Topology on Graphs

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§1. Objective

- Graphs $\Longrightarrow \cdots \Longrightarrow$ Geometric Objects

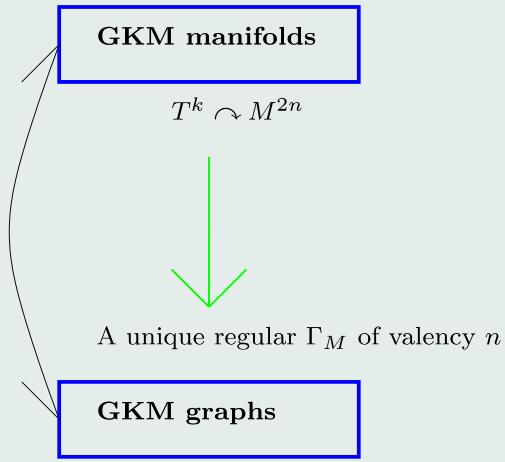
- **Two basic problems**

— — — Under what condition, is a geometric object a closed manifold?

— — — Can any closed manifold be geometrically realizable by the above way?

§2. Background

• GKM theory—by Goresky, Kottwitz and MacPherson in 1998 (see [Invent. Math. **131**, 25-83]).



A GKM manifold is a T^k-manifold M^{2n} with

- $|M^T| < +\infty$
- M having a T^k-invariant almost complex structure
- for $p \in M^T$, the weights of the isotropy representation of T^k on T_pM being pairwise linearly independent.

§3. **Coloring graphs and faces**

Let $G = (\mathbb{Z}_2)^k$.

Given a G-manifold M with $|M^G| < \infty \rightsquigarrow$ **regular graph** Γ_M with properties as follows:

$\exists$ a natural map

$$\alpha : E_{\Gamma_M} \longrightarrow \mathrm{Hom}(G, \mathbb{Z}_2)$$
$$e \longmapsto \rho$$

such that

A) for each $p \in V_{\Gamma_M}$, $\alpha(E_p)$ **spans** $\mathrm{Hom}(G, \mathbb{Z}_2)$

B) for each $e = pq \in E_{\Gamma_M}$ and $\sigma \in \alpha(E_p)$, the number of times which σ **and** $\sigma + \alpha(e)$ **occur in** $\alpha(E_p)$ is the same as that in $\alpha(E_q)$.

— Abstract definition

Let $G = (\mathbb{Z}_2)^k$.

We shall work on $H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, ..., a_k]$ ($\because H^1(BG; \mathbb{Z}_2) \cong \mathrm{Hom}(G, \mathbb{Z}_2)$).

Γ^n: a connected regular graph of valency n with $n \geq k$ and no loops.

If there is a map $\alpha : E_\Gamma \longrightarrow H^1(B(\mathbb{Z}_2)^k; \mathbb{Z}_2) - \{0\}$ s. t.

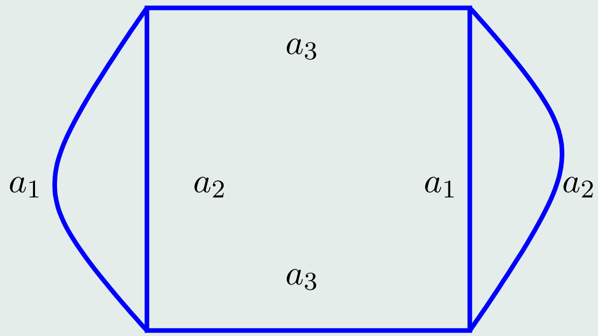
(1) for each vertex $p \in V_\Gamma$, the image $\alpha(E_p)$ spans $H^1(B(\mathbb{Z}_2)^k; \mathbb{Z}_2)$, and

(2) for each edge $e = pq \in E_\Gamma$,

$$\prod_{x \in E_p - E_e} \alpha(x) \equiv \prod_{y \in E_q - E_e} \alpha(y) \mod \alpha(e),$$

then the pair (Γ, α) is called **a coloring graph of type (k, n)**.

