The Ehrhart function for symbols and a generalization of Euler's constant.

THE EHRHART FUNCTION FOR SYMBOLS AND A GENERALIZATION OF EULER'S CONSTANT

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ABSTRACT. We derive an Ehrhart function for symbols from the Euler-MacLaurin formula with remainder.

The material in today's lecture consists of 18-th century mathematics. In fact, outside of some basic facts in the theory of analytic functions of one complex variable, facts which were well understood by the first half of the 19-th century, everything I have to say would be comprehensible to an 18-th century mathematician. Nevertheless, we were very surprised by the result when we discovered it a few weeks ago.

Contents

1. Introduction.

Let $\Delta \subset \mathbb{R}^n$ be a convex polytope whose vertices are in $\mathbb{Z}^n$ and such that the origin 0 is in the interior of Δ. Consider the expanded polytope $N \cdot \Delta$.

Ehrhart's theorem [Ehr] asserts that

$$\#\left((N \cdot \Delta) \cap \mathbb{Z}^n\right), \quad N \in \mathbb{Z}_+$$

is a polynomial in N. More generally, suppose that f is a polynomial, and let

$$p(N, f) := \sum_{\ell \in N \cdot \Delta \cap \mathbb{Z}^n} f(\ell). \tag{1}$$

Then Ehrhart's theorem asserts that $p(N, f)$ is a polynomial in N.

[Ehr] Ehrhart, E., *Sur les polyèdres rationnels homothétiques à n dimensions.* C. R. Acad. Sci. Paris **254** (1962) 616–618.

Ehrhart from Euler-MacLaurin.

In the case that Δ is a simple polytope (meaning that n edges emanate from each vertex) Ehrhart's theorem is a consequence of the Euler-MacLaurin formula, [Kh, KP, CS1, CS2, Gu, BV, DR] and one can be more explicit about the nature of the polynomial $p(N, f)$.

Here are some examples of simple and non-simple polytopes:

Simple and non-simple polytopes.

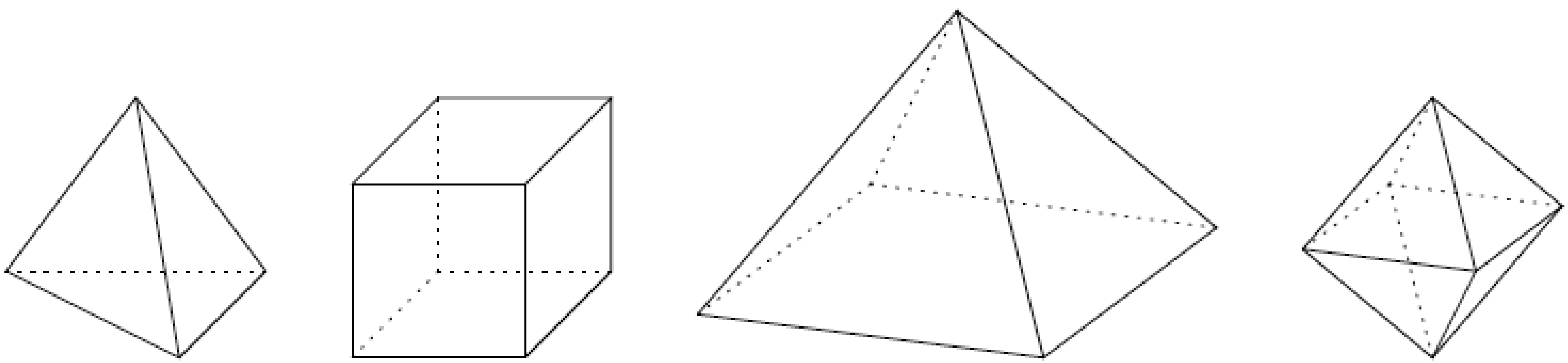


FIGURE 1. Three dimensional polytopes

Simple

Non-simple

Some references for the exact Euler-MacLaurin formula in higher dimensions.

[BV] M. Brion and M. Vergne, *Lattice points in simple polytopes*, Jour. Amer. Math. Soc. **10** (1997), 371–392.

[CS1] S. E. Cappell and J. L. Shaneson, *Genera of algebraic varieties and counting lattice points*, Bull. A. M. S. **30** (1994), 62–69.

[CS2] S. E. Cappell and J. L. Shaneson, *Euler-Maclaurin expansions for lattices above dimension one*, C. R. Acad. Sci. Paris Sr. I Math. **321** (1995), 885–890.

[DR] R. Diaz and S. Robins, *The Ehrhart Polynomial of a Lattice Polytope*, Ann. Math. (2) **145** (1997), no. 3, 503–518, and *Erratum: "The Ehrhart polynomial of a lattice polytope"*, Ann. Math. (2) **146** (1997), no. 1, 237.

[Gu] V. Guillemin, *Riemann-Roch for toric orbifolds*, J. Diff. Geom. **45** (1997), 53–73.

[Kh] A. G. Khovanski, *Newton polyhedra and toroidal varieties*, Func. Anal. Appl. **11** (1977), 289–296.

[KP] A. G. Khovanskii and A. V. Pukhlikov, *The Riemann-Roch theorem for integrals and sum of quasipolynomials on virtual polytopes*, Algebra i Analiz **4** (1992), 188–216, translation in St. Petersburg Math. J. **4** (1993), 789–812.

Regular polytopes.

Let us explain how this works in the more restrictive case where Δ is not only simple but is regular, meaning that the local cone at each vertex can be transformed by an integral unimodular affine transformation into a neighborhood of the origin in the standard orthant $\mathbb{R}^n_+$. Whereas the simple polyopes are "generic" in a suitable sense, the regular polytopes are quite special. For example every convex polygon in the plane is simple, but only special ones are regular. The original Euler-MacLauin formula in higher dimensions was given by Khovanskii and Pukhlikov for the case of regular polytopes and their formulation has the advantage that it is easy to state as we shall see.

Regular and non-regular polytopes.

