The symplectic volume of spatial polygon spaces

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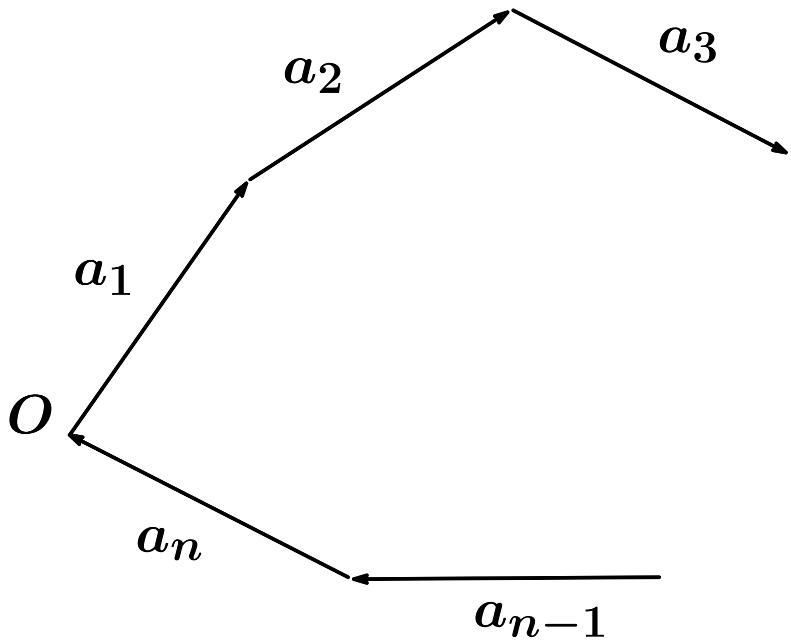
Let S^2 be the unit sphere in $\mathbb{R}^3$.

For $n \geq 3$, we define A_n by

$$A_n = \{P = (a_1, \ldots, a_n) \in (S^2)^n : \\ a_1 + \cdots + a_n = 0\}.$$



- $A_n \curvearrowleft SO(3)$ the diagonal action

- Set

$$\boxed{M_n = A_n/SO(3)}$$

Basic properties

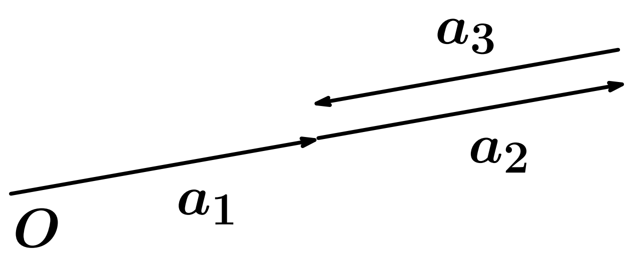
- M_n is compact.

- When n is odd or $n = 4$, M_n is a complex manifold such that $\dim_{\mathbb{C}} M_n = n - 3$.

- When n is even ≥ 6, M_n has singular points.

  $P = (a_1, \ldots, a_n) \in M_n$ is a singular point $\Leftrightarrow$

$$a_2 = \pm a_1,\ a_3 = \pm a_1,\ \ldots,\ a_n = \pm a_1.$$



Such a singular point has a neighborhood

$$C(S^{n-3} \underset{S^1}{\times} S^{n-3}).$$

Example.

(1) $M_3 = \{\text{one point}\}$.

(2) $M_4 = S^2$.

(3) M_5

$\underset{\text{biholomorphic}}{\cong}$ the surface obtained from $\mathbb{C}P^2$ by blowing up 4 points in general position

$\underset{\text{diffeo.}}{\cong}$ $\mathbb{C}P^2 \# 4\,\overline{\mathbb{C}P}^2$.

$\boxed{\text{The symplectic form}}$

We define the canonical symplectic form $\omega_n \in \Omega^2(M_n)$ as follows.
(Actually ω_n is a Kähler form.)

For $P = (a_1, \ldots, a_n) \in M_n$, $T_P M_n$ is given as follows:

$$
\begin{aligned}
T_p M_n = \{u = (u_1, \ldots, u_n) \in (\mathbb{R}^3)^n :& \\
& (u_i, a_i) = 0 \ (1 \leq i \leq n) \quad \text{and} \\
& u_1 + \cdots + u_n = 0\} / \sim .
\end{aligned}
$$

For

$$
\left.\begin{aligned}
u &= (u_1, \ldots, u_n) \\
v &= (v_1, \ldots, v_n)
\end{aligned}\right\} \in T_p M_n, \quad \text{set}
$$

$$
\boxed{\omega_n(u, v) = \sum_{i=1}^{n} \det(u_i, v_i, a_i)}
$$

We consider the following:

Question. What is the symplectic volume of M_n?