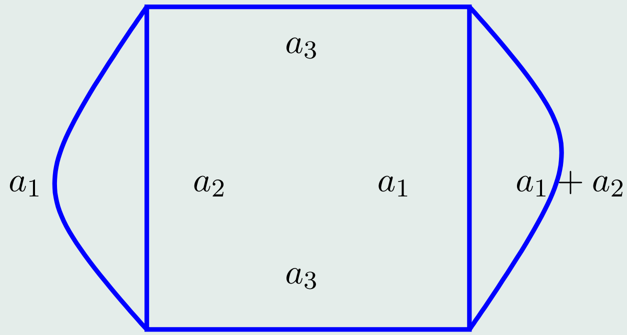
— Examples



(Γ, α_1) **is a coloring graph**

$\alpha_1 : E_\Gamma \longrightarrow H^1(B(\mathbb{Z}_2)^3; \mathbb{Z}_2)$

where $H^*(B(\mathbb{Z}_2)^3; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, a_2, a_3]$.



(Γ, α_2) **is not a coloring graph**

$\because\ a_1 a_2 \not\equiv a_1(a_1 + a_2) \mod a_3$

—Faces

(Γ, α): a coloring graph of type (k, n).

Γ^ℓ: a connected ℓ-valent subgraph of Γ where $0 \le \ell \le n$.

If $(\Gamma^\ell, \alpha|\Gamma^\ell)$ satisfies

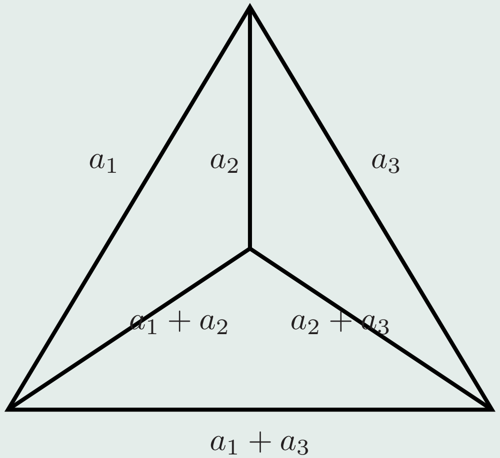
a) for any two vertices p_1, p_2 of Γ^ℓ, $\alpha((E|\Gamma^\ell)_{p_1})$ and $\alpha((E|\Gamma^\ell)_{p_2})$ span the **same subspace** of $H^1(BG; \mathbb{Z}_2)$;

b) for each edge $e = pq \in E|\Gamma^\ell$,

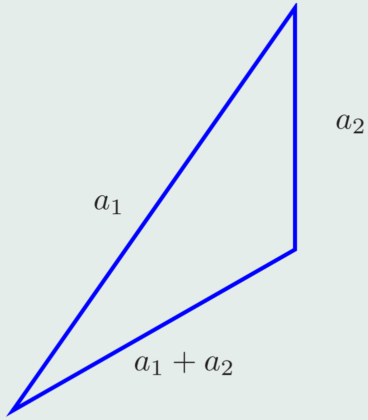
$$\prod_{x \in (E|\Gamma^\ell)_p - (E|\Gamma^\ell)_e} \alpha(x) \equiv \prod_{y \in (E|\Gamma^\ell)_q - (E|\Gamma^\ell)_e} \alpha(y) \mod \alpha(e)$$

then $(\Gamma^\ell, \alpha|\Gamma^\ell)$ is **an ℓ-face of** (Γ, α).

