Euler characteristic.

$$\chi(K) = f_0(K) - f_1(K) + f_2(K) - \dots$$

$$\chi(K^2) = \sum_{v \in K^2} \left(1 - \frac{d_v}{6}\right),$$

where d_v is the number of edges entering v.

Stiefel-Whitney classes.

$$W_n(K) = \sum_{\sigma^n \in K'} \sigma^n \pmod 2.$$

Theorem (Whitney, 1940, Halperin, Toledo, 1972) $[W_n(K)]$ is the Poincaré dual of $w_{m-n}(K)$, where $m = \dim K$.

Rational Pontrjagin classes.

Rokhlin, Swartz, Thom, 1957–1958:
Rational Pontrjagin classes are well defined for combinatorial manifolds.

Problem. Given a combinatorial manifold K construct explicitly a rational simplicial cycle $Z(K)$ representing the Poincaré dual of $p_k(K)$.

Formulae.

- Gabrielov, Gelfand, Losik, 1975, MacPherson, 1977.
  A formula for the first Pontrjagin class of any *Brouwer manifold*.

- Cheeger, 1983. Formulae for all Pontrjagin classes.
  - Include calculation of the spectrum of the Laplace operator.
  - Give only *real* cycles.

- Gelfand, MacPherson, 1992. Formulae for all Pontrjagin classes of a triangulated manifold with given smoothing or combinatorial differential (CD) structure.
  - Do not solve the above problem.

Local formulae.

$$\operatorname{link}\sigma = \{\tau \in K | \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\} .$$

$$\boxed{f_\sharp(K^m) = \sum_{\sigma^{m-n} \in K^m} f(\operatorname{link}\sigma)\sigma.}$$

f is a skew-symmetric rational-valued function on the set of isomorphism classes of oriented $(n-1)$-dimensional PL spheres.

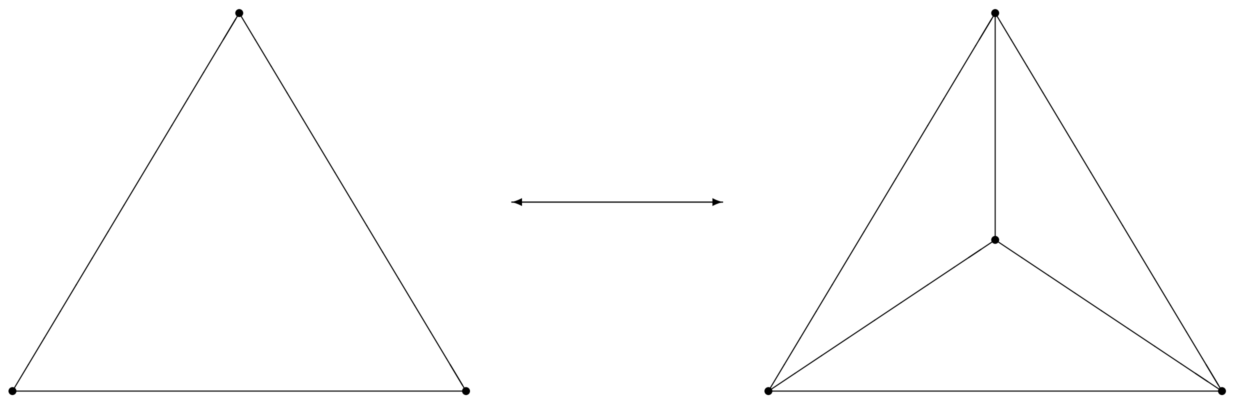
f does not depend on K.