$\boxed{\text{Main theorem}}$ (Tezuka-K).

For $n \geq 3$, set

$$v_n = \int_{M_n} \omega_n^{n-3}.$$

Then we have

$$v_n = \sum_{j=0}^{[\frac{n}{2}]-1} (-1)^j \binom{n-1}{j} (n-2-2j)^{n-3}.$$

Example.

$v_3 = 1,\ v_4 = 2,\ v_5 = 5,\ v_6 = 24,$
$v_7 = 154,\ v_8 = 1280$ and $v_9 = 13005.$

$\boxed{\text{The Duistermaat-Heckman theorem}}$

Since the main theorem is proved using the Duistermaat-Heckman theorem, we recall this.

- (X, ω) a symplectic manifold with $\dim_{\mathbb{R}} X = 2k$.
- $T^k = (S^1)^k \curvearrowright (X, \omega)$ preserving ω.
- $\mu : X \to \mathbb{R}^k$ a moment map.
- $\mathrm{Vol}(\mu(X))$ the volume of $\mu(X)$.

Then

$$\boxed{\int_X \omega^k = k!\,\mathrm{Vol}(\mu(X))}$$

Proof of the main theorem

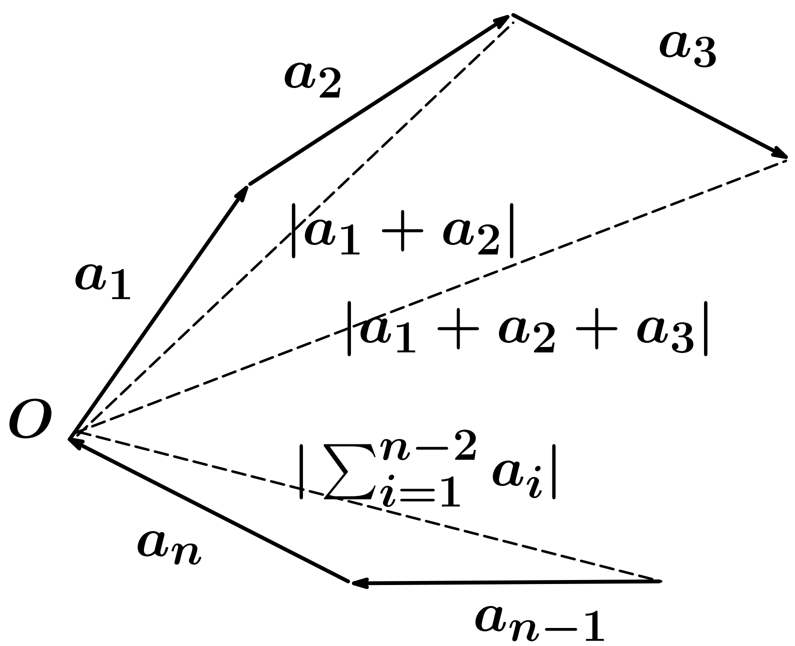
Unfortunately, we do not have an action $T^{n-3} \curvearrowright M_n$.

However, there is an open dense subspace $M_n' \subset M_n$ such that $T^{n-3} \curvearrowright M_n'$ as follows:

(1) Define a map $\mu_n : M_n \to \mathbb{R}^{n-3}$ by

$$\mu_n(P) = (|a_1+a_2|, |a_1+a_2+a_3|, \ldots, |\sum_{i=1}^{n-2} a_i|).$$

That is, the lengths of the diagonals connecting the vertices to the origin.