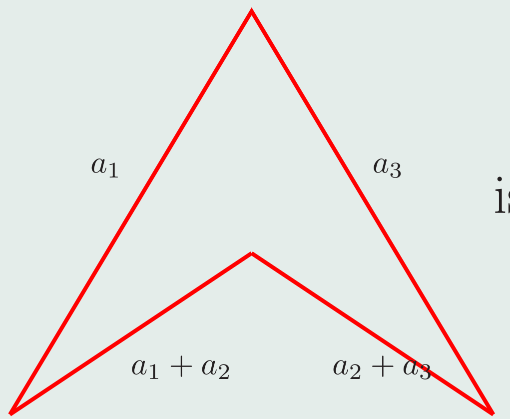
Example



a coloring graph (Γ, α)



is a 2-face



is not a 2-face

Assumption—Case: valency n of Γ = rank k of $G = (\mathbb{Z}_2)^k$

(Γ, α): a coloring graph of type (n, n) with Γ connected.

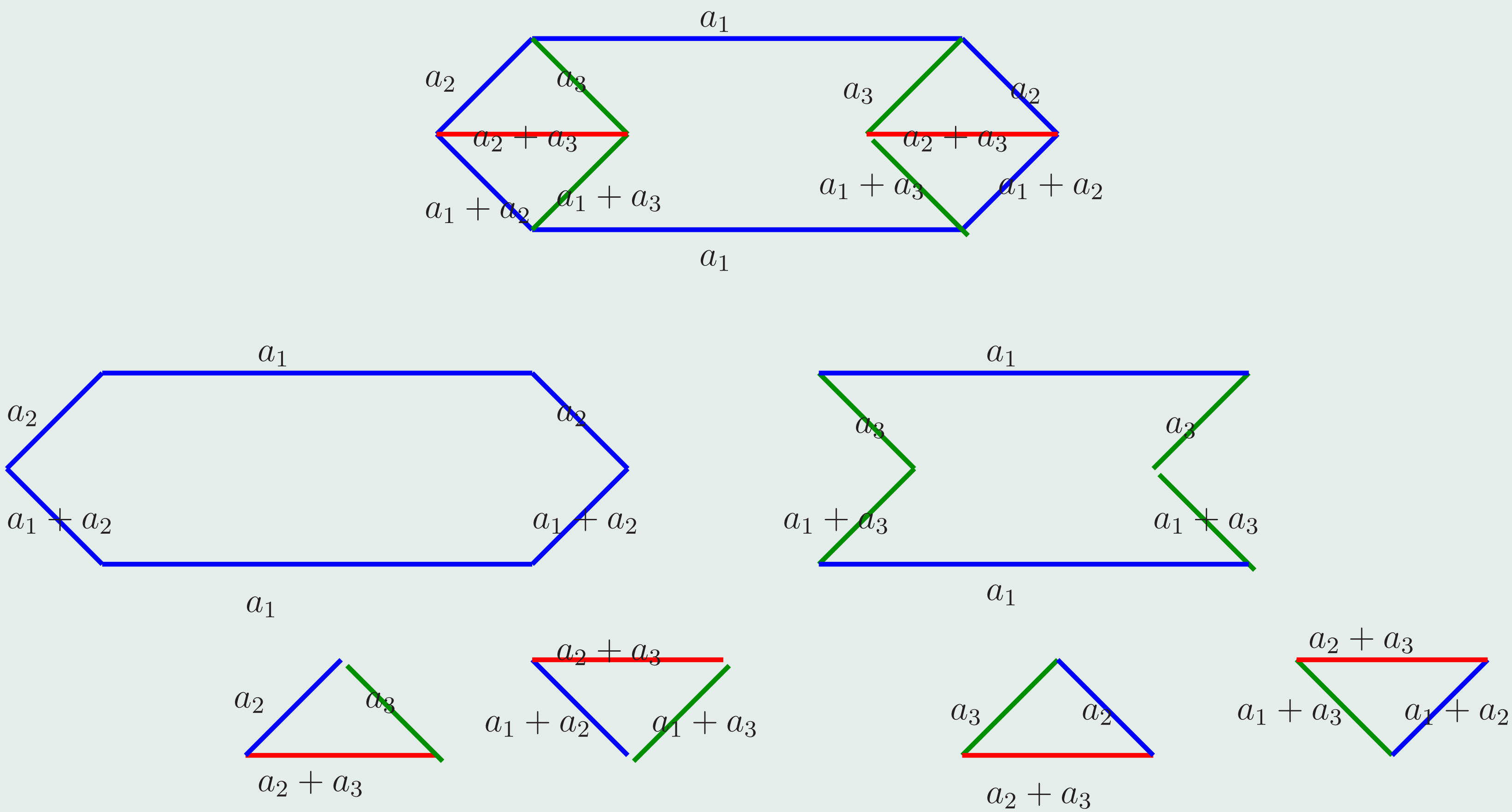
$\mathcal{F}_{(\Gamma,\alpha)}$: the set of all faces of (Γ, α).