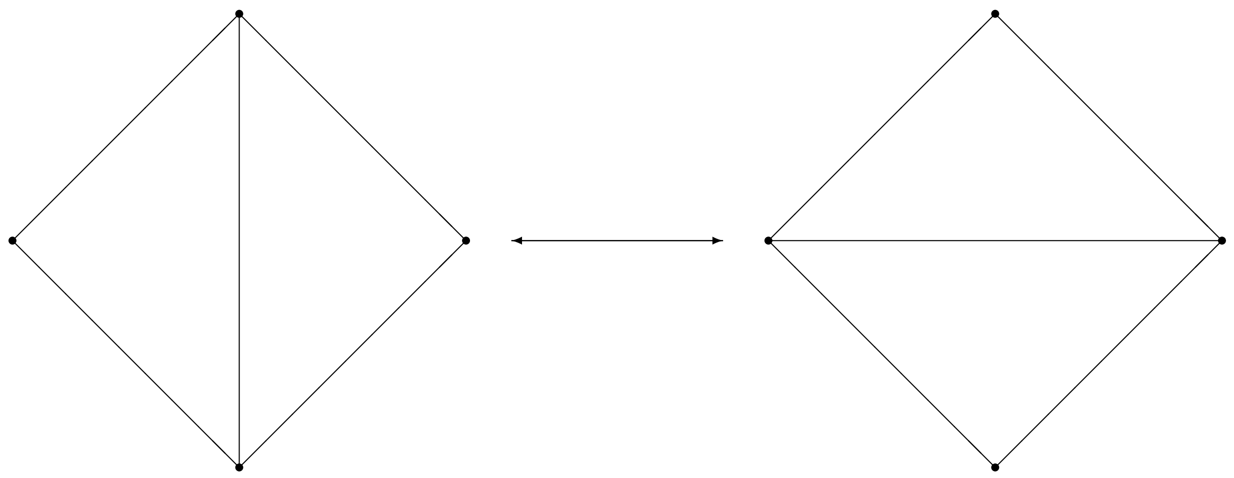
Problem. Describe all functions f such that $f_\sharp(K)$ is a cycle for every K.

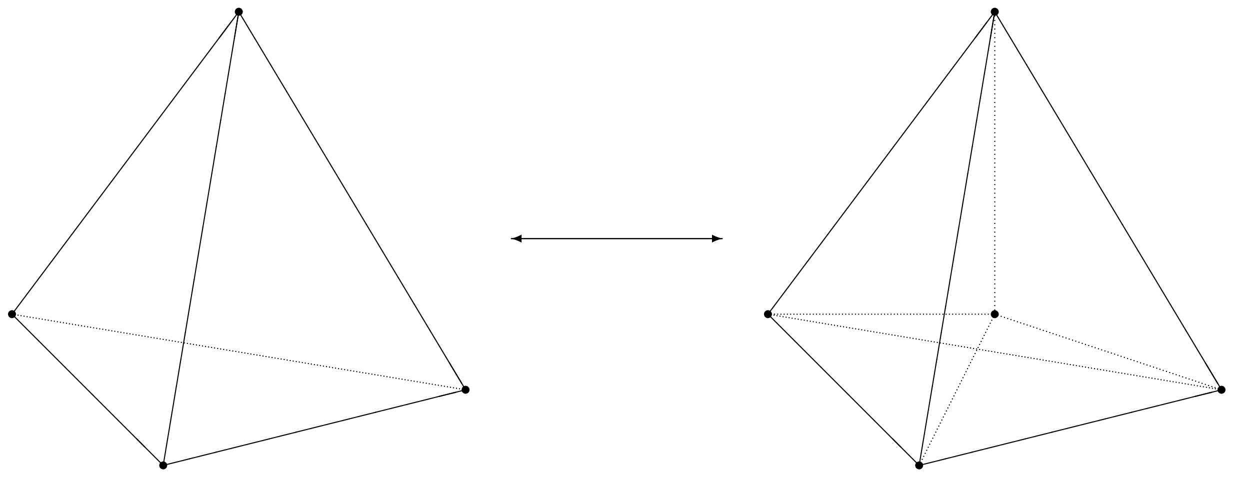
f is a *local formula* for $P \in \mathbb{Q}[p_1, p_2, \ldots]$ if $[f_\sharp(K)]$ is the Poincaré dual of $P(p_1(K), p_2(K), \ldots)$ for every K.