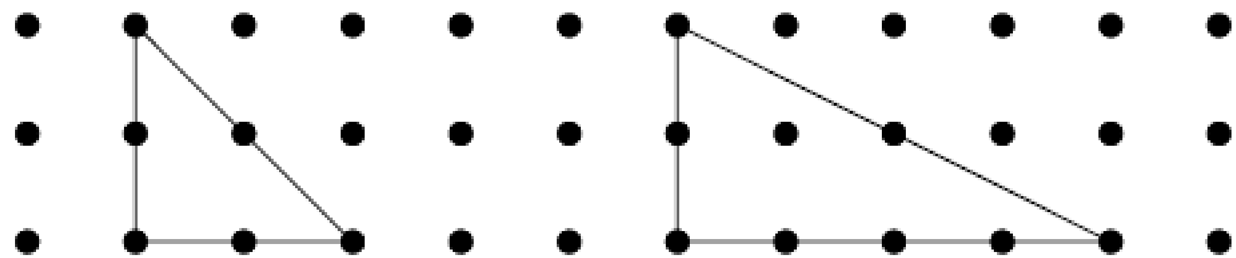


FIGURE 2. A regular polygon and a non-regular polygon

Of course, this use of the word "regular" has nothing to do with the term denoting Platonic solids. There are other names in the literature for the property we are describing, such as "smooth", "Delzant" ,"tosionfree", "unimodular" etc.. It is unfortunate that the nomenclature for polytopes with this property has not yet been standardized.

The Khovanskii-Pukhlikov formula, I.

For the case of regular polytopes the formula of Khovanskii-Pukhlikov [KP] reads as follows: The polytope Δ can be described by a set of inequalities

$$x \cdot u_i + a_i \geq 0, \qquad i = 1, \ldots, m$$

where m is the number of facets of Δ, where the u_i are primitive lattice vectors, and where the a_i are positive integers. Then for any positive number t, the expanded polytope $t \cdot \Delta$ is described by the inequalities

$$x \cdot u_i + ta_i \geq 0, \qquad i = 1, \ldots, m.$$

The Khovanskii-Pukhlikov formula, 2, moving the facets.

Let $\Delta_{t,h}$, $h = (h_1, \ldots, h_m)$ be the polytope defined by

$$x \cdot u_i + ta_i + h_i \geq 0, \qquad i = 1, \ldots, m. \tag{2}$$

Then the function

$$\tilde{p}(t, h, f) := \int_{\Delta_{t,h}} f(x) dx \tag{3}$$

is a polynomial in t and h.

The Khovanskii-Pukhlikov formula,3, the Todd operator.

The formula of Khovanskii-Pukhlikov (applied to $N \cdot \Delta$) expresses $p(N, f)$ in terms of a differential operator applied to $\tilde{p}(t, h, f)$: Explicitly, consider the infinite order constant coefficient differential operator

$$\text{Todd}\left(\frac{\partial}{\partial h}\right) = \sum_{\alpha} b_\alpha \left(\frac{\partial}{\partial h}\right)^\alpha \tag{4}$$

where $\sum_\alpha b_\alpha x^\alpha$ is the Taylor series expansion at the origin of the Todd function

$$\text{Todd}(x) = \prod_{i=1}^{m} \frac{x_i}{1 - e^{-x_i}}.$$

The Khovanskii-Pukhlikov formula.

The Khovanskii-Pukhlikov formula says that

$$p(N, f) = \operatorname{Todd}\left(\frac{\partial}{\partial h}\right) \tilde{p}(N, h, f)\bigg|_{h=0}.$$

Note that since $\tilde{p}$ is a polynomial in h the right hand side really involves only a finite order differential operator.

For purposes below it will be convenient to write the Khovanskii Pukhlikov formula in the form

$$(5)\quad p(N, f) - \tilde{p}(N, 0, f) = \left(\operatorname{Todd}\left(\frac{\partial}{\partial h}\right) - \operatorname{Id}\right) \tilde{p}(N, h, f)\bigg|_{h=0}.$$

For simple polytopes there is a more general formula due to [CS2, Gu, BV]. Our goal in this talk is to describe an analogue of (5) and its generalizations when the polynomial f is replaced by a "symbol".

Symbols.

We recall a definition from the theory of partial differential equations. A smooth function $f \in C^\infty(\mathbb{R}^n)$ is called a **symbol of order** N if for every n-tuple of non-negative integers $a := (a_1, \ldots, a_n)$, there exists a constant C_a such that

$$|\partial_1^{a_1} \ldots \partial_n^{a_n} f(x)| \leq C_a(1 + |x|)^{N-|a|}$$

where $|a| = \sum_i a_i$. In particular, a polynomial of degree N is a symbol of order N. Note that if f is a symbol of order N on $\mathbb{R}^n$ then its derivatives of order a are in L^1 if $N < |a| - n$.

Polyhomogeneous symbols.

For simplicity, we will restrict ourselves in this paper to **polyhomogeneous symbols**, meaning symbols $f \in C^\infty(\mathbb{R}^n)$ which admit asymptotic expansions of the form:

$$f(x) \sim \sum_{-\infty}^{r} f_\ell(x) \tag{6}$$

for $\|x\| >> 0$ where the $f_\ell \in C^\infty(\mathbb{R}^n \setminus \{0\})$ are homogeneous symbols of degree ℓ. The sum is over a discrete sequence of numbers tending to $-\infty$.

The order of a symbol.

$$f(x) \sim \sum_{-\infty}^{r} f_\ell(x) \tag{6}$$

for $\|x\| >> 0$ where the $f_\ell \in C^\infty(\mathbb{R}\setminus\{0\})$ are homogeneous symbols of degree ℓ. The sum is over a discrete sequence of numbers tending to $-\infty$.

"Asymptotic" means that for any j

$$\left| f(x) - \sum_{\ell=j}^{r} f_\ell(x) \right| = o(\|x\|^j)$$

as $\|x\| \to \infty$. The number r occurring in (6) is called the **order** of the asymptotic series and the collection of functions satisfying (6) will be called symbols of order r and denoted by S^r.

The Ehrhart formula.

$$(5)\quad p(N,f) - \tilde{p}(N,0,f) = \left(\operatorname{Todd}\left(\frac{\partial}{\partial h}\right) - \operatorname{Id}\right)\tilde{p}(N,h,f)\bigg|_{h=0}.$$

We will show that if f has this property then the function $p(N,f)$ given by (1) is a polyhomogeneous symbol in N and its asymptotic expansion in powers of N is given by a formula similar to (5) with two key differences:

(1) For symbols, an infinite number of differentiations occur on the right hand side of (5), i.e. the whole Todd operator must be applied. So (5) must be understood as an asymptotic series, not as an equality.

