Analogous polytopes, toric manifolds and complex cobordisms

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Our aim is to bring geometric and combinatorial methods to bear on the study omnioriented toric manifolds M, in the context of stably complex manifolds with compatible torus action.

We interpret M in terms of combinatorial data (P, Λ), where P is the combinatorial type of an oriented simple polytope, and Λ is an integral matrix whose properties are controlled by P.

By way of application, we study conditions on (P, Λ) such that the corresponding toric manifold admits special unitary and level-N structures, and develop combinatorial formulae for the evaluation of genera in terms of subcircles of the torus action. We provide a discussion of the complex cobordism ring in the term of omnioriented toric manifolds.

Prelude to toric manifolds

Consider a smooth $2n$-dimensional manifold M^{2n}, endowed with a smooth action of n-dimensional torus T^n.

The orbit space M^{2n}/T^n is a manifold with **corners**.

The most basic manifolds with corners are the **simple convex** polytopes, i.e. n-dimensional convex polytopes with exactly n facets meeting at each vertex.

A simple convex polytope is **generic**; its bounding hyperplanes are in **general position**.

Each face of a simple polytope is again a simple polytope.

Complex projective space

$$\mathbb{C}P^n = S^{2n+1}/S^1$$

where $S^{2n+1} \subset \mathbb{C}^{n+1}$ consists of vectors

$$z = (z_1, \ldots, z_{n+1}) \quad \text{with} \quad |z|^2 = \sum_{k=1}^{n+1} |z_k|^2 = 1$$

and $t_1 \in S^1$ acts by $t_1 \cdot z = (t_1 z_1, \ldots, t_1 z_{n+1})$. The **standard action** of T^{n+1} on $\mathbb{C}^{n+1}$ is

$$t \cdot z = (t_1 z_1, \ldots, t_{n+1} z_{n+1}),$$

and induces an action of T^n on $\mathbb{C}P^n$ by

$$t \cdot [z] = [t_1 z_1, \ldots, t_n z_n, z_{n+1}].$$

The orbit space is the n-**simplex** Δ^n, given by

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}, \quad x_i \geqslant 0, \quad \sum_{k=1}^{n+1} x_i = 1\}.$$

The projection $\pi : \mathbb{C}P^n \longrightarrow \Delta^n$ acts by

$$(z_1 : z_2 : \cdots : z_{n+1}) \longrightarrow (|z_1|^2, |z_2|^2, \ldots, |z_{n+1}|^2).$$

The bounded flag manifolds B_n

A point

$$U = (U_1 \subset U_2 \subset \cdots \subset U_{n+1}) \in B_n$$

is a **complete flag** in $\mathbb{C}^{n+1}$, such that each U_k, contains the subspace $\mathbb{C}^{k-1}$, spanned by the first $(k-1)$ standard basis vectors in $\mathbb{C}^{n+1}$ for $2 \leqslant k \leqslant n$.
So U is equivalent to a sequence of lines

$$\{L_k \subset \mathbb{C}_k \oplus L_{k+1},\ 1 \leqslant k \leqslant n\},$$

where $\mathbb{C}_k$ is k-th coordinate line and $L_1 = U_1,\ L_{n+1} = \mathbb{C}_{n+1}$. The standard action of T^{n+1} on $\mathbb{C}^{n+1}$ induces an action of T^n on B_n, whose orbit space is the **cube** $I^n = I \times \cdots \times I$.

The projection $\pi : B_n \longrightarrow I^n$ satisfies

$$\pi(U) = (\pi(L_1), \ldots, \pi(L_n)),$$

where $\pi(L_k) = |l_k|^2$ and l_k is the projection of a unit vector from $L_k \subset \mathbb{C}_k \oplus L_{k+1}$ into $\mathbb{C}_k$.

The manifold $B_{i,j}$

$B_{i,j}$ is a $2n$-dim smooth $\mathbb{C}P^{j-1}$-bundle over B_i, where $n = (i + j - 1)$ and $0 \leqslant i \leqslant j$.

Each point of $B_{i,j}$ is a pair (U, W), where $U \in B_i$ is a bounded flag in $\mathbb{C}^{i+1}$ and W is a line in $U_1^{\perp} \oplus \mathbb{C}^{j-i}$. The projection

$$\pi : B_{i,j} \longrightarrow B_i$$

satisfies $\pi(U, W) = U$.

So $T^n = T^i \times T^{j-1}$ acts on $B_{i,j}$ by

$$(t_1 z_1, \ldots, t_i z_i, z_{i+1}, t_{i+1} w_1, \ldots, t_n w_{j-1}, w_j).$$

The orbit space is $I^i \times \Delta^{j-1}$, and the quotient map

$$B_{i,j} \longrightarrow I^i \times \Delta^{j-1}$$

is given by $\pi(U, W) = (\pi(U), \pi(W))$.

Moment-angle manifolds

We deal only with simple polytopes, and reserve the notation $m = m(P)$ for the number facets of P. Every face of codimension k may be written uniquely as

$$F_I \;=\; F_{i_1} \cap \cdots \cap F_{i_k} \tag{1}$$

for some subset $I = \{i_1, \ldots, i_k\}$ of $[m]$.

