CATEGORICAL ASPECTS of TORIC TOPOLOGY

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OVERVIEW

Aims: (i) *to describe categorical aspects of toric objects,* **and (ii)** *to give examples of useful calculations in this framework.*

1. THE CATEGORICAL VIEWPOINT

2. TORIC OBJECTS

3. HOMOTOPY THEORY

4. FORMALITY

This is why we are all here ..

$$
\begin{array}{c}
\mathscr{Z}_K \\
\Big\downarrow{\scriptstyle T^{m-n}} \\
M_1^{2n}, \ldots, M_k^{2n} \\
\Big\downarrow{\scriptstyle T^{n}} \\
P^n
\end{array}
$$

Lower quotients are strict; $P^n = \text{Cone}(K')$.

$$
\begin{array}{c}
\mathscr{Z}_K \\
\Big\downarrow{\scriptstyle T^{m-n}} \\
M_1^{2n}, \ldots, M_k^{2n} \\
\Big\downarrow{\scriptstyle T^{n}} \\
DJ(K)
\end{array}
$$

Lower quotients are homotopy quotients.

1. THE CATEGORICAL VIEWPOINT

Some people love categories, and others hate them; but they are here to stay!

A **category** *c* has objects X, and a set of morphisms *c*(X, Y) between every pair of objects.

Some categories are large, such as *top*, the category of topological spaces and continuous maps. Others are small (finite, even!), such as the category *cat*(K) of faces of a simplicial complex K and their inclusions.

Functors are morphisms between categories, such as the singular cochain algebra functor

$$C^*(-;R)\colon \mathit{top} \longrightarrow \mathit{dga}_R$$

over a nice ring R.

Toric Topology exists within two categorical frameworks, which may seem independent . . . but they are deeply intertwined!

(i) **Local:** many toric spaces admit natural decompositions into simpler subspaces; and these are often indexed by small categories such as $cat(K)$.

(ii) **Global:** as problems vary, our spaces may lie in the category of smooth manifolds and diffeomorphisms; or CW-complexes and homotopy classes of maps; or . . . .

The local viewpoint considers toric spaces as **diagrams**, whereas the global viewpoint interprets their **invariants** as functors from geometric to algebraic categories.

As well as $\mathit{cat}(K)$, we like the small category Δ, with objects

$$(n) \;=\; \{0, 1, \ldots, n\} \;\text{ for }\; n \geq 0,$$

and morphisms the non-decreasing maps. We denote their **opposites** by cat^{op} and Δ^{op}.

We like geometric categories such as

top_+	:	pointed topological spaces
tmon	:	topological monoids.

We also like algebraic categories such as

dga_R	:	differential graded algebras
$\mathit{cdga}_{\mathbb{Q}}$	:	commutative dgas
$\mathit{dgc}_{\mathbb{Q}}$	:	differential graded colagebras,

usually with (co)augmentations. Differentials go down in dga and dgc, and up in cdga.

Given a small indexing category a, we may view diagrams in c as functors $D\colon \mathsf{a} \to \mathsf{c}$; then the collection of all such diagrams also forms a category $[\mathsf{a}, \mathsf{c}]$.

If a is Δ, then $[\Delta, \mathsf{c}]$ and $[\Delta^{op}, \mathsf{c}]$ are the categories of **cosimplicial** and **simplicial objects** in c, often denoted by cc and sc respectively. For example:

(i) the **cosimplicial simplex**

$$\Delta^{\bullet}\colon \Delta \longrightarrow \mathsf{top}$$

maps (n) to the standard n-simplex Δ^n;

(ii) the **singular chain complex**

$$C_{\bullet}(X)\colon \Delta^{op} \longrightarrow \mathsf{sset}$$

maps (n) to the set of continuous functions $f\colon \Delta^n \to X$ for any space X.

In nice categories, **pushouts** and **pullbacks** are universal objects arising from diagrams on

$$\{1\} \longleftarrow \emptyset \longrightarrow \{2\} \quad \text{and} \quad \{1\} \longrightarrow \emptyset \longleftarrow \{2\};$$

these are *cat*$(\bullet\ \bullet)$ and *cat*$^{op}(\bullet\ \bullet)$ respectively!