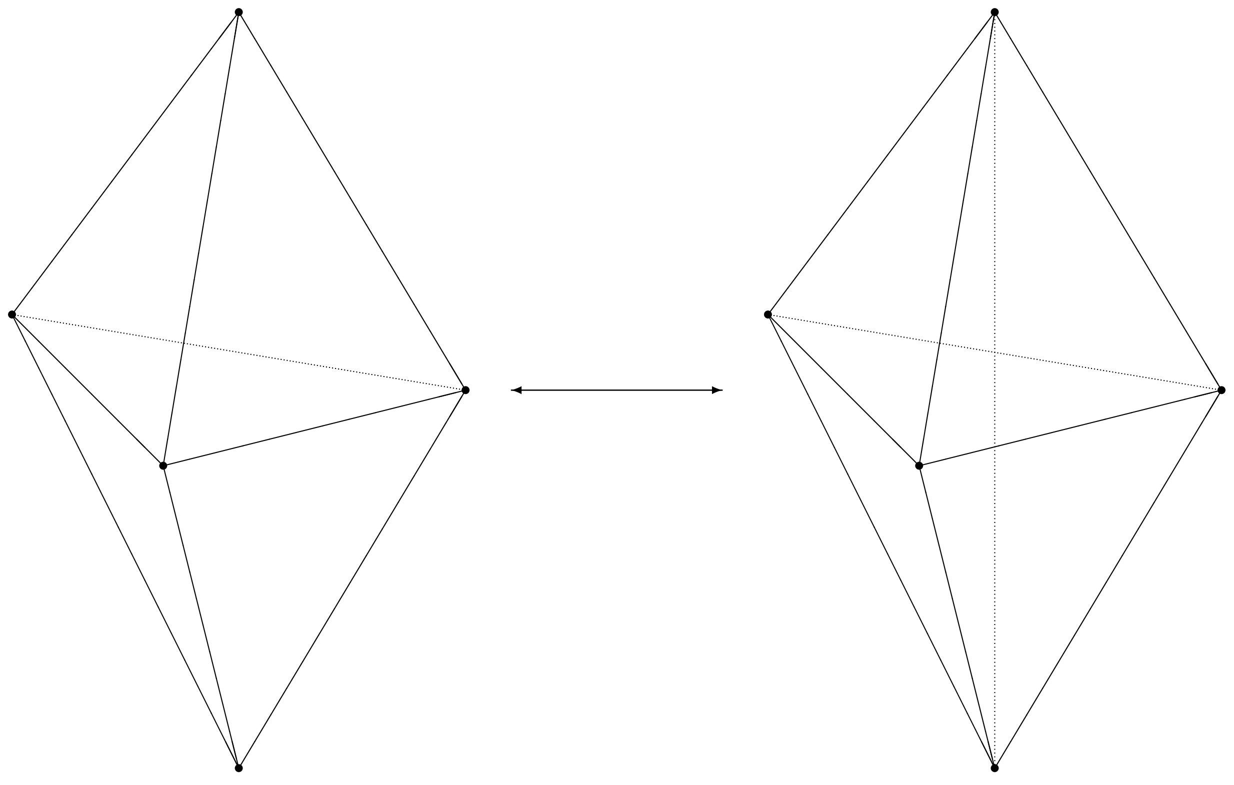
Bistellar moves.

Theorem (Pachner, 1989). Two combinatorial manifolds are PL homeomorphic iff the first can be transformed into the second by a finite sequence of bistellar moves.









Local formulae for the first Pontrjagin class.

$$f : \boxed{\begin{array}{c}\text{oriented 3-dim.}\\ \text{PL-sphere } L\end{array}} \mapsto \boxed{\begin{array}{c}\text{rational}\\ \text{number } f(L)\end{array}}$$

$$L = L_1 \xrightarrow{\beta_1} L_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_q} \partial\Delta^4$$

bistellar moves

$$L_j \xrightarrow{\beta_j} L_{j+1}, \qquad v \text{ is a vertex of } L_j$$

$$\mathrm{link}_{L_j} v \xrightarrow{\beta_{j,v}} \mathrm{link}_{L_{j+1}} v$$

Graph Γ_2.

Vertices: isomorphism classes of oriented 2-dimensional PL spheres.