(2) The formula (5) has to be corrected by adding a constant term C to the right hand side, a constant which is zero for the case of a polynomial.

More precisely, we will prove:

Theorem 1.1. *Let Δ be a regular polytope whose vertices lie in $\mathbb{Z}^n$ with 0 in the interior of Δ. Let $f \in S^r$ and $N \in \mathbb{Z}_+$ and let*

$$p(N,f) := \sum_{\ell \in N\cdot\Delta\cap\mathbb{Z}^n} f(\ell).$$

Let $\tilde{\mathrm{p}}(t,h,f)$ be defined by (3) so that

$$\tilde{p}(N,0,f) = \int_{N\cdot\Delta} f(x)dx,$$

Then $p(N,f)-\tilde{p}(N,0,f)$ is a symbol in N and has the asymptotic expansion

$$p(N,f) - \tilde{p}(N,0,f) \sim \left(\mathrm{Todd}\left(\frac{\partial}{\partial h}\right) - \mathrm{Id}\right)\tilde{p}(N,h,f)\Big|_{h=0} + C$$

where C is a constant.

The constant C .

The constant C is of interest in its own right. It can be thought of as a "regularized" version of the difference

$$\sum_{\ell \in \mathbb{Z}^n} f(\ell) - \int_{\mathbb{R}^n} f(x)dx. \tag{7}$$

Of course there is no reason why either the sum or the integral in (7) should converge. But we can "regularize" both as follows: Define the function $\langle x \rangle$ by

$$\langle x \rangle^2 := 1 + \|x\|^2.$$

For $s \in \mathbb{C}$ let

$$f(x, s) := f(x)\langle x \rangle^s.$$

$$\langle x\rangle^2 := 1 + \|x\|^2.$$

For $s \in \mathbb{C}$ let

$$f(x,s) := f(x)\langle x\rangle^s.$$

We will show that

$$C(s) := \sum_{\ell\in\mathbb{Z}^n} f(\ell,s) - \int_{\mathbb{R}^n} f(x,s)dx, \tag{8}$$

which is holomorphic for Re $s << 0$, has an analytic continuation to the entire complex plane and that the missing constant C on the right hand side of (5) is exactly $C(0)$. In particular, the constant C is independent of the particular polytope in question.

Independence of regularization.

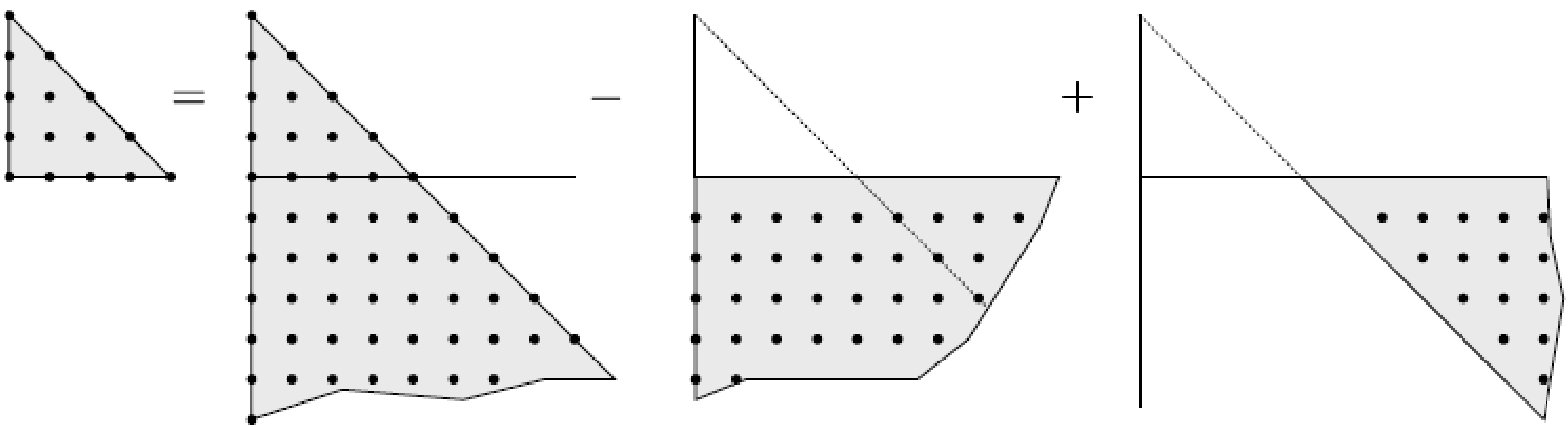
Our result is somewhat insensitive to the mode of regularization. In fact, it can be generalized as follows: Define a "gauged symbol" [Gugau] to be a function $f(x,s) \in C^\infty(\mathbb{R}^n \times \mathbb{C})$ which depends holomorphically on s and for fixed s is a symbol of order Re $s+r$. For example the function $f(x)\langle x\rangle^s$ introduced above is such a gauged symbol. We will prove that if $f(x,s)$ is a gauged symbol with

