Equivariant compactifications of reductive groups

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Abstract. We study equivariant projective compactifications of reductive groups obtained by closing the image of a group in the space of operators of a projective representation. (This is a “non-commutative generalization” of projective toric varieties.) We describe the structure and the mutual position of their orbits under the action of the doubled group by left/right multiplications, the local structure in a neighborhood of a closed orbit, and obtain some conditions of normality and smoothness of a compactification. Our approach uses the theory of equivariant embeddings of spherical homogeneous spaces and of reductive algebraic semigroups.
1. Projective embeddings of reductive groups

Let $G$ be a connected reductive complex algebraic group.

**Examples.** $G = GL_n(\mathbb{C}), SL_n(\mathbb{C}), SO_n(\mathbb{C}), Sp_n(\mathbb{C}), (\mathbb{C}^\times)^n$.

Let $G \otimes \mathbb{P}(V)$ be a faithful projective representation. It comes from a faithful rational linear representation $\tilde{G} \otimes V$, where $\tilde{G} \to G$ is a finite cover.

$$\tilde{G} \hookrightarrow \text{End } V \implies G \hookrightarrow \mathbb{P}(\text{End } V)$$

**Objective.** Describe $X = \overline{G} \subseteq \mathbb{P}(\text{End } V)$.

**Example 1.** $G = T = (\mathbb{C}^\times)^n$ an algebraic torus; $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}^n$ the eigenweights of $T = \tilde{T} \otimes V \implies X$ is a projective toric variety corresponding to the polytope $\mathcal{P} = \text{conv}\{\lambda_1, \ldots, \lambda_m\}$. 
Problem 1. *Describe* \((G \times G)\)-orbits in \(X\): dimensions, representatives, stabilizers, partial order by inclusion of closures.

Problem 2. *Describe the local structure of* \(X\).

Problem 3. *Normality of* \(X\).

Problem 4. *Smoothness of* \(X\).

**Relevant research.**

1) Affine embeddings of reductive groups = reductive algebraic semigroups (Putcha–Renner, Vinberg [Vi95], Rittatore [Ri98]).
2) Regular group compactifications: cohomology (De Concini–Procesi [CP86], Strickland [St91]), cellular decomposition (Brion–Polo [BP00]).
3) Reductive varieties (Alexeev–Brion [AB04], [AB04’]).
Notation.

\( T \subseteq G \) (resp. \( \tilde{T} \subseteq \tilde{G} \)) a maximal torus; \( \Lambda \cong \mathbb{Z}^n \) the weight lattice,

weights \( \tilde{T} \rightarrow \mathbb{C}^\times \), \( \forall \lambda = (l_1, \ldots, l_n) \in \Lambda \),

\( t \mapsto t^\lambda := t_1^{l_1} \cdots t_n^{l_n}, t = (t_1, \ldots, t_n) \in \tilde{T} \);

\( \Lambda(V) = \{ \text{eigenweights of } \tilde{T} \cap V \} \);

\( \mathcal{P} = \text{conv } \Lambda(V) \) the weight polytope;

\( \Delta = \Delta_G \subseteq \Lambda \) the set of roots (\( = \) nonzero \( T \)-eigenweights of \( \text{Lie } G \)),

\( \Delta = \Delta^+ \sqcup \Delta^- \) \( (\text{positive and negative roots}) \);

\( W = N_G(T)/T \) the Weyl group;

\((\cdot, \cdot)\) a \( W \)-invariant inner product on \( \Lambda \);

\( C = C_G = \{ \lambda \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\lambda, \alpha) \geq 0, \forall \alpha \in \Delta^+ \} \) the positive Weyl chamber. It is a fundamental domain for \( W \cap \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \).
2. Orbits

For each face $\mathcal{F}$ of $\mathcal{P}$ (of any dimension, including $\mathcal{P}$ itself) let:

- $V_\mathcal{F} \subseteq V$ be the span of $\tilde{T}$-eigenvectors with weights in $\mathcal{F}$;
- $V'_\mathcal{F} \subseteq V$ be the $\tilde{T}$-stable complement of $V_\mathcal{F}$; $V = V_\mathcal{F} \oplus V'_\mathcal{F}$;
- $E_\mathcal{F}$ = projector $V \rightarrow V_\mathcal{F}$.

**Theorem 1.** There is a 1-1 correspondence:

$$(G \times G)\text{-orbits } Y \subseteq X \longleftrightarrow \text{faces } \mathcal{F} \subseteq \mathcal{P}, \ (\text{int } \mathcal{F}) \cap C \neq \emptyset.$$ 

*Orbit representatives are:* $Y \ni y = \langle E_\mathcal{F} \rangle$.

*Stabilizers are computed.*

*Dimensions:* $\dim Y = \dim \mathcal{F} + |\Delta \setminus \langle \mathcal{F} \rangle^\perp|$.