Edges: bistellar moves.

$$\gamma = \sum_{j=1}^{q} \sum_{v \in L_j} \beta_{j,v} \in C_1(\Gamma_2; \mathbb{Z})$$

$$f(L) = \widehat{c}(\gamma), \quad \widehat{c} \in C^1(\Gamma_2; \mathbb{Q}).$$

Theorem (G., 2004) There is a cohomology class $c \in H^1(\Gamma_2; \mathbb{Q})$ such that local formulae for the first Pontrjagin class are in one-to-one correspondence with cocycles $\widehat{c} \in C^1(\Gamma_2; \mathbb{Q})$ representing c. The correspondence is given by the formula

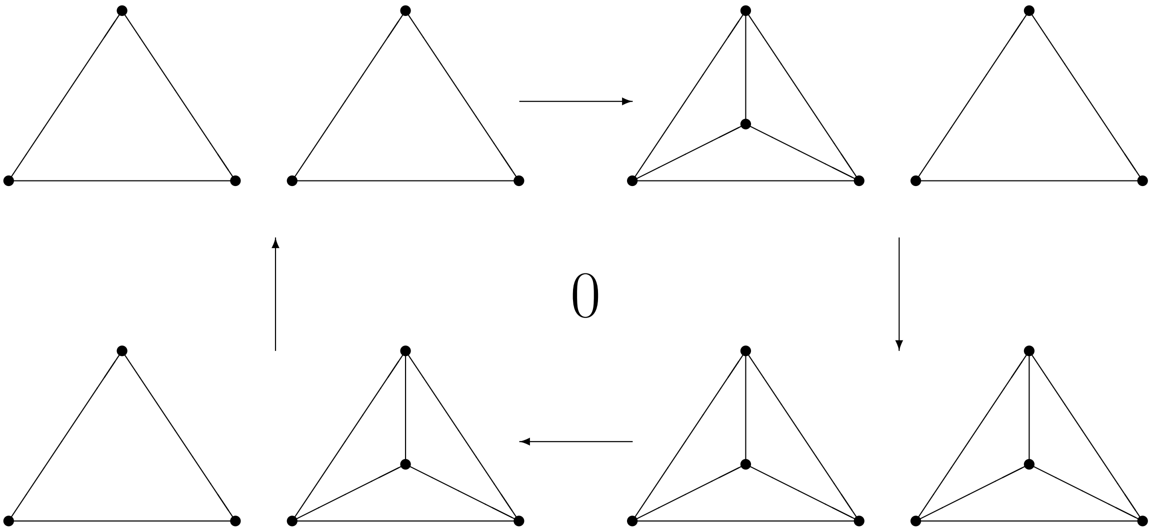
$$f(L) = \widehat{c}(\gamma)$$

Cohomology class c.

The group $H_1(\Gamma_2; \mathbb{Z})$.
Generators: 6 infinite series.
Let us give the values of c on these generators.