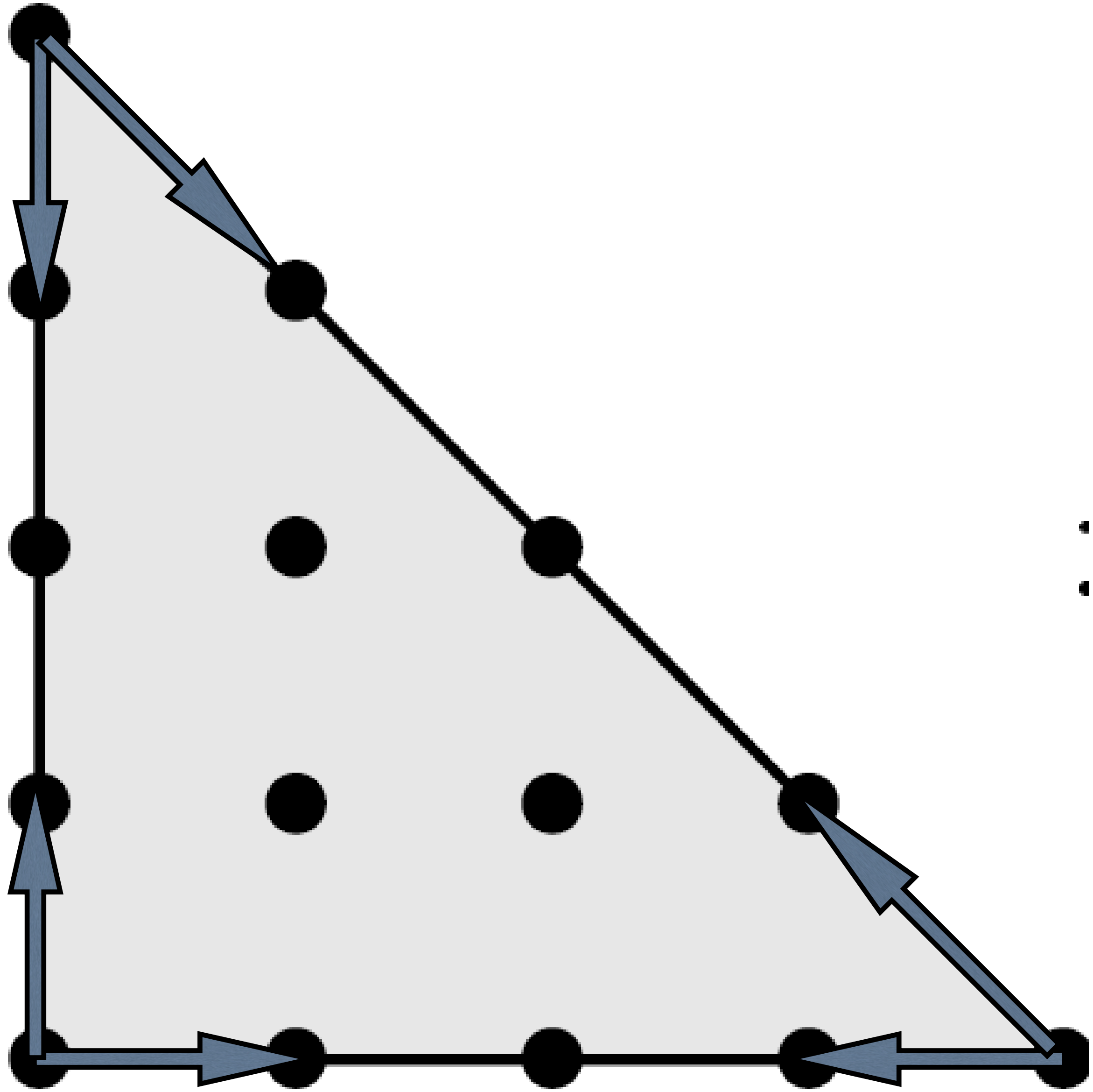
$$f(x) = f(x,0)$$

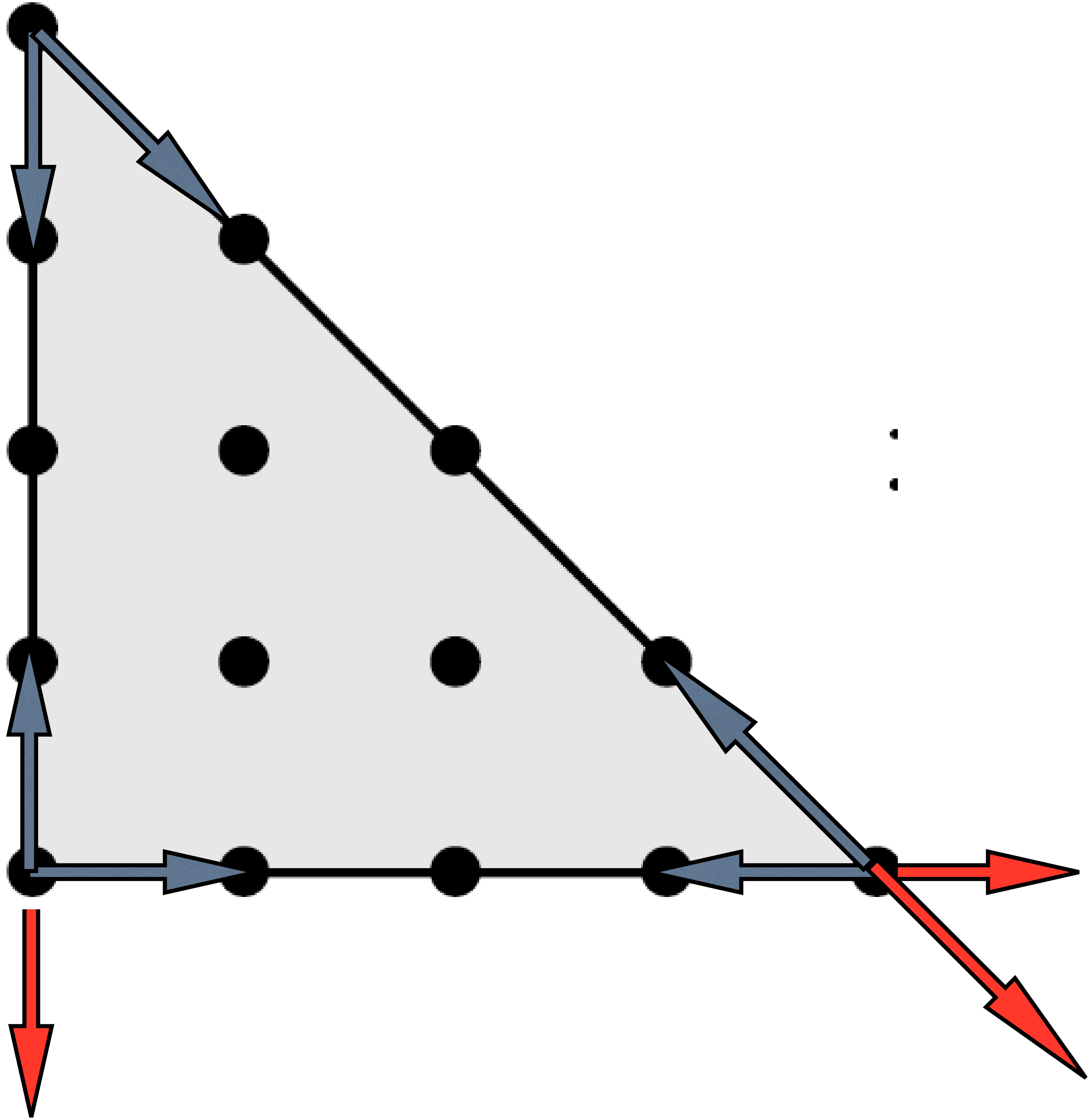
then the function given by (8) with this more general definition of $f(x,s)$ again extends holomorphically from Re $s << 0$ to the entire plane and $C = C(0)$.