In *tmon*, the pushout of the diagram

$$T_1 \longleftarrow \{1\} \longrightarrow T_2$$

of circles is the free product $T_1 \star T_2 \to T_1 \times T_2$; in *top*$_+$, the pushout of the diagram

$$BT_1 \longleftarrow * \longrightarrow BT_2$$

of classifying spaces is $BT_1 \vee BT_2 \subset BT \times BT$.

Coproducts (or sums) are special cases of pushouts, which are themselves examples of **colimits** of arbitrary diagrams in a category. Similarly, **products** are special cases of pullbacks, which are examples of **limits**.

2. TORIC OBJECTS

We start with a simplicial complex K on vertices $V = \{v_1, \ldots, v_m\}$, and construct two associated topological spaces:

- the *Davis-Januszkiewicz space* $DJ(K)$
- the *moment-angle complex* $\mathscr{Z}_K$.

The topologists amongst us are interested in their properties

up to **homotopy equivalence**,

so we have some freedom in making the constructions.

The vertices determine

- an m-torus T^V
- its classifying space $BT^V \simeq (\mathbb{C}P^\infty)^V$.

For any face $\sigma \subseteq V$ of K, there is

- a coordinate subtorus $T^\sigma \leq T^V$
- its classifying space $BT^\sigma \subseteq BT^V$
- the space $D_\sigma = (D^2)^\sigma \times T^{V \setminus \sigma}$.

So there are diagrams

- T^K: *cat*$(K) \longrightarrow$ *tmon*
- BT^K: *cat*$(K) \longrightarrow$ *top*$_+$
- D^K: *cat*$(K) \longrightarrow$ *top*$_+$

which map an inclusion $\sigma \subseteq \tau$ of faces to

- the monomorphism $T^\sigma \leq T^\tau$
- the inclusion $BT^\sigma \subseteq BT^\tau$
- the inclusion $D_\sigma \subseteq D_\tau$

respectively.

To construct our first toric **spaces**, we take colimits of diagrams. We obtain

$$\text{colim}^{tmon} T^K = Cir(K^{(1)})$$

as topological groups; and

$$\text{colim}^{top_+} BT^K = \bigcup_{\sigma\in K} BT^\sigma \cong DJ(K)$$

and

$$\text{colim}^{top_+} D^K = \bigcup_{\sigma\in K} D_\sigma \cong \mathscr{Z}_K$$

as pointed topological spaces.

We may also define a diagram $T^{V\setminus K}$ by mapping $\sigma \subseteq \tau$ to the projection

$$T^{V\setminus\sigma} \longrightarrow T^{V\setminus\tau}.$$

In this case,

$$\text{colim}^{top_+} T^{V\setminus K} = \{1\}$$

is a single point.

For algebraic purposes, we write the vertices $v_1, \ldots, v_m$ as 2-dimensional variables; their desuspensions $u_1, \ldots, u_m$ are 1-dimensional. In either case, we denote the commutative monomials $\prod_\alpha w_i$ by w_α, for any multiset $\alpha\colon V \to \mathbb{N}$.

The symmetric algebra $S_R(V)$ is **polynomial** over R, with basis elements v_α. The symmetric algebra $\wedge_R(U)$ is **exterior**, with basis elements u_α for genuine subsets $\alpha \subseteq U$.

With $d = 0$, both are objects of *cdga*; and so is $\Lambda(U) \otimes S(\sigma)$, with $du_i = v_i$ for all $v_i \in \sigma$.

The graded duals $S_R(V)'$ and $\wedge_R(U)'$ have dual basis elements v^α and u^α over R. In either case, their coproducts satisfy

$$\delta(w^\alpha) = \sum_{\alpha_1 \sqcup \alpha_2 = \alpha} w^{\alpha_1} \otimes w^{\alpha_2}$$

With $d = 0$, both are objects of *cdgc*.

