Equivariant compactifications of reductive groups

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Abstract. We study equivariant projective compactifications of reductive groups obtained by closing the image of a group in the space of operators of a projective representation. (This is a "non-commutative generalization" of projective toric varieties.) We describe the structure and the mutual position of their orbits under the action of the doubled group by left/right multiplications, the local structure in a neighborhood of a closed orbit, and obtain some conditions of normality and smoothness of a compactification. Our approach uses the theory of equivariant embeddings of spherical homogeneous spaces and of reductive algebraic semigroups.

1. Projective embeddings of reductive groups

Let G be a connected reductive complex algebraic group.
Examples. $G = GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_n(\mathbb{C})$, $(\mathbb{C}^\times)^n$.

Let $G \circlearrowleft \mathbb{P}(V)$ be a faithful projective representation. It comes from a faithful rational linear representation $\widetilde{G} \circlearrowleft V$, where $\widetilde{G} \to G$ is a finite cover.

$$\widetilde{G} \hookrightarrow \operatorname{End} V \implies G \hookrightarrow \mathbb{P}(\operatorname{End} V)$$

Objective. *Describe* $X = \overline{G} \subseteq \mathbb{P}(\operatorname{End} V)$.

Example 1. $G = T = (\mathbb{C}^\times)^n$ an algebraic torus; $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}^n$ the eigenweights of $T = \widetilde{T} \circlearrowleft V \implies X$ is a projective toric variety corresponding to the polytope $\mathcal{P} = \operatorname{conv}\{\lambda_1, \ldots, \lambda_m\}$.

Problem 1. *Describe $(G \times G)$-orbits in X: dimensions, representatives, stabilizers, partial order by inclusion of closures.*

Problem 2. *Describe the local structure of X.*

Problem 3. *Normality of X.*

Problem 4. *Smoothness of X.*

Relevant research.

1) Affine embeddings of reductive groups = reductive algebraic semigroups (Putcha–Renner, Vinberg [Vi95], Rittatore [Ri98]).
2) Regular group compactifications: cohomology (De Concini–Procesi [CP86], Strickland [St91]), cellular decomposition (Brion–Polo [BP00]).
3) Reductive varieties (Alexeev–Brion [AB04], [AB04']).

Notation.

$T \subseteq G$ (resp. $\tilde{T} \subseteq \tilde{G}$) a maximal torus; $\Lambda \simeq \mathbb{Z}^n$ the weight lattice,

$$\text{weights} \quad \tilde{T} \to \mathbb{C}^\times, \qquad \forall \lambda = (l_1, \ldots, l_n) \in \Lambda,$$
$$t \mapsto t^\lambda := t_1^{l_1} \cdots t_n^{l_n}, \qquad t = (t_1, \ldots, t_n) \in \tilde{T};$$

$\Lambda(V) = \{$eigenweights of $\tilde{T} \circlearrowleft V\}$;

$\mathcal{P} = \operatorname{conv} \Lambda(V)$ the weight polytope;

$\Delta = \Delta_G \subset \Lambda$ the set of roots ($=$ nonzero T-eigenweights of $\operatorname{Lie} G$),

$\Delta = \Delta^+ \sqcup \Delta^-$ (positive and negative roots);

$W = N_G(T)/T$ the Weyl group;

$(\cdot, \cdot)$ a W-invariant inner product on Λ;

$C = C_G = \{\lambda \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\lambda, \alpha) \geq 0, \ \forall \alpha \in \Delta^+\}$ the positive Weyl chamber. It is a fundamental domain for $W \circlearrowleft \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

2. Orbits

For each face $\mathcal{F}$ of $\mathcal{P}$ (of any dimension, including $\mathcal{P}$ itself) let:
$V_{\mathcal{F}} \subseteq V$ be the span of $\widetilde{T}$-eigenvectors with weights in $\mathcal{F}$;
$V'_{\mathcal{F}} \subseteq V$ be the $\widetilde{T}$-stable complement of $V_{\mathcal{F}}$; $V = V_{\mathcal{F}} \oplus V'_{\mathcal{F}}$;
$E_{\mathcal{F}} =$ projector $V \to V_{\mathcal{F}}$.