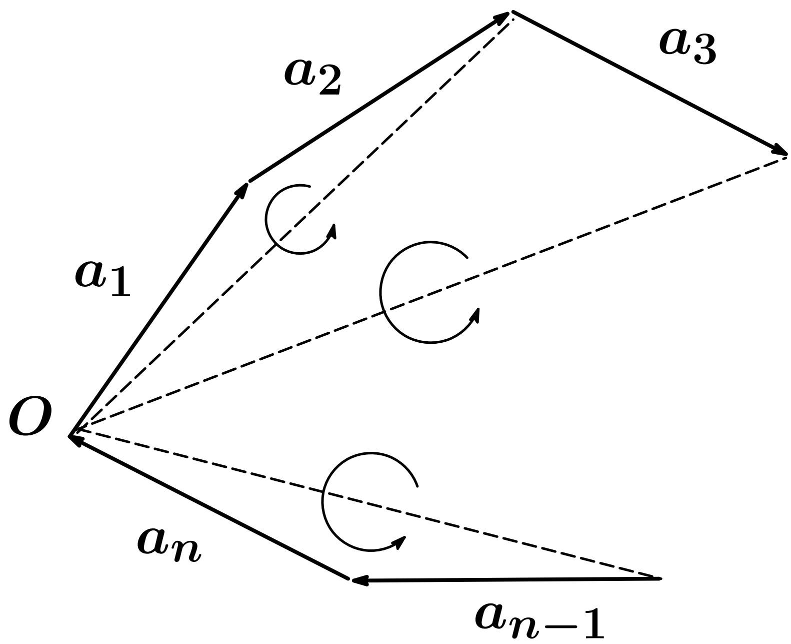


(2) Set

$M'_n = \{P \in M_n$: none of
these $n-3$ lengths vanishes$\}$.

- M'_n is open and dense in M_n.

(3) Define $T^{n-3} \curvearrowright M'_n$ as follows: The i-th circle acts by rotating the part of a polygon, formed by the first $i+1$ edges, around the i-th diagonal.



Our situation is as follows.

- (M'_n, ω_n) a symplectic manifold.

- $T^{n-3} \curvearrowright M'_n$ preserving ω_n.

- There is a map

$$\mu_n | M'_n : M'_n \longrightarrow \mathbb{R}^{n-3}.$$

<u>Theorem</u> (Kapovich-Millson).

$\mu | M'_n$ is a moment map for the action

$T^{n-3} \curvearrowright (M'_n, \omega_n)$.

We can apply the Duistermaat-Heckman theorem and we have

$$\int_{M_n'} \omega_n^{n-3} = (n-3)!\,\mathrm{Vol}(\mu_n(M_n')).$$

We set

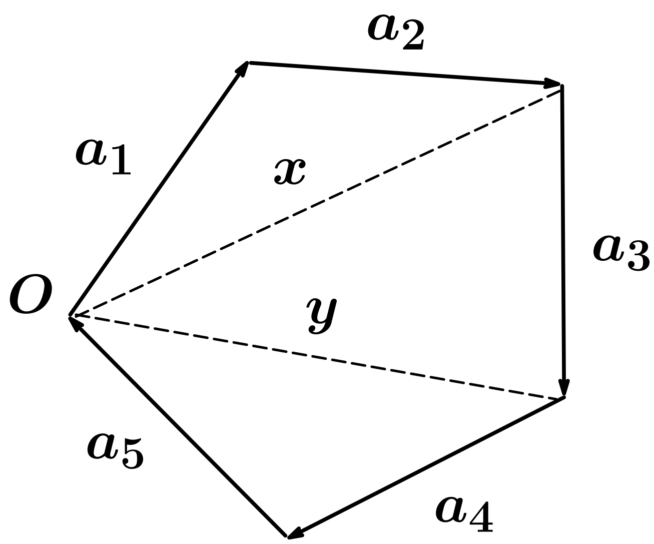
$$\Delta_n = \mu_n(M_n).$$

Since M_n' is open and dense in M_n, we have

$$\boxed{v_n = (n-3)!\,\mathrm{Vol}(\Delta_n)}$$

Note that $\Delta_n \subset \mathbb{R}^{n-3}$ is a convex polytope which is defined by triangle inequalities. Hence we can calculate the right-hand side by multiple integral.

Example. Consider the case $n = 5$.



$x = |a_1 + a_2|$, $y = |a_1 + a_2 + a_3|$ and

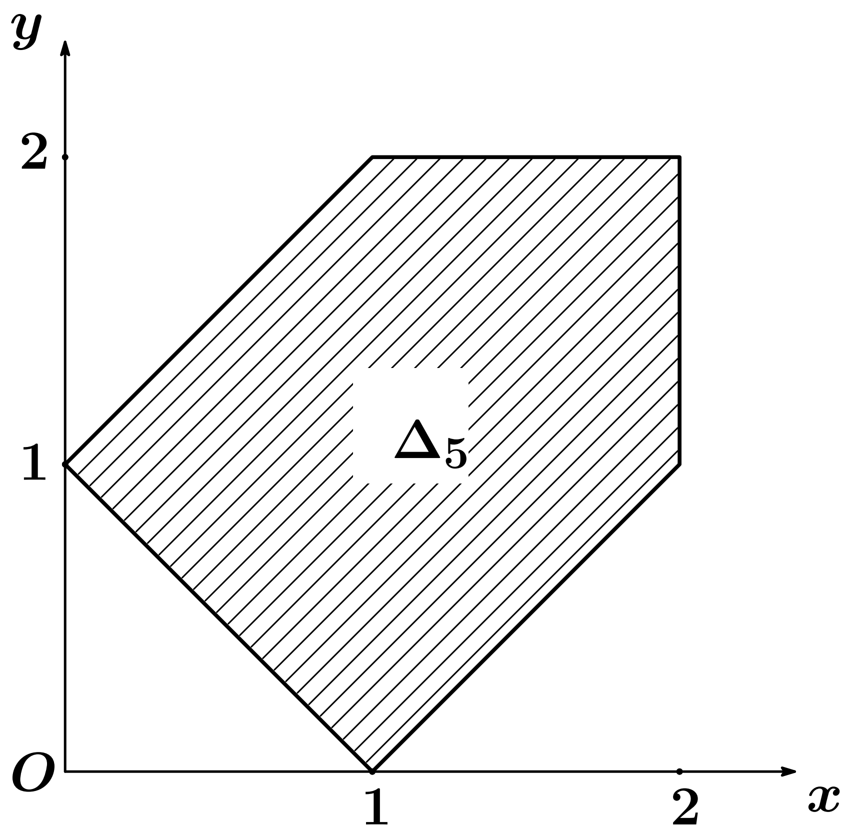
$\mu_5(P) = (x, y)$.

