Invariant complex structures on generalized symmetric spaces

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1. General facts

$M$ - closed, oriented manifold of even dimension;

**Almost complex structure** - field $J$ of endomorphisms $J_p$ on $TM$ such that $J_p^2 = -I_d$ for any $p \in M$;

$J$ is integrable - if there exists complex atlas on $M$ producing $J_p, p \in M$;

We consider homogeneous spaces $G/H$ - restrict ourselves to description of **invariant** almost complex structures $J$ - meaning invariance under the action of $G$ on $G/H$;

$J$ can be identified with complex structure on $Te(G/H)$ that commutes with isotropy representation for $H$;
2. Existence

**Thm 1 (Borel-Hirzebruch)**

If $H$ is centralizer of some element $g \in G$ of odd order, than $G/H$ admits an invariant almost complex structure. In particular, if $H$ is centralizer of a torus in $G$ than $G/H$ admits an invariant almost complex structure.

**Thm 2 (Wang)**

$G/H$ admits an invariant complex structure if and only if semisimple part for $H$ coincide with semisimple part of centralizer of some toral subgroup in $G$.

We distinguish two classes of homogeneous complex spaces:

1. If $\text{rk} \ H = \text{rk} \ G$ - than $H$ is centralizer of a toral subgroup in $G$;

   In this case : $b_2(G/H) \neq 0$ and $G/H$ is Kaehler.
Remark. This implies an existence of 7 symmetric spaces which are invariant almost complex, but not invariant complex:

\[ G_2/A_2 = S^6, \quad F_4/A_2 \times A_2, \quad E_6/A_2 \times A_2 \times A_2, \ldots \]

2. \( \text{rk } H < \text{rk } G \) - than semisimple part for \( H \) coincide with semisimple part of some stabilizer from the above case.

There is explicit list of all such semisimple ”parts” for compact simple Lie groups - Wang (*).

Proof for 2. implies any such space is homeomorphic with some homogeneous complex space \( K/L_c \), for \( c \in \mathbb{Q} \), with \( \text{rk } K = \text{rk } L_c \). By varying \( c \) one, in fact, gets infinitely many invariant complex structures on \( G/H \) (which are not Kaehler).
Therefore, we consider only the case $\text{rk } H = \text{rk } G$.

**Thm 3 (Wolf-Gray)**

Let $H$ be subgroup of $G$ with $\text{rk } H = \text{rk } U$. Then $G/H$ admits an invariant complex structure if and only if $H$ is a fixed point set for some finite group of odd order of inner automorphisms of $G$.

Roughly - only complex homogeneous spaces are generalized symmetric spaces - $H$ is fixed point subgroup of some automorphism for $G$ of finite order $k$.

**Remark.** There is explicit list-classification of all generalized symmetric spaces for compact simple Lie groups (Terzic). Comparing with Wang’s list (*), a lot of them do not admit any invariant complex structure.
3. Root description of an invariant (almost) complex structures

$\mathfrak{g}, \mathfrak{h}$ - Lie algebra for $G$ and $H$ respectively;

t - Cartan algebra for $\mathfrak{g}$ and $\mathfrak{h}$, $\{\beta\}$ - roots for $\mathfrak{g}$ related to $t$.

$$\mathfrak{g} = \mathfrak{h} \oplus T_e(G/H) = t \oplus \sum \mathfrak{g}_\beta = t \oplus \sum \mathfrak{h}_{\beta_i} \oplus \sum \mathfrak{g}_{\beta_j},$$

where $\{\beta\} = \{\beta_i\} \cup \{\beta_j\}$ and $\{\beta_i\}$ are the roots for $\mathfrak{h}$ related to $t$.

$\{\beta_j\}$ - are called complementary roots for $G$ related to $H$;

$$T_e(G/H) = \oplus \mathfrak{g}_{\beta_j}$$

$J$ induces complex structures on $\mathfrak{g}_{\beta_j}$ - because it commutes with isotropy representation;

$\beta_j \rightarrow \varepsilon_j = \pm 1$ depending if $(v, \text{Ad}_t(v))$ and $(v, J(v))$ for $v \in \mathfrak{g}_{\beta_j}$ define the same orientation in $\mathfrak{g}_{\beta_j}$ or not;

$\varepsilon_j\beta_j$ - the roots of almost complex structure $J$;
4. On the number of invariant (almost) complex structures

**Thm 4 (Borel-Hirzebruch)**

Isotropy representation for $G/H$ decomposes into $t$ irreducible summands $\Rightarrow G/H$ admits exactly $2^t$ invariant almost complex structures.

**Thm 5 (Wolf-Gray)**

Let $\beta_1, \ldots, \beta_k$ - be the set of complementary positive roots for $G$ related to $H$. Let $t$- be the number of linear functionals we get as a restriction of the complementary roots to the center of $H$. The number of invariant almost complex structures on $G/H$ is $2^t$. 
5. Deciding integrability using root theory

Canonical coordinates - coordinates for $G$ and $H$ on $\mathfrak{t}$ in which the roots for $G$ and $H$ are in expressed in standard way.