$$\rho(p, q) = \frac{q - p}{(p + q + 2)(p + q + 3)(p + q + 4)}$$

$$\lambda(p) = \frac{1}{(p + 2)(p + 3)}$$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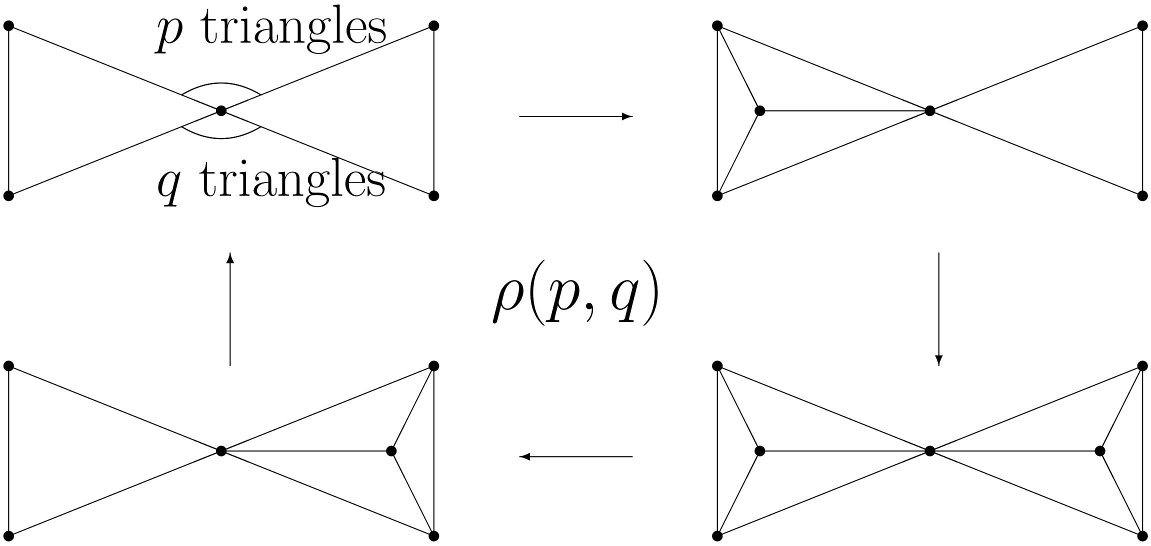
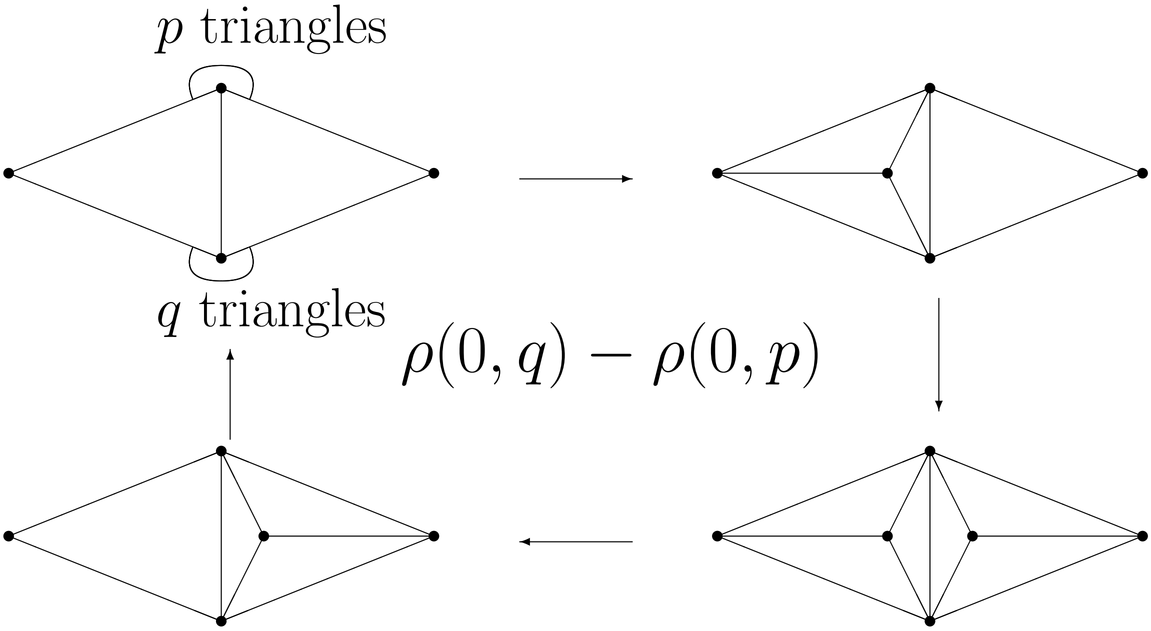
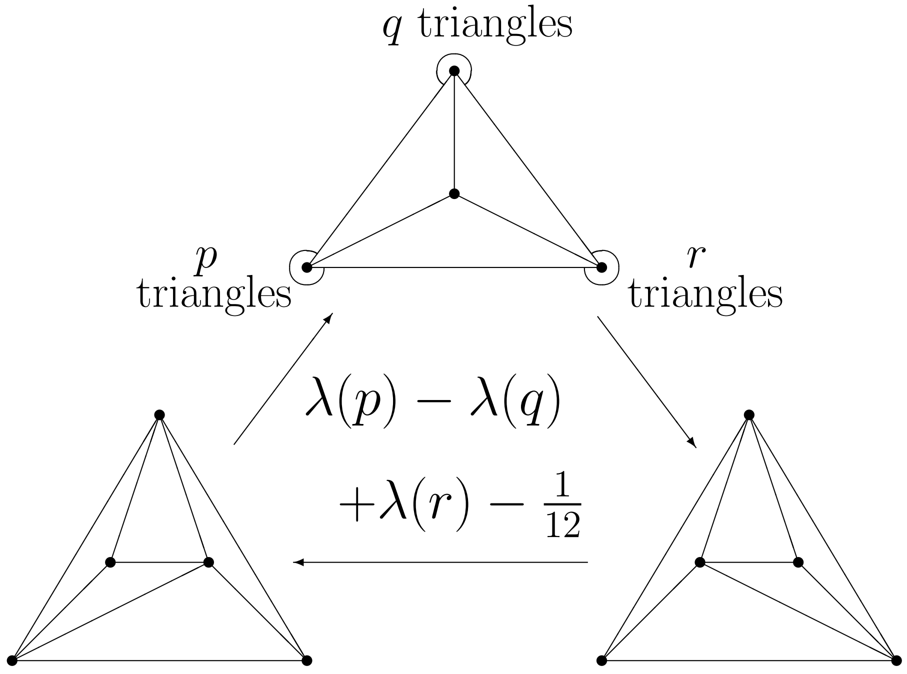
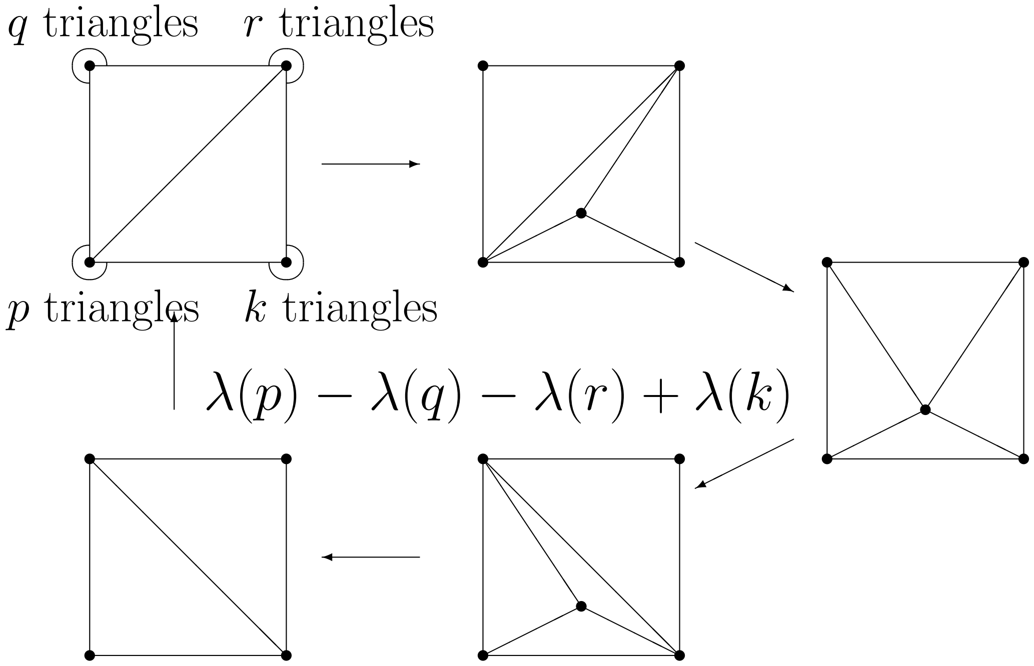