Theorem 1. *There is a 1-1 correspondence:*

$$(G \times G)\text{-orbits } Y \subseteq X \quad \longleftrightarrow \quad \text{faces } \mathcal{F} \subseteq \mathcal{P},\ (\operatorname{int} \mathcal{F}) \cap C \neq \emptyset.$$

Orbit representatives are: $Y \ni y = \langle E_{\mathcal{F}} \rangle$.
Stabilizers are computed.
Dimensions: $\dim Y = \dim \mathcal{F} + \left| \Delta \setminus \langle \mathcal{F} \rangle^{\perp} \right|$.
Partial order: $Y_1 \subset \overline{Y_2} \iff \mathcal{F}_1 \subset \mathcal{F}_2$.

Example 2. $G = PGL_n$, $V = \mathbb{C}^n \implies X = \mathbb{P}(\mathsf{Mat}_n) = \mathbb{P}^{n^2-1}$. Here $\widetilde{G} = SL_n$, $T = \{$diagonal matrices$\}$,

$$\Lambda(V) = \{\varepsilon_1, \ldots, \varepsilon_n\} = \text{the standard basis of } \mathbb{Z}^n,$$
$$\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}, \quad \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\},$$
$$C = \{\lambda = (l_1, \ldots, l_n) \mid l_1 \geq \cdots \geq l_n\}$$

We see that $\mathcal{P} = \mathrm{conv}\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a simplex, the faces of $\mathcal{P}$ whose interior intersects C are

$$\mathcal{F}_r = \mathrm{conv}\{\varepsilon_1, \ldots, \varepsilon_r\}, \qquad r = 1, \ldots, n,$$

the respective projectors are

$$E_{\mathcal{F}_r} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0),$$

and the orbits are

$$Y_r = \mathbb{P}(\text{matrices of rank } r).$$

Proof. **Step 1.** Let $K \subset G$ be a maximal compact subgroup; then $G = KTK$ (Cartan decomposition) $\implies X = K\overline{T}K \implies \overline{T}$ intersects all $(G \times G)$-orbits in X.

Step 2. By toric geometry, there is a 1-1 correspondence:

$$T\text{-orbits in } \overline{T} \quad \longleftrightarrow \quad \text{all faces } \mathcal{F} \subseteq \mathcal{P},$$

which respects partial order, $\langle E_{\mathcal{F}} \rangle$ being the orbit representatives.

Step 3. Compute the stabilizers of $\langle E_{\mathcal{F}} \rangle$ in $G \times G$. In particular, this yields the orbit dimensions.

Step 4. Given $y = \langle E_{\mathcal{F}} \rangle \in Y = GyG$, the structure of $(G \times G)_y$ implies that $Y^{\operatorname{diag} T} = \bigcup_{w_1, w_2 \in W} T w_1 y w_2$? It follows that

$$Y_1 = Y_2 \iff \mathcal{F}_1 = w\mathcal{F}_2 \qquad \text{for some } w \in W.$$

There exists a unique $\mathcal{F}^+ = w\mathcal{F}$ such that $(\operatorname{int} \mathcal{F}^+) \cap C \neq \emptyset$. $\square$

3. Local structure

Fix a closed orbit $Y_0 \subset X$. What is the structure of X in a neighborhood of Y_0?

W.l.o.g. V is assumed to be a multiplicity-free G-module. By Theorem 1, $Y_0 = Gy_0G$, $y_0 = \langle E_{\lambda_0} \rangle$, $\lambda_0 \in \mathcal{P}$ a vertex, E_{λ_0} is the projector $V = \mathbb{C}v_{\lambda_0} \oplus V'_{\lambda_0} \to \mathbb{C}v_{\lambda_0}$, where v_{λ_0} is the (unique) eigenvector of weight λ_0 (a highest weight vector).

Associated parabolic subgroups: $P^+ = G_{\langle v_{\lambda_0} \rangle}$, $P^- \subseteq G$.

Levi decomposition: $P^\pm = P^\pm_{\mathrm{u}} \rtimes L$, $L = P^+ \cap P^- \supseteq T$.

Theorem 2. *$\mathring{X} = \{x = \langle A \rangle \in X \mid Av_{\lambda_0} \notin V'_{\lambda_0}\}$ is a $(P^- \times P^+)$-stable neighborhood of y_0 in X.*

$$\mathring{X} \simeq P_u^- \times Z \times P_u^+, \qquad \textit{where } Z = \overline{L} \subseteq \mathrm{End}(V'_{\lambda_0} \otimes (-\lambda_0)).$$

Proof is based on the local structure of $P^- \circlearrowleft V$ in a neighborhood of v_{λ_0} (Brion, Luna, Vust).

Remark. Z is a reductive algebraic semigroup with 0 (corresponding to y_0), called the slice semigroup.