— An application for the n-connectedness of a graph.

Theorem (Whitney) *A graph Γ with at least $n + 1$ vertices is n-connected if and only if every subgraph of Γ, obtained by omitting from Γ any $n - 1$ or fewer vertices and the edges incident to them, is connected.*

Theorem (Z. Lü and M. Masuda). *Suppose that (Γ, α) is a coloring graph of type (n, n) with Γ connected. If the intersection of any two faces of dimension ≤ 2 in $\mathcal{F}_{(\Gamma,\alpha)}$ is either connected or empty, then Γ is n-connected.*

Example



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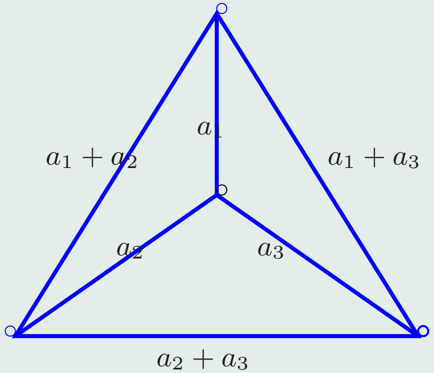
§4. Geometric realization

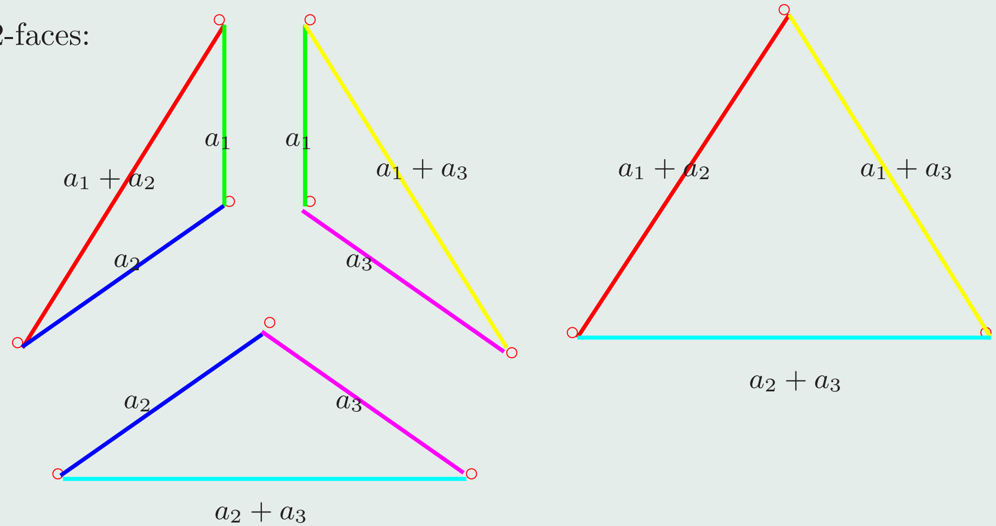
(Γ, α):a coloring graph of type $(n, n) \Longrightarrow \mathcal{F}_{(\Gamma,\alpha)} \Longrightarrow |\mathcal{F}_{(\Gamma,\alpha)}|$

Example 1.