p triangles
q triangles
$\rho(p, q)$

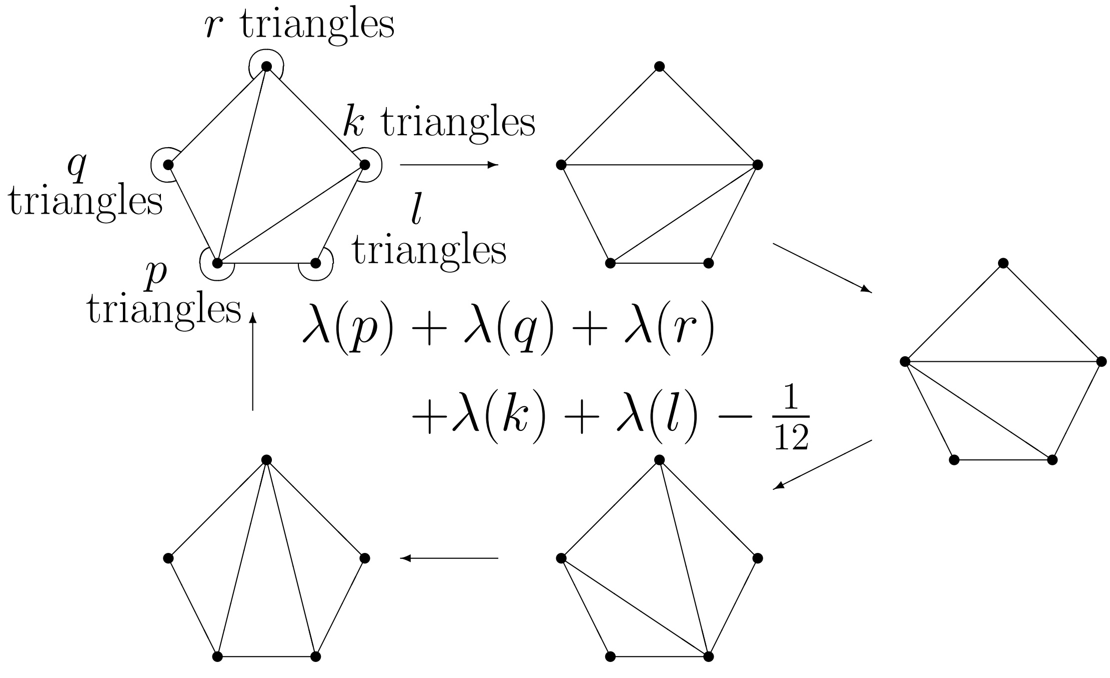

p triangles
q triangles
$\rho(0, q) - \rho(0, p)$



q triangles
p triangles
r triangles
$\lambda(p) - \lambda(q)$
$+\lambda(r) - \frac{1}{12}$



q triangles
r triangles
p triangles
k triangles
$\lambda(p) - \lambda(q) - \lambda(r) + \lambda(k)$



r triangles
k triangles
q triangles
l triangles
p triangles
$\lambda(p) + \lambda(q) + \lambda(r)$
$+\lambda(k) + \lambda(l) - \frac{1}{12}$

The cochain complex $\mathcal{T}^*(\mathbb{Q})$.

$\mathcal{T}^n(\mathbb{Q})$ is the vector space of all skew-symmetric rational-valued functions on the set of isomorphism classes of oriented $(n-1)$-dimensional PL spheres.

$$\delta : \mathcal{T}^n(\mathbb{Q}) \to \mathcal{T}^{n+1}(\mathbb{Q});$$

$$(\delta f)(L) = \sum_{v \in L} f(\operatorname{link} v); \quad \delta^2 = 0.$$

$f_\sharp(K)$ is a cycle for every $K \Leftrightarrow f$ is a cocycle.

$f_\sharp(K)$ is a boundary for every $K \Leftrightarrow f$ is a coboundary.

Existence and uniqueness.

- $H^*(\mathcal{T}^*(\mathbb{Q})) \cong \mathbb{Q}[p_1, p_2, \ldots],\ \deg p_i = 4i.$
- Each cocycle of $\mathcal{T}^*(\mathbb{Q})$ is a local formula for some polynomial in rational Pontrjagin classes.
- A local formula for a polynomial in rational Pontrjagin classes exists and is unique up to a coboundary.
  (The existence strengthens a result of Levitt and Rourke, 1978.)
- We describe explicitly the cohomology class $\phi \in H^4(\mathcal{T}^*(\mathbb{Q}))$ such that $\alpha(\phi) = p_1$.
- We describe explicitly the cohomology classes $\psi_i \in H^{4i}(\mathcal{T}^*(\mathbb{Q}))$ such that $\alpha(\psi_i) = L_i(p_1, \ldots, p_i)$.

Denominators.

For $f \in \mathcal{T}^n(\mathbb{Q})$, by $\mathrm{den}_l(f)$ we denote the least common multiple of the denominators of the values $f(L)$, where L runs over all $(n-1)$-dimensional oriented PL spheres with not more than l vertices.

- $\forall\, \psi \in H^*(\mathcal{T}^*(\mathbb{Q}))$ there exist a cocycle f representing ψ and an integer constant C such that $\mathrm{den}_l(f)$ is a divisor of $C(l+1)!$ for any l.

- Suppose f is a local formula for the first Pontrjagin class. Then $\mathrm{den}_l(f)$ is divisible by the least common multiple of the numbers $1, 2, \ldots, l-3$ for any even $l \geq 10$.

- $H^4(\mathcal{T}^*(G)) = 0$ for any subgroup $G \subsetneq \mathbb{Q}$. Recall that $H^4(\mathcal{T}^*(\mathbb{Q})) = \mathbb{Q}$.