The above results will be proved for the more general case of simple integral polytopes in §2. The proof is largely based on the Euler-MacLaurin formula with remainder as proved in [KSW] and motivated by an argument of Hardy on “Ramanujan regularization”, [Hardy]. Ramanujan’s key idea was to use the classical Euler-MacLaurin formula in one variable to regularize (7) by providing “counter terms” in passing to infinity in the difference between sum and integral in one dimension.

The polar decomposition theorem.







The construction in general.

fix a vector $\xi \in \mathbb{R}^n$ such that

$$\alpha_{i,p} \cdot \xi \neq 0 \quad \forall\, p \text{ and } i. \tag{10}$$

We then define

$$\alpha^{\sharp}_{i,p} \quad := \quad \begin{cases} \alpha_{i,p} & \text{if} \quad \alpha_{i,p} \cdot \xi > 0 \\ -\alpha_{i,p} & \text{if} \quad \alpha_{i,p} \cdot \xi < 0 \end{cases} \tag{11}$$

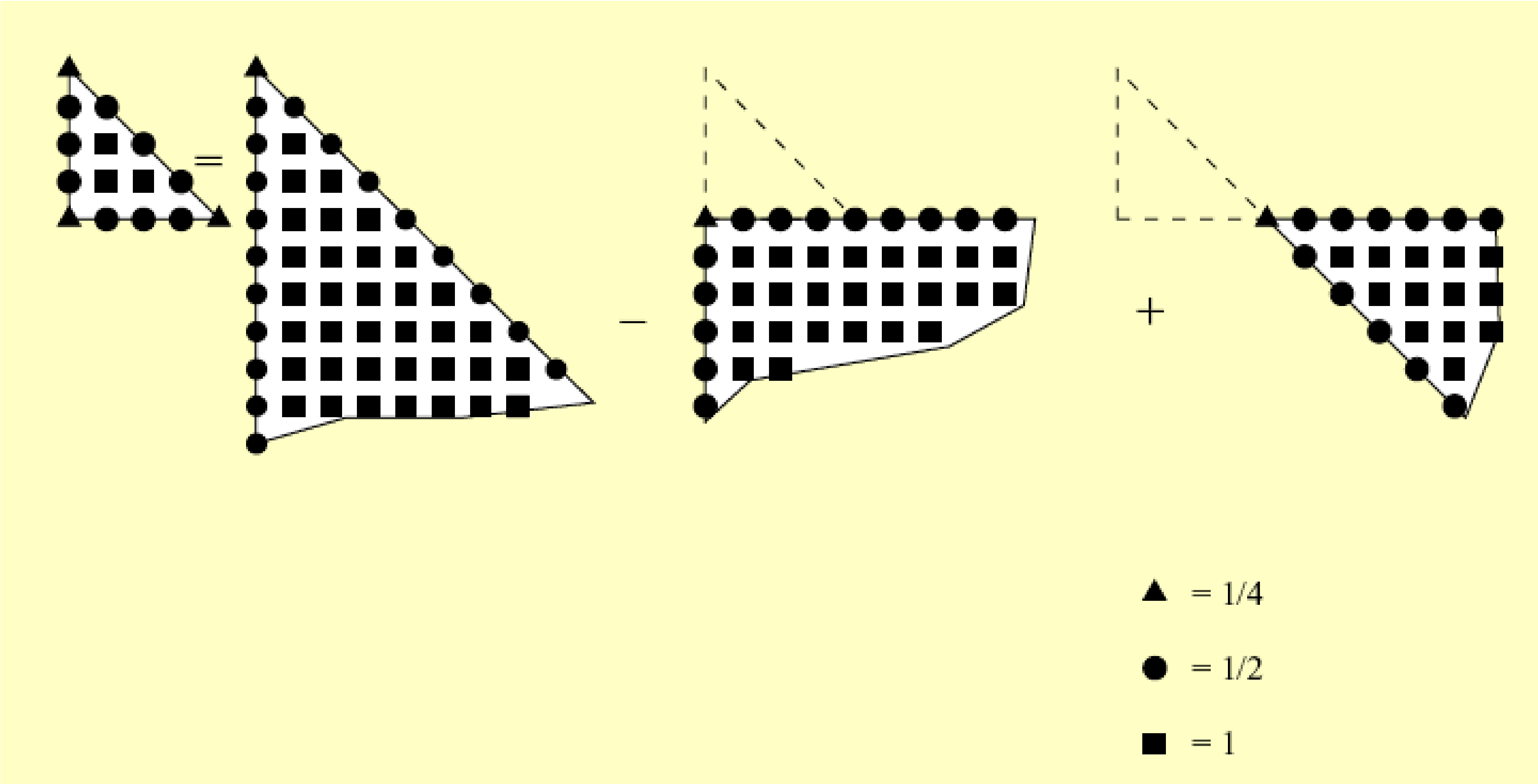
$$(-1)^p \quad := \quad \prod_{i=1}^{n} \frac{\alpha^{\sharp}_{i,p} \cdot \xi}{\alpha_{i,p} \cdot \xi} \tag{12}$$

and

$$C_{p,t} \quad := \quad \left\{ tp + \sum_{i=1}^{n} t_i \alpha^{\sharp}_{i,p}, \quad t_i \geq 0 \right\}. \tag{13}$$

