International conference on Toric topology
Osaka, 29.05 - 03.06. 2006.

Invariant complex structures on generalized symmetric spaces

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1. General facts

M - closed, oriented manifold of even dimension;

Almost complex structure - field J of endomorphisms J_p on TM such that $J_p^2 = -I_d$ for any $p \in M$;

J **is integrable** - if there exists complex atlas on M producing J_p, $p \in M$;

We consider homogeneous spaces G/H - restrict ourselves to description of **invariant** almost complex structures J - meaning invariance under the action of G on G/H;

J can be identified with complex structure on $T_e(G/H)$ that commutes with isotropy representation for H;

2. Existence

Thm 1 *(Borel-Hirzebruch)*

If H is centralizer of some element $g \in G$ of odd order, than G/H admits an invariant almost complex structure. In particular, if H is centralizer of a torus in G than G/H admits an invariant almost complex structure.

Thm 2 *(Wang)*

G/H admits an invariant complex structure if and only if semisimple part for H coincide with semisimple part of centralizer of some toral subgroup in G.

We distinguish two classes of homogeneous complex spaces:

1. If $\mathrm{rk}\, H = \mathrm{rk}\, G$ - than H is centralizer of a toral subgroup in G;

   In this case : $b_2(G/H) \neq 0$ and G/H is Kaehler.

Remark. This implies an existence of 7 symmetric spaces which are invariant almost complex, but not invariant complex:

$$G_2/A_2 = S^6, \quad F_4/A_2 \times A_2, \quad E_6/A_2 \times A_2 \times A_2, \ldots$$

2. rk $H <$ rk G - than semisimple part for H coincide with semisimple part of some stabilizer from the above case.

   There is explicit list of all such semisimple "parts" for compact simple Lie groups - Wang (*).

Proof for 2. implies any such space is homeomorphic with some homogeneous complex space K/L_c, for $c \in \mathbb{Q}$, with rk $K =$ rk L_c. By varying c one, in fact, gets infinitely many invariant complex structures on G/H (which are not Kaehler).

Therefore, we consider only the case $\mathrm{rk}\, H = \mathrm{rk}\, G$.

Thm 3 *(Wolf-Gray)*

Let H be subgroup of G with $\mathrm{rk}\, H = \mathrm{rk}\, U$. Then G/H admits an invariant complex structure if and only if H is a fixed point set for some finite group of odd order of inner automorphisms of G.

Roughly - only complex homogeneous spaces are generalized symmetric spaces - H is fixed point subgroup of some automorphism for G of finite order k.

Remark. There is explicit list-classification of all generalized symmetric spaces for compact simple Lie groups (Terzic). Comparing with Wang's list (*), a lot of them do not admit any invariant complex structure.

3. Root description of an invariant (almost) complex structures

$\mathfrak{g}$, $\mathfrak{h}$ - Lie algebra for G and H respectively;

$\mathfrak{t}$ - Cartan algebra for $\mathfrak{g}$ and $\mathfrak{h}$, $\{\beta\}$ - roots for $\mathfrak{g}$ related to $\mathfrak{t}$.

$\mathfrak{g} = \mathfrak{h} \oplus T_e(G/H) = \mathfrak{t} \oplus \sum \mathfrak{g}_\beta = \mathfrak{t} \oplus \sum \mathfrak{h}_{\beta_i} \oplus \sum \mathfrak{g}_{\beta_j}$,

where $\{\beta\} = \{\beta_i\} \cup \{\beta_j\}$ and $\{\beta_i\}$ are the roots for $\mathfrak{h}$ related to $\mathfrak{t}$.

$\{\beta_j\}$ - are called complementary roots for G related to H;

$T_e(G/H) = \oplus\, \mathfrak{g}_{\beta_j}$

J induces complex structures on $\mathfrak{g}_{\beta_j}$ - because it commutes with isotropy representation;

$\beta_j \to \varepsilon_j = \pm 1$ depending if $(v, Adt(v))$ and $(v, J(v))$ for $v \in \mathfrak{g}_{\beta_j}$ define the same orientation in $\mathfrak{g}_{\beta_j}$ or not;

$\varepsilon_j \beta_j$ - the roots of almost complex structure J;

4. On the number of invariant (almost) complex structures

Thm 4 *(Borel-Hirzebruch)*

Isotropy representation for G/H decomposes into t irreducible summands $\Rightarrow$ G/H admits exactly 2^t invariant almost complex structures.

Thm 5 *(Wolf-Gray)*

Let $\beta_1, \ldots, \beta_k$ - be the set of complementary positive roots for G related to H. Let t- be the number of linear functionals we get as a restriction of the complementary roots to the center of H. The number of invariant almost complex structures on G/H is 2^t.