We can define a diagram *cat*$(K) \to$ *dga* by:

- $\wedge^K$ maps $\sigma \subseteq \tau$ to the monomorphism

$$\wedge(\sigma) \longrightarrow \wedge(\tau),$$

and diagrams *cat*$^{op}(K) \to$ *cdga* by:

- S_K maps $\tau \supseteq \sigma$ to the epimorphism

$$S(\tau) \longrightarrow S(\sigma),$$

- $\wedge \otimes S_K$ maps $\tau \supseteq \sigma$ to the epimorphism

$$\wedge(U) \otimes S(\tau) \longrightarrow \wedge(U) \otimes S(\sigma).$$

. . . and a diagram *cat*$(K) \to$ *cdgc* by:

- $(S^K)'$ maps $\tau \subseteq \sigma$ to the monomorphism

$$S(\tau)' \longrightarrow S(\sigma)'.$$

To define **algebraic** toric objects, we consider

$$\text{colim}^{\mathit{dga}} \wedge^K \;\cong\; T(u_1, \ldots, u_m)\,/\,I,$$
$$\text{where } I \;=\; \left(u_h^2,\, [u_i, u_j] : \forall h,\, \{i,j\} \in K\right);$$

also

$$\lim^{\mathit{cdga}} S_K \;\cong\; R[K],$$

the **Stanley-Reisner algebra** of K; and

$$\lim^{\mathit{cdga}} \wedge \otimes S_K \;\cong\; (\wedge \otimes R[K], d),$$

where $du_i = v_i$ for $1 \leq i \leq m$; and

$$\text{colim}^{\mathit{cdgc}} (S^K)' \;\cong\; R\langle K\rangle,$$

the **Stanley-Reisner coalgebra** of K.

We can also define a diagram $\wedge_{U\backslash K}$ by mapping $\tau \supseteq \sigma$ to the monomorphism

$$\wedge(U \setminus \tau) \longrightarrow \wedge(U \setminus \sigma).$$

In this case,

$$\lim^{\mathit{cdga}} \wedge_{U\backslash K} \;\cong\; R$$

is simply the ground ring in dimension 0.

3. HOMOTOPY THEORY

Classical homotopy theory does not interact well with limits and colimits!

Taking colimits in *top*$_+$, we have that

$$\operatorname{colim} D^K = \mathscr{Z}_K \quad \text{and} \quad \operatorname{colim} T^{V\setminus K} = \{1\}$$

However, the projections

$$D_\sigma = (D^2)^\sigma \times T^{V\setminus\sigma} \longrightarrow T^{V\setminus\sigma},$$

induce a morphism $D^K \to T^{V\setminus K}$, which is a **homotopy equivalence** for each face σ of K.

The simplest example of this case is $P^1 = \Delta^1$, so $K = \bullet\ \bullet$. Then D^K is the pushout diagram

$$T_1 \times D_2^2 \longleftarrow T_1 \times T_2 \longrightarrow D_1^2 \times T_2,$$

and $\operatorname{colim} D^K \cong S^3$.

But $T^{V\setminus K}$ is the pushout

$$T_1 \longleftarrow T_1 \times T_2 \longrightarrow T_2,$$

and $\operatorname{colim} T^{V\setminus K} = \{1\}$.

Algebraically, we take limits in *cdga* and find:

$$\lim \wedge \otimes S_K = \wedge \otimes R[K]$$

$$\text{and} \quad \lim \wedge_{U \setminus K} = R$$

However, the monomorphisms

$$\wedge(U \setminus \sigma) \longrightarrow \wedge(U) \otimes S(\sigma)$$

induce a morphism $\wedge_{U \setminus K} \to \wedge \otimes S_K$, which is a **quasi-isomorphism** for each face σ of K. Both have cohomology $\wedge(U \setminus \sigma)$.

Again, the simplest example is $K = \bullet\ \bullet$, for which $\wedge \otimes S_K$ is the pullback diagram

$$\wedge(u_1, u_2) \otimes S(v_2) \longrightarrow \wedge\,(u_1, u_2) \longleftarrow \wedge(u_1, u_2) \otimes S(v_1),$$

and $\lim \wedge \otimes S_K = \wedge(u_1 v_2)$.