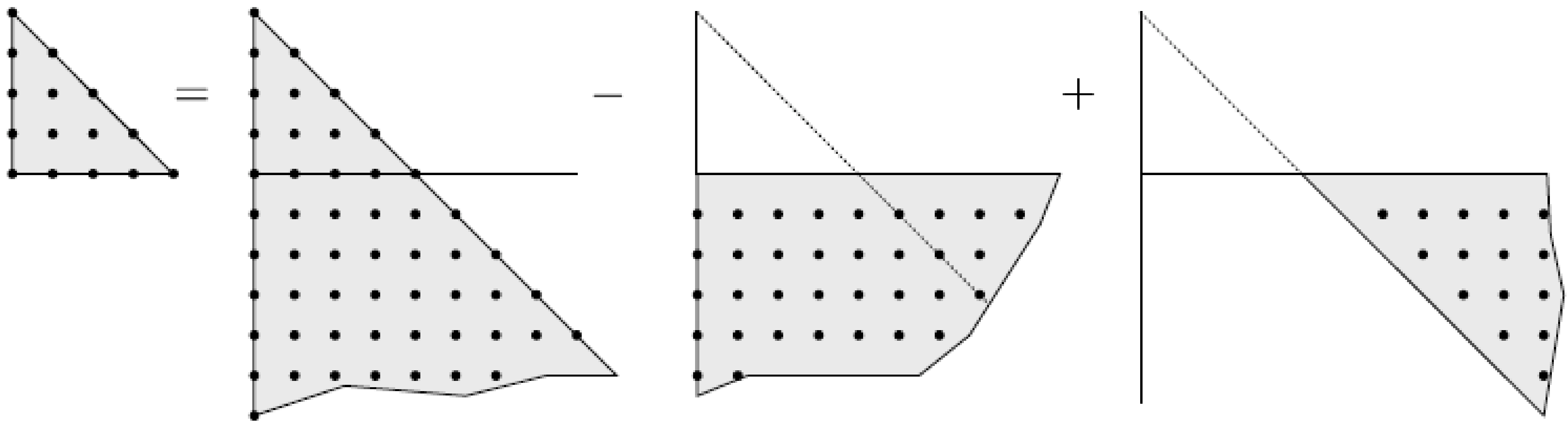
Polar decomposition.

Then, as in the the figure of the triangle, the polytope $t \cdot \Delta$ "is" an alternating sum sum of the cones $C_{p,t}$. There is a slight problem with overcounting the points on the boundary of the cones. As far as the integral is concerned this makes no difference as these sets have measure zero. But if we are summing over lattice points, there will be lattice points on the boundary.



Here are two (of many) ways to handle this problem in our triangle example: We could assign weight one to points in the interior of the triangle or cones, weight $\frac{1}{2}$ to points on the relative interior of the edges of the triangle or cones, and weight $\frac{1}{4}$ to the vertices. Then the alternating sum with this weighting matches up correctly.

Using unweighted sums.



Or if we insist on weight one, then we may have to "shift" the integer summation over the cones.

A general theory of weightings is presented in [AW].

[AW] J. Agapito and J Weitsman *The weighted Euler-MacLaurin formula for a simple integral polytope.* Asian Journal of Mathematics,9 2005 199-212

The Euler-MacLaurin formula with remainder.

2.1. **The Euler-MacLaurin formula for symbols.** We want to apply the Euler-MacLaurin formula with remainder, [KSW]. In [KSW] one dealt with a weighted sum where points in the interior of the polytope are given weight 1, points on the relative interior of a facet are given weight $w(x) := \frac{1}{2}$, and, more generally, points in the relative interior of a face of codimension k are given weights $w(x) := \frac{1}{2^k}$. The weighted sum $p_{\frac{1}{2}}(N, f)$ is then defined as

$$p_{\frac{1}{2}}(N, f) := \sum_{\ell \in (N \cdot \Delta) \cap \mathbb{Z}^n} w(\ell) f(\ell).$$

Theorem 3 of [KSW] gives an Euler-MacLaurin formula with remainder for weighted sums of symbols.

[KSW] Y. Karshon, S. Sternberg, and J. Weitsman, *Euler-MacLaurin with remainder for a simple integral polytope* Duke Mathematical Journal **130** 2005, 401- 434

More generally, [AW] consider the Euler-MacLaurin formula with remainder for more general weightings including the unweighted sum we considered in §1. We refer to equations (28) and (29) in [AW] for the definition of a general weighting, w, and we will denote the corresponding weighted sum here by $p_w(N, f)$. They stated their formula with remainder for smooth functions of compact support, but the passage from the case of smooth functions of compact support to that of symbols is exactly the same as in [KSW].

There is a certain infinite order differential operator $\mathbf{M}$ (depending on the weighting) in the variables $h_1, \ldots, h_m$ with constant term 1 whose truncation of order k is given by the sum in equation (89) of [KSW] (for weight $\frac{1}{2}$) and the sum in equation (56) in [AW] (for general weights) such that for any symbol $f \in S^r$ and $k > n + r$

(14)

$$p_w(N,f) - \tilde{p}(N,0,f) = \left((\mathbf{M}^{[k]}) \left(\frac{\partial}{\partial h} \right) - \mathrm{Id} \right) \tilde{p}(N,h,f) \Big|_{h=0} + R^k(f,N)$$

where $\mathbf{M}^{[k]}$ denotes the truncation of $\mathbf{M}$ at order k and

$$R^k(f,N) = \sum_p (-1)^p \int_{C_{p,N}} \left(\sum_{|\gamma|=k}^{|\gamma|=nk} \phi^p_{\alpha,k} D^\alpha f \right) dx \tag{15}$$

A very rough outline of the proof.

(14)

$$p_w(N,f)-\tilde{p}(N,0,f) = \left((\mathrm{M}^{[k]}) \left(\frac{\partial}{\partial h} \right) - \mathrm{Id} \right) \tilde{p}(N,h,f) \bigg|_{h=0} + R^k(f,N)$$

The idea of the proof of (14) is to use the polar decomposition to reduce the Euler-MacLaurin formula for polytopes to a corresponding formula for cone. To prove the formula for cones one transforms the cone to the standard orthant $\mathbb{R}^n_+$ and then use derive a formula for $\mathbb{R}^n_+$ by reducing it to some "twisted" Euler-MacLaurin formuls in one dimension. The actual details are somewhat complicated.

The limiting form of the remainder.