- $G/H$ - homogeneous complex and $\beta_1, \ldots, \beta_k$ roots defining this complex structure. Then there exists ordering on the canonical coordinates for $G$ such that the above system is positive and closed.

- The opposite is also true. Let $\Theta$ be the system of positive roots for $H$ and $\Psi$ some closed system of roots for $G$ such that $\Theta \cup \Psi$ forms the system of positive roots for $G$. Then there exists on $G/H$ invariant complex structure such that $\Psi$ is its root system.

- Let $\Psi$ and $\Psi'$ be the root systems of invariant complex structures on $G/H$. If there is automorphism on $\mathfrak{t}$ carrying $\Psi$ into $\Psi'$ and leaving the root system for $H$ invariant than the corresponding complex structures are equivalent (under some diffeomorphism of $G/H$).
For explicit description (classification) of all invariant (almost) complex structures on $G/H$ one needs relation between canonical coordinates for $G$ and $H$.

For generalized symmetric spaces such relations are obtained earlier → complete description of invariant complex structures.

**Remark.** M. Nishiyama (Osaka J. Math.) gave the explicit formulas for the number of equivalent invariant complex structures for all complex homogeneous spaces with $G$ simple compact Lie group.
6. Chern classes through the root theory on homogeneous spaces

There are seven equivalent definition of Chern classes (Borel-Hirzebruch).

Definition using representation theory can be used as follows.

\( J \) - an invariant almost complex structure on \( G/H \);

\( \iota \) - isotropy representaion of \( H \);

\( J \) gives rise to complex linear representation \( \iota_c \) of \( H \) in \( \mathbb{C}^n \), \( \dim G/H = 2n \);

**Thm 6** \( c(J) = \prod (1 + w_i) \), where \( w_i \) runs through the weights of the complex isotropy representation \( \iota_c \).

\( \{\varepsilon_i \beta_i\} \) - the roots defining \( J \);

If \( \text{rk} H = \text{rk} G \) - \( \{\varepsilon_i \beta_i\} \) are the weights for \( \iota_c \);

\( c(J) = \prod (1 + \varepsilon_i \beta_i) \) - total Chern class for \( G/H \);

\( c_k = \sum_{i_1 < \ldots < i_k} \beta_{i_1} \ldots \beta_{i_k} \).
7. On non-invariance of Chern numbers

Hirzebruch (1954) - To what extent Chern numbers are topological invariant (for projective algebraic manifolds)?

Calabi (1958) - Chern classes of a complex 3-fold are not determined by the topology of the underlying manifold - his examples say nothing on Chern numbers - all have vanishing Chern numbers;

For almost complex structures hardly - even on the same manifold simply because there may be a lot of them;

First example Borel-Hirzebruch (1959): 10-dimensional homogeneous space with two invariant complex structures for which $c_1^5$ are different;
Immediately true for:

- top Chern number - Euler number;

- some combination of Chern numbers which give Pontryagin numbers (orientation is fixed, dimension divisible by four)

\[ p_k(E) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \ldots \pm 2c_1(E)c_{2k-1}(E) + 2c_{2k}(E) \]

under orientation preserving diffeomorphism.
8. Generalized Borel-Hirzebruch example

\[ SU(n+1)/S(U(1) \times U(1) \times U(n-1)) \] - 3 - symmetric space;

\( n = 1 \) - 2-sphere,

\( n = 2 \) - complex three-dimensional flag manifold;

\( n = 3 \) - Borel-Hirzebruch example;

\[ A_n/\mathfrak{t}^2 \oplus A_{n-2} \] - corresponding 3-symmetric Lie algebra;
**Thm 7** 3 - symmetric homogeneous space $SU(n+1)/S(U(1) \times U(1) \times U(n-1))$ has up to automorphisms two integrable and one non integrable invariant almost complex structure.

**Thm 8** Chern numbers (complex) for $SU(n+1)/S(U(1) \times U(1) \times U(n-1))$ are not topological invariants of underlying manifold (even not diffeomorphic invariant).

**Cor 1** For $n = 3$ the above complex structures have the following Chern numbers

1. $c_1^5 = 4500, \quad c_1^3 c_2 = 2148, \quad c_1^2 c_3 = 612,$
   $c_1 c_2^2 = 1028, \quad c_1 c_4 = 108, \quad c_2 c_3 = 292, \quad c_5 = 12$

2. $c_1^5 = 4860, \quad c_1^3 c_2 = 2268, \quad c_1^2 c_3 = 612,$
   $c_1 c_2^2 = 1068, \quad c_1 c_4 = 108, \quad c_2 c_3 = 128, \quad c_5 = 12$.