*Partial order:* $Y_1 \subset Y_2 \iff \mathcal{F}_1 \subset \mathcal{F}_2$. 
Example 2. $G = PGL_n$, $V = \mathbb{C}^n \implies X = \mathbb{P}(\text{Mat}_n) = \mathbb{P}n^{2-1}$. Here $\tilde{G} = SL_n$, $T = \{\text{diagonal matrices}\}$,

$$\Lambda(V) = \{\varepsilon_1, \ldots, \varepsilon_n\} = \text{the standard basis of } \mathbb{Z}^n,$$
$$\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}, \quad \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\},$$
$$C = \{\lambda = (l_1, \ldots, l_n) \mid l_1 \geq \cdots \geq l_n\}$$

We see that $P = \text{conv}\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a simplex, the faces of $P$ whose interior intersects $C$ are

$$\mathcal{F}_r = \text{conv}\{\varepsilon_1, \ldots, \varepsilon_r\}, \quad r = 1, \ldots, n,$$

the respective projectors are

$$E_{\mathcal{F}_r} = \text{diag}((1, \ldots, 1, 0, \ldots, 0)_{r}),$$

and the orbits are

$$Y_r = \mathbb{P}(\text{matrices of rank } r).$$
Proof. Step 1. Let $K \subset G$ be a maximal compact subgroup; then $G = KTK$ (Cartan decomposition) $\implies X = K\overline{T}K \implies \overline{T}$ intersects all $(G \times G)$-orbits in $X$.

Step 2. By toric geometry, there is a 1-1 correspondence:

$$T\text{-orbits in } \overline{T} \longleftrightarrow \text{all faces } \mathcal{F} \subseteq \mathcal{P},$$

which respects partial order, $\langle E_\mathcal{F} \rangle$ being the orbit representatives.

Step 3. Compute the stabilizers of $\langle E_\mathcal{F} \rangle$ in $G \times G$. In particular, this yields the orbit dimensions.

Step 4. Given $y = \langle E_\mathcal{F} \rangle \in Y = G_yG$, the structure of $(G \times G)_y$ implies that $Y^{\text{diag}T} = \bigcup_{w_1, w_2 \in W} Tw_1yw_2$? It follows that

$$Y_1 = Y_2 \iff \mathcal{F}_1 = w\mathcal{F}_2 \quad \text{for some } w \in W.
$$

There exists a unique $\mathcal{F}^+ = w\mathcal{F}$ such that $(\text{int } \mathcal{F}^+) \cap C \neq \emptyset$. \qed
3. Local structure

Fix a closed orbit $Y_0 \subset X$. What is the structure of $X$ in a neighborhood of $Y_0$?

W.l.o.g. $V$ is assumed to be a multiplicity-free $G$-module. By Theorem 1, $Y_0 = G y_0 G$, $y_0 = \langle E_{\lambda_0} \rangle$, $\lambda_0 \in \mathcal{P}$ a vertex, $E_{\lambda_0}$ is the projector $V = \mathbb{C} v_{\lambda_0} \oplus V'_{\lambda_0} \rightarrow \mathbb{C} v_{\lambda_0}$, where $v_{\lambda_0}$ is the (unique) eigenvector of weight $\lambda_0$ (a highest weight vector).

Associated parabolic subgroups: $P^+ = G_{\langle v_{\lambda_0} \rangle}$, $P^- \subseteq G$.

Levi decomposition: $P^\pm = P_u^\pm \rtimes L$, $L = P^+ \cap P^- \supseteq T$.

Theorem 2. $\hat{X} = \{ x = \langle A \rangle \in X \mid A v_{\lambda_0} \notin V'_{\lambda_0} \}$ is a $(P^- \times P^+)$-stable neighborhood of $y_0$ in $X$.

$$\hat{X} \simeq P_u^- \times Z \times P_u^+,$$

where $Z = \bar{L} \subseteq \text{End}(V'_{\lambda_0} \otimes (\lambda_0))$. 
Proof is based on the local structure of $P^- \cap V$ in a neighborhood of $v_{\lambda_0}$ (Brion, Luna, Vust).

**Remark.** $Z$ is a reductive algebraic semigroup with 0 (corresponding to $y_0$), called the slice semigroup.

**Regular case:** $\lambda_0 \in \text{int} C \implies L = T \implies Z$ affine toric variety.

**Example 3.** In the notation of Example 2,

$$Y_0 = \mathbb{P} (\text{matrices of rank 1}), \quad \lambda_0 = \varepsilon_1, \quad v_{\lambda_0} = e_1,$$

$$P_u^+ = \begin{bmatrix} 1 & \ast \\ 0 & \vdots \\ 0 & E \end{bmatrix}, \quad P_u^- = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \ast & \vdots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ast \end{bmatrix}, \quad L = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ \vdots & & \ast \end{bmatrix} \cong GL_{n-1},$$

$$V'_{\lambda_0} = \langle e_2, \ldots, e_n \rangle, \quad Z = \text{Mat}_{n-1}. $$
4. Normality

Does $X$ have normal singularities? It suffices to consider singularities in a neighborhood of a closed orbit $Y_0$. By Theorem 2 it suffices to study the singularity of $Z$ at 0.

