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Torus actions of complexity one

Let $T = (\mathbb{C}^\times)^r$ be a complex algebraic torus acting on an algebraic variety X. The *complexity* $c(T, X)$ is the codimension of generic T-orbits in X. Toric varieties are exactly those of complexity 0. We give a combinatorial description of torus actions of complexity 1 in the language of convex geometry in the same spirit as for toric varieties.

We restrict our consideration to *normal* T-varieties. This restriction, common in toric geometry, is not very essential since every T-variety admits a T-equivariant normalization. Without loss of generality we may assume that the action $T \circlearrowleft X$ is faithful, i.e., generic T-orbits have trivial stabilizers.

By Sumihiro's theorem $X = \bigcup X_i$ is covered by finitely many affine open T-stable subvarieties. Hence a description of X amounts to **2 problems**: (1) Describe the affine T-varieties X_i; (2) Indicate how to patch them together.

To solve the 1-st problem, we may assume that X itself is affine. It is determined by its coordinate algebra $\mathbb{C}[X]$. The latter is a finitely generated integrally closed T-algebra, whence $\mathbb{C}[X] = \bigcap_{v=v_D} \mathcal{O}_v$ over all T-stable prime divisors $D \subset X$ (*T-divisors* in short), where v_D denotes the valuation of the field of rational functions $\mathbb{C}(X)$ corresponding to D.

Now we describe T-invariant discrete valuations of $\mathbb{C}(X)$ taking values in $\mathbb{Q}$ (*T-valuations* in short). It is easy to see that they are completely determined by the restriction to the multiplicative group of T-eigenfunctions $\mathbb{C}(X)^{(T)}$, and $\mathbb{C}(X)^{(T)} \simeq (\mathbb{C}(X)^T)^\times \times \Lambda$, where $\mathbb{C}(X)^T$ is the field of T-invariant functions and Λ is the weight lattice of T. Since $c(T, X) = 1$, we have $\mathbb{C}(X)^T \simeq \mathbb{C}(C)$ for some smooth projective curve C. Restricting a valuation to $\mathbb{C}(X)^T$ and Λ, in turn, we deduce:

Proposition. *The T-valuations are in a 1–1 correspondence with the triples (z, h, γ), $z \in C$, $h \in \mathbb{Q}_+$, $\gamma \in \mathcal{Z} := \mathrm{Hom}(\Lambda, \mathbb{Q})$, modulo the equivalence relation $(z_1, 0, \gamma) \equiv (z_2, 0, \gamma)$, $\forall z_1, z_2 \in \mathbb{C}$. Hence the set of T-valuations is $\mathcal{V} = \bigcup_{z\in C} \mathcal{V}_z$, where the half-spaces $\mathcal{V}_z = \mathbb{Q}_+ \times \mathcal{Z}$ are patched together along $\mathcal{Z}$.*

Definition. A *hypercone* in $\mathcal{V}$ is a union $\mathcal{C} = \bigcup \mathcal{C}_z$ of rational polyhedral cones $\mathcal{C}_z \subset \mathcal{V}_z$ such that: (1) $\mathcal{C}_z \cap \mathcal{Z} =: \mathcal{K}$ does not depend on $z \in C$; (2) $\mathcal{C}_z = \mathbb{Q}_+ \times \mathcal{K}$ for all but finitely many z;

(3) Let $\mathcal{P}_z$ be the projections of $\mathcal{C}_z \cap (\{1\} \times \mathcal{Z})$ to $\mathcal{Z}$; then $\mathcal{P} = \sum_{z\in C} \mathcal{P}_z := \{\sum \gamma_z \mid \gamma_z \in \mathcal{P}_z, \gamma_z = 0 \text{ for all but finitely many } z\} \subset \mathcal{K} \setminus \{0\}$. ($\mathcal{P}$ may be empty!)

(4) For any face $\mathcal{K}_0 \subset \mathcal{K}$, $\mathcal{K}_0 \cap \mathcal{P} \neq 0$, and $\forall \lambda \in \Lambda$, $\langle \lambda, \mathcal{K}_0 \rangle = 0$, $\langle \lambda, \mathcal{K} \rangle \geq 0$, put $\ell_z = \min\langle \lambda, \mathcal{P}_z \rangle$; then a multiple of $\sum_z \ell_z \cdot z$ is a principal divisor on C.

Note: Condition (4) holds automatically if $C = \mathbb{P}^1$, i.e., if X is rational, because $\sum \ell_z = 0$

Theorem 1. *The normal affine T-varieties of complexity 1 are in a 1–1 correspondence with the hypercones. The T-divisors on X correspond to the edges of the $\mathcal{C}_z$'s not intersecting $\mathcal{P}$.*

Next we address the 2-nd problem. By a *hyperface* $\mathcal{C}' \subseteq \mathcal{C}$ we mean a hypercone $\mathcal{C}'$ such that $\mathcal{C}'_z$ is a face of $\mathcal{C}_z$, $\forall z \in C$.

Theorem 2. *Affine T-varieties X_i can be patched together giving a (possibly non-affine) T-variety X of complexity 1 iff the respective hypercones $\mathcal{C}_i$ intersect exactly in their common hyperfaces.*

Conclusion: Normal T-varieties of complexity 1 are in a 1–1 correspondence with finite collections of hypercones intersecting in their common hyperfaces, called *hyperfans*.

In terms of a hyperfan, there are a description of all orbits, a criterion for smoothness, etc.

References

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