By triangle inequalities, we have

$$0 \leq x,\ y \leq 2$$

and

$$\begin{cases} 1 \leq x + y \\ x \leq 1 + y \\ y \leq 1 + x \end{cases}$$



$$\mathbf{Vol}(\Delta_5) = 4 - 3 \cdot \left(\frac{1}{2}\right)$$
$$= \frac{5}{2}$$

Hence

$$v_5 = 2!\ \mathbf{Vol}(\Delta_5)$$
$$= 5.$$

Another proof of the main theorem

Hereafter n is odd and all cohomology groups are with $\mathbb{R}$-coefficients.

We can calculate $v_n = \int_{M_n} \omega_n^{n-3}$ from the cohomology ring $H^*(M_n)$.

Let

$$z_1, \ldots, z_n$$

be the generators of $H^2(M_n)$.

- They are essentially the generators of $H^2(\underbrace{S^2 \times \cdots \times S^2}_{n})$.

- The cohomology classes $z_1, \ldots, z_n$ generate the cohomology ring $H^*(M_n)$. (Similarly to toric varieties, the cohomology ring $H^*(M_n)$ is generated by 2-dimensional classes.)

Theorem (Hausmann-Knutson).

$$[\omega_n] = z_1 + \cdots + z_n \in H^2(M_n).$$

Theorem (Tezuka-K).

For all $i_1, \ldots, i_n \geq 0$ with $i_1 + \cdots + i_n = n - 3$, the intersection pairings

$$\int_{M_n} z_1^{i_1} \ldots z_n^{i_n}$$

are determined.

In particular, we can calculate

$$\boxed{v_n = \int_{M_n} (z_1 + \cdots + z_n)^{n-3}}$$

Example. Consider the case $n = 5$.

$$\int_{M_5} z_i z_j = \begin{cases} 1 & i \neq j \\ -3 & i = j. \end{cases}$$

Hence

$$\begin{aligned} v_5 &= -3 \cdot 5 + 1 \cdot 20 \\ &= 5. \end{aligned}$$

$\boxed{\text{Cobordism}}$

Since we can calculate all the Chern numbers of M_n, it is possible to describe M_n in the complex cobordism group Ω^U_{2n-6}.

For example,

(i) In Ω^U_4, $M_5 = 4\,(\mathbb{C}P^1)^2 - 3\,\mathbb{C}P^2$.

(ii) In Ω^U_8,

$$\begin{aligned} M_7 = &- 9\,(\mathbb{C}P^1)^4 + 33\,(\mathbb{C}P^1)^2 \times \mathbb{C}P^2 \\ &- 33\,\mathbb{C}P^1 \times \mathbb{C}P^3 + 10\,\mathbb{C}P^4. \end{aligned}$$

(iii) In Ω^U_{12},

$$\begin{aligned} M_9 = &\, 3123\,(\mathbb{C}P^1)^6 - 10196\,(\mathbb{C}P^1)^4 \times \mathbb{C}P^2 \\ &+ 0 \cdot (\mathbb{C}P^2)^3 + 0 \cdot \mathbb{C}P^2 \times \mathbb{C}P^4 \\ &+ \text{(omit 6 terms)} - 35\,\mathbb{C}P^6. \end{aligned}$$

<u>Remark</u>. Let $n = 2m+1$.

Then in Ω^{SO}_{2n-6}, we have

$$M_n = (-1)^{m+1}\binom{2m-1}{m}\mathbb{C}P^{n-3}.$$