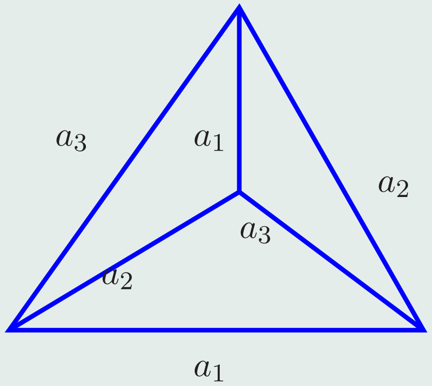
(Γ, α):a coloring graph of type $(3, 3)$

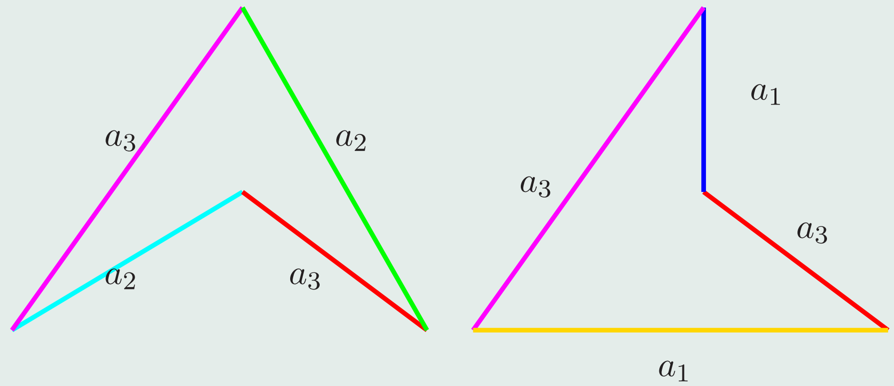
four 2-faces:



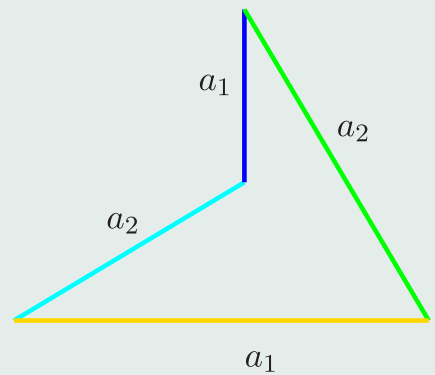
The geometric realization $|\mathcal{F}_{(\Gamma,\alpha)}| = S^2$

Example 2.





(Γ, α):a coloring graph of type $(3, 3)$.



The geometric realization $|\mathcal{F}_{(\Gamma,\alpha)}| = \mathbb{R}P^2$

Generally,

Fact. $\mathcal{F}_{(\Gamma,\alpha)}$ *forms a simplicial poset of rank* n *with respect to reversed inclusion with* (Γ, α) *as smallest element.*

$$\Downarrow$$

$$\boxed{|\mathcal{F}_{(\Gamma,\alpha)}| \text{ is a pseudo manifold.}}$$

poset means partially ordered set

A poset $\mathcal{P}$ is simplicial if it contains a smallest element $\hat{0}$ and for each $a \in \mathcal{P}$ the segment $[\hat{0}, a]$ is a boolean algebra (i.e., the face poset of a simplex with empty set as the smallest element).

$\mathcal{P}$: a simplicial poset

$$\Downarrow$$

a simplicial cell complex $\mathbb{K}_{\mathcal{P}}$ in the following way:

for each $a \neq \hat{0}$ in $\mathcal{P}$, one obtains a geometrical simplex such that its face poset is $[\hat{0}, a]$, and then one glues all obtained geometrical simplices together according to the ordered relation in $\mathcal{P}$, so that one can get a cell complex as desired.

By $|\mathcal{P}|$ one denotes the underlying space of this cell complex, and one calls $|\mathcal{P}|$ *the geometric realization* of $\mathcal{P}$.

Basic problems:

(I). Under what condition, is the geometric realization $|\mathcal{F}_{(\Gamma,\alpha)}|$ a closed topological manifold?