But $\wedge_{U \setminus K}$ is the pullback

$$\wedge(u_1) \longrightarrow \wedge(u_1, u_2) \longleftarrow \wedge(u_2),$$

and $\lim \wedge_{U \setminus K} = R$.

In both geometric and algebraic contexts, we learn that *objectwise weak equivalences do not preserve colimits or limits.*

In order to understand this situation properly, we follow Quillen's inspired ideas for axiomatising categories in which we can "do homotopy theory".

This is the world of **model category** theory. In any such category, three classes of special morphism are defined; the **fibrations**, the **cofibrations**, and the **weak equivalences**. They obey axioms that are suggested by the properties of *top*, and allow us to pass to a **homotopy category**, where the weak equivalences are invertible.

The beauty of the axioms is that many algebraic categories also admit natural model structures, as well as more obvious geometric examples such as top_+ and *sset*.

Given a model category *mc* and a nice indexing category *a*, the category $[a, mc]$ admits a canonical model structure, and

a weak equivalence of diagrams
is an objectwise weak equivalence.

Recent results show that lim and colim may always be replaced by more subtle functors **holim** and **hocolim** : $[a, mc] \to mc$.

To construct $\text{hocolim}^{mc} D$ for any diagram $D\colon a \to mc$ we:

(i) replace the objects $D(a)$ by nicer $D'(a)$
(ii) replace the diagram D' by nicer D''
(iii) form $\text{colim}^{mc} D''$.

Then $\text{hocolim}^{mc} D$ is preserved (up to weak equivalence in *mc*) by weak equivalences of diagrams; and by certain functors which do not preserve colim.

Some diagrams are *given* in the form D'', so lim and colim are weakly equivalent to holim and hocolim. Examples are BT^K and S_K, for which there is also an isomorphism

$$H^*(\text{colim}^{\textit{top}_+} BT^K; R) \cong \lim^{\textit{cdga}} S_K.$$

This is better known as

$$H^*(DJ(K); R) \cong R[K] \,!$$

The source of our weak equivalence $D^K \to T^{V\backslash K}$ is of the form D'', but the target is not. So we have a zig-zag

$$\mathscr{Z}_K \simeq \text{colim}\, D^K \longrightarrow \cdots \longleftarrow \text{hocolim}\, T^{V\backslash K}$$

of weak equivalences in $\textit{top}_+$.

The target of our weak equivalence $\wedge_{U\backslash K} \to \wedge \otimes S_K$ is also of the form D'', but the source is not. So we have a zig-zag

$$\text{holim} \wedge_{U\backslash K} \longrightarrow \cdots \longleftarrow \lim \wedge \otimes S_K \simeq C^*(\mathscr{Z}_K; \mathbb{Q})$$

of weak equivalences in $\textit{cdga}_{\mathbb{Q}}$.

Now consider the problem of describing the loop space $\Omega DJ(K)$ as a colimit.

There are homorphisms $T^{\sigma} \to \text{colim}^{tmon} T^{K}$, which combine to give a homotopy homomorphism

$$\Omega\, \text{colim}^{top_{+}} BT^{K} \longrightarrow \text{colim}^{tmon} T^{K}.$$

When K is flag, this is a weak equivalence

$$\Omega DJ(K) \xrightarrow{\simeq} Cir(K^{(1)});$$

but not in general.

There is also a homotopy homomorphism

$$\Omega DJ(K) \xrightarrow{\simeq} \text{hocolim}^{tmon} T^{K},$$

which is a weak equivalence for *all* K. So

looping preserves homotopy colimits.

Almost all our algebraic categories admit model structures, in which weak equivalences are the quasi-isomorphisms.

So we hope we can describe structures like the Pontrjagin ring $H_*(\Omega DJ(K); R)$ as the homology of an appropriate hocolim in *dga*.

When K is flag, there is a zig-zag of weak equivalences

$$C_*(\Omega DJ(K); \mathbb{Q}) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} \mathrm{colim}^{dga_{\mathbb{Q}}} \wedge^K;$$

in general, there is a zig-zag

$$C_*(\Omega DJ(K); \mathbb{Q}) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} \mathrm{hocolim}^{dga_{\mathbb{Q}}} \wedge^K.$$

The proof uses Adams's cobar construction

$$\Omega_* \colon dgc_{\mathbb{Q}} \longrightarrow dga_{\mathbb{Q}},$$

which is the algebraic analogue of taking loops.