(14)

$$p_w(N,f)-\tilde{p}(N,0,f) = \left((\mathrm{M}^{[k]}) \left(\frac{\partial}{\partial h} \right) - \mathrm{Id} \right) \tilde{p}(N,h,f) \Big|_{h=0} + R^k(f,N)$$

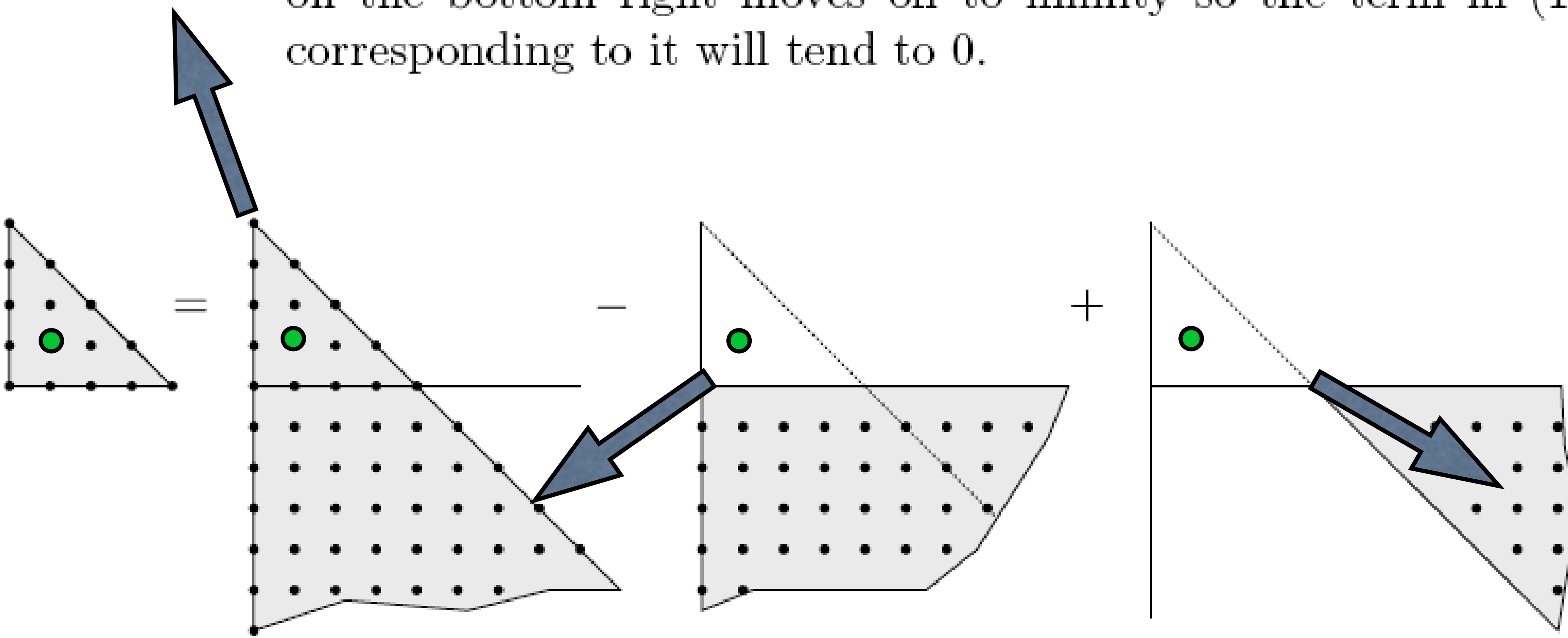
where $\mathrm{M}^{[k]}$ denotes the truncation of M at order k and

$$R^k(f,N) = \sum_p (-1)^p \int_{C_{p,N}} \left(\sum_{|\gamma|=k}^{|\gamma|=nk} \phi^p_{\alpha,k} D^\alpha f \right) dx \tag{15}$$

We want to investigate the behavior of this remainder as $N \to \infty$. Once again, let us examine the picture of what is happing to our decomposition of the triangle. Remember that the origin is in the interior of the triangle.

The upper vertex is moving upward and leftward. So the corresponding cone fills up all of the plane. The term in (15) corresponding to it will tend to an integral over all of space.

The bottom left vertex is moving down and to the left. So the cone corresponding to it will move off to infinity, so the term in (15) corresponding to it will tend to 0. Similarly, the cone on the bottom right moves off to infinity so the term in (15) corresponding to it will tend to 0.



So the remainder term will end to a certain integral over the whole plane.

Here is the argument in general:

Since the $\alpha^\sharp_{i,p}$ form a basis of $\mathbb{R}^n$ we can write

$$p = \sum_{i=1}^{n} a_{i,p}\alpha^\sharp_{i,p}, \tag{16}$$

where at least one $a_{i,p} \neq 0$ since we are assuming that the origin lies in the interior of the polytope. We may assume by relabeling that

$$a_{1,p} \geq a_{2,p} \geq \cdots \geq a_{n,p}.$$

If $a_{1,p} > 0$ then the cone $C_{p,t}$ is contained in the half space

$$\left\{ x = \sum_i x_i \alpha^\sharp_{i,p}, \quad x_1 \geq a_{1,p}t \right\}$$

and so the p-th summand in (15) tends to zero as $N \to \infty$.

On the other hand, if $a_1 = \cdots = a_q = 0$ and $a_i < 0$ for $i > q$ then as $N \to \infty$ the p-th summand in (15) tends to

$$(-1)^p \int_{W_p} \left(\sum_{|\alpha|=k}^{|\gamma|=nk} \phi^p_{\gamma,k} D^\gamma f \right) dx \tag{17}$$

where W_p is the set of all points of the form

$$x = \sum_{i \leq q} x_i \alpha^\sharp_{i,p} + \sum_{j>q} y_j \alpha^\sharp_{j,p}$$

where

$$0 \leq x_i < \infty \quad \text{and} \quad -\infty < y_j < \infty.$$

(14)
$$p_w(N,f)-\tilde{p}(N,0,f) = \left((\mathrm{M}^{[k]})\left(\frac{\partial}{\partial h}\right)-\mathrm{Id}\right)\tilde{p}(N,h,f)\bigg|_{h=0}+R^k(f,N)$$

It follows from (14) that the sum of these limiting values, i.e.

$$\sum_p(-1)^p\int_{W_p}\left(\sum_{|\alpha|=k}^{|\gamma|=nk}\phi^p_{\gamma,k}D^\gamma f\right)dx \tag{18}$$

is independent of k for k sufficiently large. We shall interpret this limiting value C using regularization in the next section. If f is a polynomial, so that we choose k to be greater than the degree of f, we see from (18) that $C=0$, as it must be from the classical Ehrhart theorem.