$L$ is reductive $\implies$ $L$-modules are completely reducible.

Simple $L$-modules $V = V_L(\lambda) \iff$ highest weights $\lambda \in \Lambda \cap C_L$,

$\lambda + \alpha \notin \Lambda(V), \forall \alpha \in \Delta^+_L$.

$V_L(\lambda) \otimes V_L(\mu) \cong V_L(\lambda + \mu) \oplus \cdots \oplus V_L(\lambda + \mu - \alpha_1 - \cdots - \alpha_k) \oplus \cdots$

$(\alpha_i \in \Delta^+_L)$.

Definition. Weights $\mu_1, \ldots, \mu_k$ $L$-generate a semigroup $\Sigma \subset \Lambda$ if

$\Sigma = \{ \mu \mid V_L(\mu) \hookrightarrow V_L(\mu_{i_1}) \otimes \cdots \otimes V_L(\mu_{i_N}) \}$.

Observation: “generate” (in a usual sense) $\implies$ “$L$-generate”; $\iff$ fails in general.
**Theorem 3.** Let $\lambda_0, \lambda_1, \ldots, \lambda_m$ be the highest weights of the simple $G$-submodules in $V$ and $\alpha_1, \ldots, \alpha_r$ be the simple roots in $\Delta_G^+ \setminus \Delta_L^+$. Put $\Sigma = (\text{director cone of } \mathcal{P} \cap C \text{ at } \lambda_0) \cap \Lambda$. Consider the following conditions:

1. $X$ is normal along $Y_0$;
2. $T$ is normal at $y_0$;
3. $\Sigma$ is $L$-generated by $\lambda_i - \lambda_0, -\alpha_j$.
4. $\Sigma$ is generated by $\lambda - \lambda_0$, $\forall \lambda \in \Lambda(V)$.

Then (1) $\iff$ (3) $\implies$ (2) $\iff$ (4).

**Remark.** (3) $\iff$ $\lambda_i - \lambda_0, -\alpha_j$ $L$-generate a saturated semi-group.
Example 4. \( G = Sp_4, \ V = S^3\mathbb{C}^4 \oplus S^2_0(\Lambda^2_0 \mathbb{C}^4), \) the highest weights \( \{\lambda_0 = 3\omega_1, \lambda_1 = 2\omega_2\} \) (\( \omega_i \) denote the fundamental weights, \( \alpha_i \) the simple roots). Here \( L \simeq SL_2 \times \mathbb{C}^\times, \ \Delta_L = \{\pm \alpha_2\}. \)
\( \lambda_1 - \lambda_0, -\alpha_1 \) \( L \)-generate \{bold dots\} (Clebsch–Gordan formula).
\( \implies X \) non-normal along \( Y_0; \) becomes normal if we add \( \lambda_2 = 2\omega_1. \)
5. Smoothness

**Theorem 4.** $X$ is smooth along $Y_0$ $\iff$ (1)$\&$(2)$\&$(3):

1. $L = GL_{n_1} \times \cdots \times GL_{n_p}$.
2. $L \odot V \otimes (-\lambda_0)$ is polynomial.
3. $[L \odot V \otimes (-\lambda_0)] \leftarrow [GL_{n_i} \odot \mathbb{C}^{n_i}], \forall i$.

**Remark.** (1), (2), (3) are reformulated in terms of $\Lambda(V)$.

**Idea of the proof.** $X$ is smooth $\iff$ $Z$ is smooth $\iff$ $Z \cong \text{Mat}_{n_1} \times \cdots \times \text{Mat}_{n_p}$.

**Example 5.** $G = SO_{2m+1}$, $V = V_G(\omega_i)$ ($\omega_i$ are the fundamental weights).

- a) $i < m$: $V = \bigwedge^i \mathbb{C}^{2m+1}$, $L \cong GL_i \times SO_{2m+1-2i} \implies X$ singular.
- b) $i = m$: $V = \text{spinor module} \cong \bigwedge^\bullet \mathbb{C}^m \otimes \omega_m$ over $L \cong GL_m \implies X$ smooth.
6. “Small” compactifications

Let $G$ be a simple Lie group. We take a closer look at $X = \overline{G} \subseteq \mathbb{P}(\text{End } V)$ for “small” $V = V_G(\omega_i)$ ($\omega_i$ are the fundamental weights) or $V = \text{Lie } G$ (adjoint representation).

**Results:** 1) $(G \times G)$-orbits, their dimensions, Hasse diagrams of partial order.
2) Non-normal: $(SO_{2m+1}, \omega_i), i < m; (Sp_{2m}, \omega_m); (G_2, \omega_2); (F_4, \omega_i), i = 3, 4$.
3) Smooth: $(SL_n, \omega_i), i = 1, n-1; (SL_n, \text{Ad}), n \leq 3; (SO_{2m+1}, \omega_m); (Sp_4, \omega_1); (Sp_4, \text{Ad}); (G_2, \omega_1)$.
References


