \in \mathsf{Aut}(P)$ $\left(\Rightarrow \ \psi^T : \begin{tikzcd} T_P^2 \arrow[rr, "\cong"] \arrow[dr, "\pi_T"'] & \arrow[d, phantom, "\circlearrowright"] & T_P^2 \arrow[dl, "\pi_T"] \\ & B & \end{tikzcd} \right)$

(ii) $\psi^X : X_1 \cong X_2$: a homeo

such that the following commutes

$$\begin{tikzcd}
T_P^2 \arrow[rr, "\psi^T"] \arrow[dr, "\nu_1"] \arrow[dd, "\pi_T"'] & & T_P^2 \arrow[dr, "\nu_2"] \arrow[dd, "\pi_T"] & \\
& X_1 \arrow[rr, "\psi^X"] \arrow[dl, "\mu_1"] & & X_2 \arrow[dl, "\mu_2"] \\
B \arrow[rr, equal, "\mathsf{id}_B"] & & B &
\end{tikzcd} .$$

Theorem 3.4.(Y) *(1) For any twisted toric manifold $\{X,\nu,\mu\}$, there is a primitive rank one sublattice bundle $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$ of $\pi_{\mathbb{Z}} : \mathbb{Z}^2_P\big|_{\mathcal{S}^{(1)}B} \to \mathcal{S}^{(1)}B$ which is determined uniquely by $\{X,\nu,\mu\}$.*

(2) Fix $\pi_P : P \to B$. Then there is a one-to-one correspondence

$$\frac{\left\{\{X,\nu,\mu\} : \begin{array}{c}\textit{twisted toric mfd}\\ \textit{ass.with } \pi_P : P \to B\end{array}\right\}}{\textit{top iso}} \overset{1:1}{\longleftrightarrow} \frac{\left\{\begin{array}{ccc}\mathcal{L} & & \mathbb{Z}^2_P\big|_{\mathcal{S}^{(1)}B} \\ \pi_{\mathcal{L}}\downarrow & \subset & \pi_{\mathbb{Z}}\downarrow \\ \mathcal{S}^{(1)}B & & \mathcal{S}(1)B\end{array} : \begin{array}{c}\textit{primitive rank one}\\ \textit{sublattice bundle}\end{array}\right\}}{\mathrm{Aut}(P)}.$$

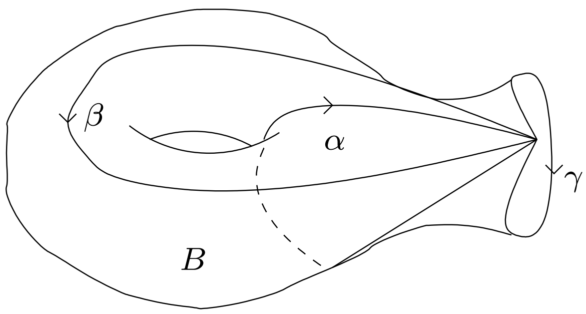
§4 Topology

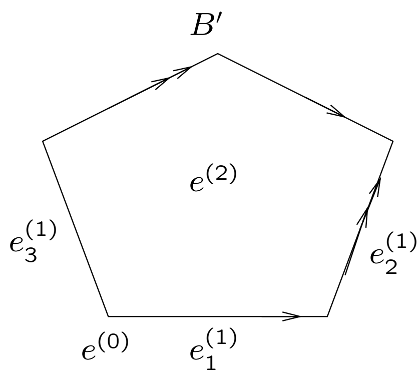
4.1 Fundamental groups

Theorem 4.1.(Y) *If B has at least one corner point, then $\pi_1(X) \cong \pi_1(B)$.*

4.2 Cohomology groups

Cell decomposition of B:





$X^{(q)} = \mu^{-1}(B^{(q)})$

$\{(E_X)_r^{p,q}, d_r\}$:spectral sequence w.r.t.