5. Deciding integrability using root theory

Canonical coordinates - coordinates for G and H on $\mathfrak{t}$ in which the roots for G and H are in expressed in standard way.

- G/H - homogeneous complex and $\beta_1, \ldots, \beta_k$ roots defining this complex structure. Then there exists ordering on the canonical coordinates for G such that the above system is positive and closed.

- The opposite is also true. Let Θ be the system of positive roots for H and Ψ some closed system of roots for G such that $\Theta \cup \Psi$ forms the system of positive roots for G. Then there exists on G/H invariant complex structure such that Ψ is its root system.

- Let Ψ and $\Psi^{'}$ be the root systems of invariant complex structures on G/H. If there is automorphism on $\mathfrak{t}$ carrying Ψ into $\Psi^{'}$ and leaving the root system for H invariant than the corresponding complex structures are equivalent (under some diffeomorphism of G/H).

For explicit description (classification) of all invariant (almost) complex structures on G/H one needs relation between canonical coordinates for G and H.

For generalized symmetric spaces such relations are obtained earlier $\rightarrow$ complete description of invariant complex structures.

Remark. M. Nishiyama (Osaka J. Math.) gave the explicit formulas for the number of equivalent invariant complex structures for all complex homogeneous spaces with G simple compact Lie group.

6. Chern classes through the root theory on homogeneous spaces

There are seven equivalent definition of Chern classes (Borel-Hirzebruch).

Definition using representation theory can be used as follows.

J - an invariant almost complex structure on G/H;

ι - isotropy representaion of H;

J gives rise to complex linear representation ι_c of H in $\mathbb{C}^n$, $\dim G/H = 2n$;

Thm 6 $c(J) = \prod(1 + w_i)$, *where* w_i *runs through the weights of the complex isotropy representation* ι_c.

$\{\varepsilon_i \beta_i\}$ - the roots defining J;

If $\mathrm{rk}\, H = \mathrm{rk}\, G$ - $\{\varepsilon_i \beta_i\}$ are the weights for ι_c;

$c(J) = \prod(1 + \varepsilon_i \beta_i)$ - total Chern class for G/H;

$c_k = \sum_{i_1 < \ldots < i_k} \beta_{i_1} \cdots \beta_{i_k}$.

7. On non-invariance of Chern numbers

Hirzebruch (1954) - To what extent Chern numbers are topological invariant (for projective algebraic manifolds)?

Calabi (1958) - Chern classes of a complex 3-fold are not determined by the topology of the underlying manifold - his examples say nothing on Chern numbers - all have vanishing Chern numbers;

For almost complex structures hardly - even on the same manifold simply because there may be a lot of them;

First example Borel-Hirzebruch (1959): 10-dimensional homogeneous space with two invariant complex structures for which c_1^5 are different;

Immediately true for:

- top Chern number - Euler number;
- some combination of Chern numbers which give Pontryagin numbers (orientation is fixed, dimension divisible by four)

$$p_k(E) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \ldots \pm 2c_1(E)c_{2k-1}(E) \mp 2c_{2k}(E)$$

under orientation preserving diffeomorphism.

8. Generalized Borel-Hirzebruch example

$SU(n+1)/S(U(1)\times U(1)\times U(n-1))$ - 3 - symmetric space;

$n=1$ - 2-sphere,

$n=2$ - complex three-dimensional flag manifold;

$n=3$ - Borel-Hirzebruch example;

$A_n/\mathfrak{t}^2\oplus A_{n-2}$ - corresponding 3-symmetric Lie algebra;

Thm 7 *3 - symmetric homogeneous space* $SU(n+1)/S(U(1)\times U(1)\times U(n-1))$ *has up to automorphisms two integrable and one non integrable invariant almost complex structure.*

Thm 8 *Chern numbers (complex) for* $SU(n+1)/S(U(1)\times U(1)\times U(n-1))$ *are not topological invariants of underlying manifold (even not diffeomorphic invariant) .*

Cor 1 *For* $n = 3$ *the above complex structures have the following Chern numbers*

1. $c_1^5 = 4500, \quad c_1^3 c_2 = 2148, \quad c_1^2 c_3 = 612,$

   $c_1 c_2^2 = 1028, \quad c_1 c_4 = 108, \quad c_2 c_3 = 292, \quad c_5 = 12$

2. $c_1^5 = 4860, \quad c_1^3 c_2 = 2268, \quad c_1^2 c_3 = 612,$

   $c_1 c_2^2 = 1068, \quad c_1 c_4 = 108, \quad c_2 c_3 = 128, \quad c_5 = 12$.