Regular case: $\lambda_0 \in \operatorname{int} C \implies L = T \implies Z$ affine toric variety.

Example 3. In the notation of Example 2,

$$Y_0 = \mathbb{P}(\text{matrices of rank } 1), \qquad \lambda_0 = \varepsilon_1, \qquad v_{\lambda_0} = e_1,$$

$$P_{\mathrm{u}}^+ = \left[\begin{array}{c|c} 1 & * \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & E \end{array}\right], \quad P_{\mathrm{u}}^- = \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline * & E \end{array}\right], \quad L = \left[\begin{array}{c|c} * & 0 \cdots 0 \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & * \end{array}\right] \simeq GL_{n-1},$$

$$V'_{\lambda_0} = \langle e_2, \dots, e_n \rangle, \qquad Z = \mathrm{Mat}_{n-1}\,.$$

4. Normality

Does X have normal singularities? It suffices to consider singularities in a neighborhood of a closed orbit Y_0. By Theorem 2 it suffices to study the singularity of Z at 0.

$$L \text{ is reductive} \implies L\text{-modules are completely reducible.}$$

$$\text{Simple } L\text{-modules } V = V_L(\lambda) \longleftrightarrow \text{highest weights } \lambda \in \Lambda \cap C_L,$$
$$\lambda + \alpha \notin \Lambda(V), \ \forall \alpha \in \Delta_L^+.$$

$$V_L(\lambda) \otimes V_L(\mu) \simeq V_L(\lambda+\mu) \oplus \cdots \oplus V_L(\lambda+\mu-\alpha_1-\cdots-\alpha_k) \oplus \cdots$$
$$(\alpha_i \in \Delta_L^+).$$

Definition. Weights $\mu_1, \ldots, \mu_k$ L-generate a semigroup $\Sigma \subset \Lambda$ if

$$\Sigma = \{\mu \mid V_L(\mu) \hookrightarrow V_L(\mu_{i_1}) \otimes \cdots \otimes V_L(\mu_{i_N})\}.$$

Observation: "generate" (in a usual sense) $\implies$ "L-generate"; $\impliedby$ fails in general.

Theorem 3. *Let* $\lambda_0, \lambda_1 \dots, \lambda_m$ *be the highest weights of the simple* G*-submodules in* V *and* $\alpha_1, \dots, \alpha_r$ *be the simple roots in* $\Delta_G^+ \setminus \Delta_L^+$*. Put* $\Sigma =$ *(director cone of* $\mathcal{P} \cap C$ *at* λ_0*)*$\cap \Lambda$*. Consider the following conditions:*

(1) X *is normal along* Y_0*;*

(2) $\overline{T}$ *is normal at* y_0*;*

(3) Σ *is* L*-generated by* $\lambda_i - \lambda_0, -\alpha_j$*.*

(4) Σ *is generated by* $\lambda - \lambda_0$*,* $\forall \lambda \in \Lambda(V)$*.*

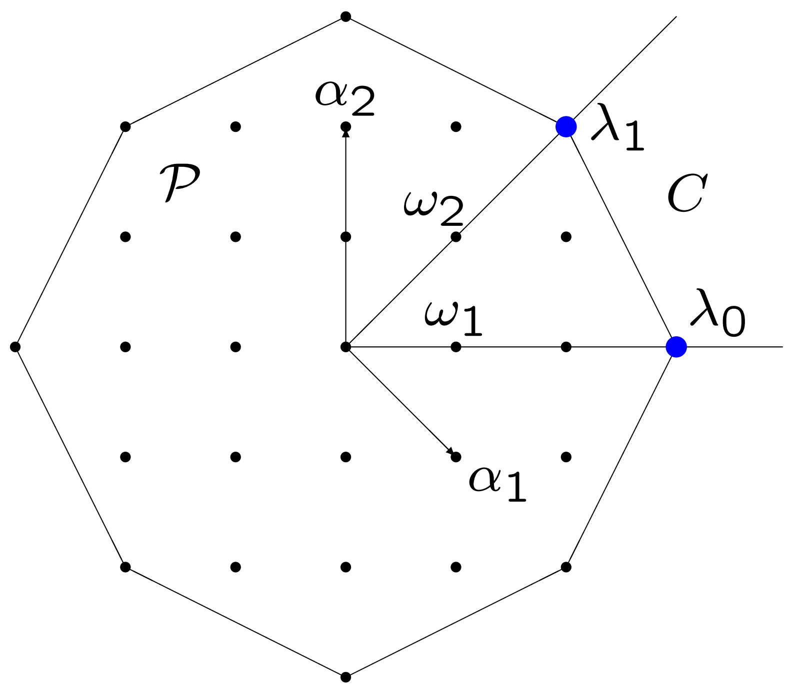
Then (1) $\iff$ (3) $\implies$ (2) $\iff$ (4)*.*