$$S^*(X;\mathbb{Z}) \supset S^*(X, X^{(0)};\mathbb{Z}) \supset S^*(X, X^{(1)};\mathbb{Z}) \supset S^*(X, X^{(2)};\mathbb{Z}) = 0.$$

$$C^p(B;\mathcal{H}_X^q) = \left\{ u \in C^p(B;\mathcal{H}_T^q) \colon u(e_\lambda^{(p)}) \in \mathrm{Im}\{\nu^* : H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \hookrightarrow H^q(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})\} \right\}$$

Theorem 4.2.(Y) *For* $\forall\ p, q,\ \delta : C^p(B;\mathcal{H}_T^q) \to C^{p+1}(B;\mathcal{H}_T^q)$ *sends* $C^p(B;\mathcal{H}_X^q)$ *to* $C^{p+1}(B;\mathcal{H}_X^q)$*, and*

$$(E_X)_1^{p,q} \cong C^p(B;\mathcal{H}_X^q),\ d_1 = \delta\Big|_{C^p(B;\mathcal{H}_X^q)}.$$

If $\partial B \neq \phi$ *and* $B^{(0)} \subset \partial B$*, then* $\{(E_X)_r^{p,q}, d_r\}$ *is degenerate at the* E_2*-term.*

Corollary 4.3.(Y) $\chi(X) = \#$ *of corner points of* B*.*

Case of Ex.2.6.

$X' = B' \times T^2 / \sim$

$$(\xi, \theta) \sim (\xi', \theta') \overset{\text{def}}{\Leftrightarrow} \ \xi' = \xi \text{ and}$$
$$\begin{cases} \theta' = \theta & \text{if } \xi \in e^{(2)} \cup e_1^{(1)} \cup e_2^{(1)} \\ \theta' - \theta \in 0 \times S^1 & \text{if } \xi \in e_3^{(1)} \\ \xi \in e^{(0)} & \end{cases}.$$

$X = X' / \sim_{\pi_1}$

$$(\xi, \theta) \sim_{\pi_1} (\xi', \theta')$$
$$\overset{\text{def}}{\Leftrightarrow} \ \exists \omega \in \pi_1(B) \text{ s.t. } \xi' = \xi \cdot \omega, \ \theta' = \rho(\omega)^{-1} \theta.$$

$q = 0.$

$$C^p(B;\mathcal{H}^0_X) = C^p(B;\mathbb{Z}) \ \Rightarrow\ (E_X)_2^{p,0} = H^p(B;\mathbb{Z})$$

$q = 1.$

$$C^p(B;\mathcal{H}^1_X) = \begin{cases} 0 & p = 0 \\ \mathbb{Z}^{\oplus 5} & p = 1 \\ \mathbb{Z}^{\oplus 2} & p = 2 \end{cases}$$

$$\delta : C^1 \to C^2 \ \delta = \begin{pmatrix} -1 & 1 & -2 & 2 & 3 \\ -1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow\ (E_X)_2^{p,1} = \begin{cases} \mathbb{Z}^{\oplus 3} & p = 1 \\ 0 & \text{others} \end{cases}$$

$q = 2.$

$$C^p(B;\mathcal{H}^2_X) = \begin{cases} 0 & p = 0 \\ \mathbb{Z}^{\oplus 2} & p = 1 \\ \mathbb{Z} & p = 2 \end{cases}$$

$$\delta : C^1 \to C^2 \ \delta = 0$$

$$\Rightarrow\ (E_X)_2^{p,2} = \begin{cases} \mathbb{Z}^{\oplus 2} & p = 1 \\ \mathbb{Z} & p = 2 \\ 0 & \text{others} \end{cases}$$

$$
\begin{array}{c|ccc}
q & & & \\
2 & 0 & \mathbb{Z}^2 & \mathbb{Z} \\
1 & 0 & \mathbb{Z}^3 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^2 & 0 \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}
$$

$$
\therefore \ H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 4 \\ \mathbb{Z}^{\oplus 2} & k = 1, 3 \\ \mathbb{Z}^{\oplus 3} & k = 2 \\ 0 & \text{others} \end{cases}
$$