2.2. **Regularization.** Suppose that we replace f by a gauged symbol $f(x,s)$ with $f(x,0) \in S^r$. Then the remainder term (15) applied to $f_s = f(\cdot, s)$ is well defined if Re $s < -r-n+k$. Moreover, if $a_{1,p} > 0$ the p-th summand on the right of (15) is of order $O(N^{\mathrm{Re}\, s+r+n-k})$ and if $a_{1,p} \leq 0$ the p-th summand of (15) differs from the integral

$$(-1)^p \int_{W_p} \left(\sum_{|\alpha|=k}^{|\alpha|=nk} \phi^p_{\alpha,k} D^\alpha_x f(x,s) \right) dx \tag{19}$$

by a term of order $O(N^{\mathrm{Re}\, s+r+n-k})$. Thus the gauged version of (14) is

$$p_w(N, f_s) - \tilde{p}(N, 0, f_s) =$$
$$\left(\mathbf{M}^{[k]} - \mathrm{Id}\right)\left(\frac{\partial}{\partial h}\right) \tilde{p}(N, h, f_s)\Big|_{h=0} + C_k(s) + O(N^{\mathrm{Re}\, s+r+n-k})$$
$$\mathrm{Re}\, s < -n-r+k$$

Letting $k \to \infty$ we conclude that on this half-plane we have
(21)

$$p_w(N, f_s) - \tilde{p}(N, 0, f_s) \sim \left(\mathbf{M}\left(\frac{\partial}{\partial h}\right) - \mathrm{Id} \right) \tilde{p}(N, h, f_s) \Big|_{h=0} + C(s)$$

where

$$C(s) - C_k(s) = O(N^{\mathrm{Re}\, s + r + n - k}). \tag{22}$$

Since the $C_k(s)$ are holomorphic of the half-plane $\mathrm{Re}\, s < -n - r + k$, it follows that $C(s)$ is holomorphic on the whole plane.

Moreover, in the asymptotic series on the right of (21) all the terms are of order at most $\mathrm{Re}\, s + r + n$. Hence for $\mathrm{Re}\, s < -r - n$ these terms tend to zero and we get

$$\begin{aligned} C(s) &= \lim_{N \to \infty} (p(N, f_s) - \tilde{p}(N, 0, f_s)) \\ &= \sum_{\ell \in \mathbb{Z}^n} f(\ell, s) - \int_{\mathbb{R}^n} f(x, s) dx, \end{aligned}$$

$$
\begin{aligned}
C(s) &= \lim_{N\to\infty} \left(p(N, f_s) - \tilde{p}(N, 0, f_s)\right) \\
&= \sum_{\ell\in\mathbb{Z}^n} f(\ell, s) - \int_{\mathbb{R}^n} f(x, s)dx,
\end{aligned}
$$

and both the sum and the integral converge absolutely. So if we set $s = 0$ we obtain

(23)

$$
p_w(N, f_s) - \tilde{p}(N, 0, f) \sim \left(\mathbf{M}\left(\frac{\partial}{\partial h}\right) - \mathrm{Id}\right)\tilde{p}(N, h, f)\Big|_{h=0} + C
$$

where $f(x) = f(x, 0)$ and $C = C(0)$. So we can think of C as a "regularization" of (7). To summarize: We have proved

Theorem 2.1. *Let Δ be a simple polytope whose vertices lie in $\mathbb{Z}^n$ with 0 in the interior of Δ. Let $f \in S^r$ and $N \in \mathbb{Z}_+$ and let*

$$p_w(N,f) := \sum_{\ell \in N\cdot\Delta\cap\mathbb{Z}^n} w(\ell) f(\ell).$$

Let $\tilde{p}(t,h,f)$ be defined by (3) so that

$$\tilde{p}(N,0,f) = \int_{N\cdot\Delta} f(x)dx,$$

Then $p(N,f) - \tilde{p}(N,0,f)$ is a symbol in N and has the asymptotic expansion

$$p_w(N,f) - \tilde{p}(N,0,f) \sim \left(\mathbf{M}\left(\frac{\partial}{\partial h}\right) - \mathrm{Id}\right)\tilde{p}(N,h,f)\bigg|_{h=0} + C$$

where C is a constant.

Furthermore, if $f(x,s)$ *is a gauged symbol with* $f(x,0) = f(x)$ *then* $C = C(0)$ *where* $C(s)$ *is the entire function given by (21) and (22). For* $\operatorname{Re} s < -r - n$

$$C(s) = \sum_{\ell \in \mathbb{Z}^n} f(\ell, s) - \int_{\mathbb{R}^n} f(x,s)dx.$$

Hence $C(s)$ *and in particular* $C = C(0)$ *is independent of the polytope.*

Relation to Euler's gamma.

Suppose we look at the one dimensional case where our initial polytope is the interval $[-1, 1]$. Consider a function f such that

$$f(x) = \frac{1}{|x|} \quad \text{for} \quad |x| \geq 1$$

and we modify the function so that it is smooth and equals zero in some neighborhood of the origin. Then

$$p(N, f) = 2 \sum_{j=1}^{N} \frac{1}{j}$$

while

$$\tilde{p}(N, f) = 2 \int_{1}^{N} \frac{1}{x} dx + \int_{-1}^{1} f(x) dx.$$

So basically, we are considering the limit of

$$\sum_{j=1}^{N} \frac{1}{j} - \int_{1}^{N} \frac{1}{x} dx$$

which approaches a constant γ (known as Euler's constant). Euler proved this by applying the Euler-MacLaurin formula and in fact gave an ingenious method for computing γ to many decimal places. See [Kn] for example for an exposition of this computation.

[Kn] K. KNOPP, "Chapter 14: Euler's summation formula and asymptotic expansions" in *Theory and Application of Infinite Series*, translated from the 2nd German ed., Blackie, London, 1928, 518 – 555. 406, 409

Relation of *C(s)* to the zeta function.

Suppose we consider the function

$$\int_1^\infty \frac{1}{x^z}dx + \int_1^\infty f(x)dx = \zeta(z) - \frac{1}{z-1}$$

which is initially defined for $\operatorname{Re} z > 1$. As is well known,

$$\zeta(z) - \frac{1}{z-1}$$

is holomorphic on the entire complex plane, and in fact Euler used the Euler-MacLaurin formula to extend $\zeta(z) - \frac{1}{z-1}$ to (real) values of $z < 1$.

So if we set $s = -z$ (and so are considering the function x^s instead of x^{-z}) our $C(s)$ is the analogue of $\zeta(z) - \frac{1}{z-1}$. This, of course, leads to many interesting questions.