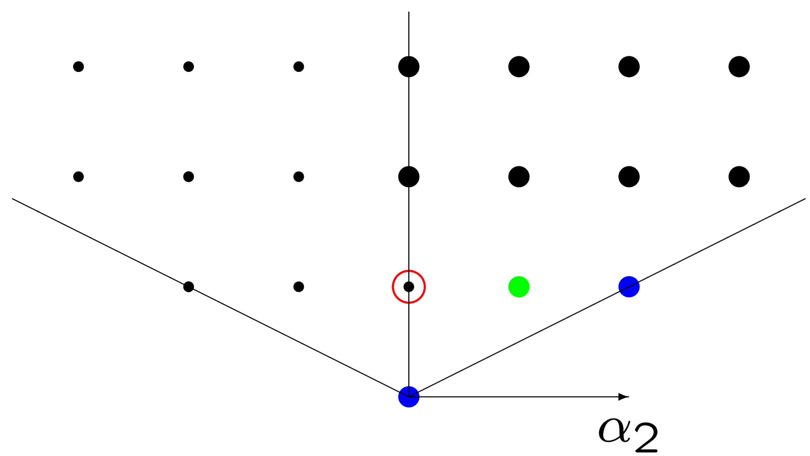
Remark. (3) $\iff$ $\lambda_i - \lambda_0$, $-\alpha_j$ L-generate a saturated semigroup.

Example 4. $G = Sp_4$, $V = S^3\mathbb{C}^4 \oplus S_0^2(\bigwedge_0^2 \mathbb{C}^4)$, the highest weights $\{\lambda_0 = 3\omega_1, \lambda_1 = 2\omega_2\}$ (ω_i denote the fundamental weights, α_i the simple roots). Here $L \simeq SL_2 \times \mathbb{C}^\times$, $\Delta_L = \{\pm\alpha_2\}$.

$\lambda_1 - \lambda_0, -\alpha_1$ L-generate {bold dots} (Clebsch–Gordan formula).

$\Longrightarrow$ X non-normal along Y_0; becomes normal if we add $\lambda_2 = 2\omega_1$.





5. Smoothness

Theorem 4. *X is smooth along Y_0 $\iff$ (1)&(2)&(3):*
(1) $L = GL_{n_1} \times \cdots \times GL_{n_p}$.
(2) $L \circlearrowleft V \otimes (-\lambda_0)$ is polynomial.
(3) $[L \circlearrowleft V \otimes (-\lambda_0)] \hookleftarrow [GL_{n_i} \circlearrowleft \mathbb{C}^{n_i}]$, $\forall i$.
Remark. (1), (2), (3) are reformulated in terms of $\Lambda(V)$.

Idea of the proof. X is smooth $\iff$ Z is smooth $\iff$ $Z \simeq \mathrm{Mat}_{n_1} \times \cdots \times \mathrm{Mat}_{n_p}$. □

Example 5. $G = SO_{2m+1}$, $V = V_G(\omega_i)$ (ω_i are the fundamental weights).
a) $i < m$: $V = \bigwedge^i \mathbb{C}^{2m+1}$, $L \simeq GL_i \times SO_{2m+1-2i} \implies X$ singular.
b) $i = m$: $V =$ spinor module $\simeq \bigwedge^\bullet \mathbb{C}^m \otimes \omega_m$ over $L \simeq GL_m \implies X$ smooth.

6. "Small" compactifications

Let G be a simple Lie group. We take a closer look at $X = \overline{G} \subseteq \mathbb{P}(\mathrm{End}\, V)$ for "small" $V = V_G(\omega_i)$ (ω_i are the fundamental weights) or $V = \mathrm{Lie}\, G$ (adjoint representation).

Results: 1) $(G \times G)$-orbits, their dimensions, Hasse diagrams of partial order.
2) Non-normal: (SO_{2m+1}, ω_i), $i < m$; (Sp_{2m}, ω_m); (G_2, ω_2); (F_4, ω_i), $i = 3, 4$.
3) Smooth: (SL_n, ω_i), $i = 1, n-1$; (SL_n, Ad), $n \leq 3$; (SO_{2m+1}, ω_m); (Sp_4, ω_1); (Sp_4, Ad); (G_2, ω_1).

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