4. FORMALITY

Sullivan's PL-forms functor $A^*\colon$ *top* $\to$ *dga*$_{\mathbb{Q}}$ provides a very good representation of the rational homotopy category as an algebraic model category. Understanding $A^*(X)$ in *dga*$_{\mathbb{Q}}$ is as good as understanding $X_{\mathbb{Q}}$ in *top*$_{\mathbb{Q}}$.

Certain nice spaces X (such as Eilenberg-Mac Lane spaces, or classifying spaces of Lie groups) are **formal**, because there exists a zig-zag of quasi-isomorphisms

$$(A^*(X), d) \longrightarrow \cdots \longleftarrow (H^*(X; \mathbb{Q}), 0)$$

in *cdga*$_{\mathbb{Q}}$. In fact $DJ(K)$ is formal for every K.

Some X are also **integrally formal**, because a similar zig-zag of quasi-isomorphisms

$$(C^*(X; \mathbb{Z}), d) \longrightarrow \cdots \longleftarrow (H^*(X; \mathbb{Z}), 0)$$

exists in *dga*, for the singular cochain algebra $C^*(X; \mathbb{Z})$. This is also true for every $DJ(K)$.

Now we can discuss **toric manifolds** M^{2n}!

We let K be the boundary of a simplicial convex n-polytope, and search for a **linear system of parameters** $t_1, \ldots, t_n$ in $\mathbb{Z}[K]$. For any such system t, an M is defined (up to weak equivalence) as the pullback of

$$DJ(K) \xrightarrow{t} BT^n \xleftarrow{p} ET^n,$$

in *top*. So T^n acts freely on M.

Moreover, t induces $t_*\colon T^V \to T^n$, and each subtorus $t_*(T^\sigma) < T^n$ is an isotropy subgroup $T(\sigma)$. So there is a *cat*(K)-diagram $T^{n/K}$, which maps each $\sigma \subseteq \tau$ to the projection

$$T^n/T(\sigma) \longrightarrow T^n/T(\tau).$$

Then M is hocolim$^{top} T^{n/K}$, and the orbit space M/T^n is the **nerve** of *cat*(K), or P^n.

We may verify that $A^*(M)$ is weakly equivalent to the homotopy colimit of

$$A^*(DJ(K)) \longleftarrow A^*(BT^n) \longrightarrow A^*(ET^n).$$

Similarly, $H^*(M;\mathbb{Q})$ is weakly equivalent to the homotopy colimit of

$$H^*(DJ(K);\mathbb{Q}) \longleftarrow H^*(BT^n;\mathbb{Q}) \longrightarrow \mathbb{Q},$$

because t is a linear system of parameters.

As $DJ(K)$ is formal and ET^n is contractible, the two diagrams are related by a zig-zag of weak equivalences.

So their homotopy colimits are related by a zig-zag of weak equivalences in *cdga*$_{\mathbb{Q}}$, and M is formal.

Quillen studies $X_{\mathbb{Q}}$ by means of the model category *dgl*$_{\mathbb{Q}}$ of differential graded Lie algebras. The homology of $(L_*(X_{\mathbb{Q}}), d)$ gives $\pi_*(\Omega X) \otimes \mathbb{Q}$, equipped with the Whitehead product bracket.

Then X is **coformal** if there is a zig-zag of quasi-isomorphisms

$$(L_*(X_{\mathbb{Q}}), d) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} (\pi_*(\Omega X) \otimes \mathbb{Q}, 0)$$

in *dgl*$_Q$.

By restricting to primitive elements, we find that such a zig-zag exists for $DJ(K)$ when and only when there is a zig-zag

$$\Omega_* \mathbb{Q}\langle K \rangle \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} H_*(\Omega DJ(K); \mathbb{Q}).$$

So $DJ(K)$ is coformal if and only if K is flag.