(II). For any closed topological manifold M^n, is there a coloring graph (Γ, α) of type $(n+1, n+1)$ such that $M^n \approx |\mathcal{F}_{(\Gamma,\alpha)}|$?

Basic problem (I)

(Γ, α): a coloring graph of type (n, n) with Γ connected.

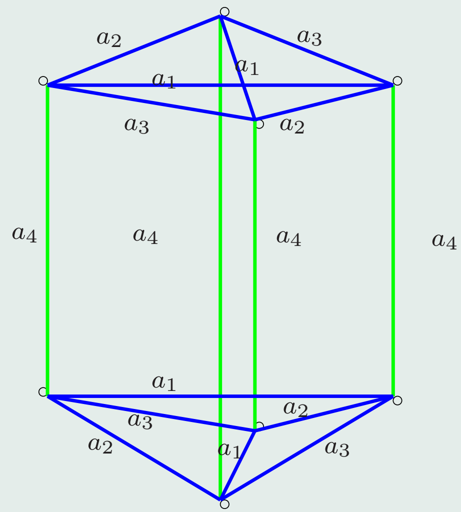
The case $n = 1$: $|\mathcal{F}_{(\Gamma,\alpha)}| \approx S^0$

The case $n = 2$: it is easy to see that for any coloring graph (Γ, α) of type $(2, 2)$, the geometric realization $|\mathcal{F}_{(\Gamma,\alpha)}|$ is always a circle.

The case $n = 3$:

Fact. $|\mathcal{F}_{(\Gamma,\alpha)}|$ *is a closed surface* S.

Generally, if $n > 3$, the geometric realization $|\mathcal{F}_{(\Gamma,\alpha)}|$ is not a closed topological manifold. For example, see the following coloring graph (Γ, α) of type $(4, 4)$.



$$\chi(|\mathcal{F}_{(\Gamma,\alpha)}|) = 5 - 12 + 16 - 8 = 1 \neq 0$$

so $|\mathcal{F}_{(\Gamma,\alpha)}|$ is not a closed topological 3-manifold.

The case $n = 4$.

Write $v = |V_\Gamma|$ and $e = |E_\Gamma|$ so $2v = e$.

f: the number of all 2-faces in $\mathcal{F}_{(\Gamma,\alpha)}$

f_3: the number of all 3-faces in $\mathcal{F}_{(\Gamma,\alpha)}$

Theorem. ***Let $n = 4$. $|\mathcal{F}_{(\Gamma,\alpha)}|$ is a closed connected topological 3-manifold $\iff f = f_3 + v$.***

Problem: for $n > 4$, to give a sufficient (and necessary) condition that $|\mathcal{F}_{(\Gamma,\alpha)}|$ is a closed connected topological manifold.

Basic problem (II)

M^n:n-dim closed connected topological manifold

1-dim case: $M^1 \approx S^1$.

S^1 is realizable by **any** coloring graph of type $(2,2)$.

2-dim case:

Prop. *Any closed surface can be realized by some coloring graph of type* $(3,3)$.

3-dim case:

Conjecture: ***Any closed 3-manifold*** M^3 ***is geometrically realizable by a coloring graph*** (Γ, α) ***of type*** $(4, 4)$***, i.e.,***

$$M^3 \approx |\mathcal{F}_{(\Gamma,\alpha)}|.$$

4-dim case: It is well known that there exist closed topological 4-manifolds that **don't admit** any triangulation.

$$\Downarrow$$

$\exists$ closed topological 4-manifolds that **cannot** be realized by any coloring graph of type $(5, 5)$.

Proposition. *Let* M^n *be a closed manifold. If* M^n *admits a simplicial cell decomposition with at least* $n+2$ *vertices, then* M^n *can be geometrically realizable by a coloring graph.*

Restatement

Proposition. *Suppose that* Γ *is a 3-valent graph and is at least 2-connected. Then* Γ *is planar if and only if* Γ *admits a coloring* α *of type* $(3,3)$ *such that* $|\mathcal{F}_{(\Gamma,\alpha)}| \approx S^2$.