We denote the i-th coordinate subcircle of the standard m–torus T^m by T_i for every $1 \leqslant i \leqslant m$. Given any subset $I \subseteq [m]$, we define the subgroup by

$$T_I = \prod_{i \in I} T_i \;\subset\; T^m,$$

in particular, $T_\varnothing$ is the trivial subgroup $\{1\}$.

Every point p of P lies in the interior of a unique face F_{I_p}, where $I_p = \{i : p \in F_i\}$, and it is convenient to abbreviate F_{I_p} to $F(p)$ and T_{I_p} to $T(p)$.
If p is a vertex, then $T(p)$ has dimension n (the maximum possible), and if p is an interior point of P, then $T(p)$ is trivial.

We now construct the identification space

$$\mathscr{Z}_P = T^m \times P/\sim \qquad (2)$$

where $(t_1, p) \sim (t_2, p)$ if and only if

$$t_1^{-1} t_2 \in T(p).$$

So $\mathscr{Z}_P$ is an $(m+n)$-dim **manifold** with a canonical left T^m-action, whose **isotropy subgroups** are precisely the subgroups $T(p)$.

Many important examples of manifolds in topology and geometry arise as factor spaces of $\mathscr{Z}_P$ by an action of an appropriate subtorus $T^k \subset T^m$.

Define $s = s(P)$ to be the maximal dimension for which there exists a subgroup $H \cong T^s$ in T^m acting freely on $\mathscr{Z}_P$. The number $s(P)$ is obviously a combinatorial invariant of P.

We have

$$1 \leqslant s(P) \leqslant m - n.$$

In the case $s(P) = m - n$ the quotient space

$$M^{2n} = \mathscr{Z}_P / T^{m-n}$$

is smooth and called **toric manifold**.

Any nonsingular compact toric variety is a toric manifold.

Dicharacteristic

In order to construct toric manifolds over P, we need one further set of data.
This consists of a homomorphism $\ell\colon T^m \to T^n$, whose properties are controlled by P (cf. Davis and Januszkiewicz), namely

$$F_I \text{ a face of codim } k \Longrightarrow \ell \text{ monic on } T_I. \quad (3)$$

Any such ℓ is called a **dicharacteristic**; condition (3) ensures that the kernel $K(\ell)$ of ℓ is isomorphic to an $(m-n)$–dim subtorus of T^m.
Wherever possible we abbreviate $K(\ell)$ to K.

We write the subcircle $\ell(T_i) \subset T^n$ as $T(F_i)$ for any $1 \leqslant i \leqslant m$, and the subgroup $\ell(T_I)$ as $T(F_I)$ for any face F_I.
For each point p in P let $S(p)$ denote the subgroup $T(F(p))$; it is, of course, $\ell(T(p))$.
For example, $S(w) = T^n$ for any vertex w, and $S(p) = \{1\}$ for any point p in the interior of P.

Refined form of dicharacteristic matrix

Applied to the initial vertex $v_\star = F_1 \bigcap \cdots \bigcap F_n$, (3) ensures that the restriction of ℓ to $T_1 \times \cdots \times T_n$ is an isomorphism.
So we may use the circles $T(F_1)$, ..., $T(F_n)$ to define a basis for the Lie algebra of T^n, and represent the homomorphism induced by ℓ by an $n \times m$ integral matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}. \quad (4)$$

Given any other vertex $v = F_{j_1} \cap \cdots \cap F_{j_n}$, (3) implies that the corresponding columns λ_{j_1}, ..., λ_{j_n} form a basis for $\mathbb{Z}^n$, and have determinant ± 1. We refer to (4) as the **refined form** of the dicharacteristic matrix.

A model for toric manifolds

Since $K = K(\ell)$ acts freely on $\mathscr{Z}_P$ there is a principal K-bundle

$$\pi_\ell\colon \mathscr{Z}_P \to M,$$

whose base space is a $2n$–dim manifold. By construction,

$$M = T^n \times P/\!\sim \tag{5}$$

where $(s_1, p) \sim (s_2, p)$ if and only if

$$s_1^{-1} s_2 \in S(p).$$

Furthermore, M admits a canonical T^n-action α, which is locally isomorphic to the **standard action** on $\mathbb{C}^n$, and has quotient map

$$\pi\colon M \to P.$$

Note that $\pi \cdot \pi_\ell$ is the natural projection $\varrho_P : \mathscr{Z}_P \to P$.

The fixed points of α project to the vertices of P, and we refer to $\pi^{-1}(v_\star)$ as the **initial fixed point** $x_\star$.

Then (5) identifies a neighbourhood of $x_\star$ with $\mathbb{C}^n$, on which α is **standard**;
its representation at other fixed points $\pi^{-1}(v)$ may be read off from the corresponding columns of Λ.

The quadruple (M, α, π, P) is an example of Davis and Januszkiewicz's **toric manifolds**.

Any manifold with a similarly well-behaved torus action over P is **equivariantly diffeomorphic** to one of the form (5).

Facial submanifold structure

The **facial submanifolds** M_i of M are defined as the inverse images of the facet F_i under π, for $1 \leqslant i \leqslant m$. Every M_i has codim 2, with isotropy subgroup $T(F_i) \subset T^n$.
The quotient map

$$\mathscr{Z}_P \times_K \mathbb{C}_i \longrightarrow M \tag{6}$$

defines a canonical complex **line-bundle** ρ_i, whose restriction to M_i is isomorphic to the **normal bundle** ν_i of its embedding in M.

The submanifolds M_i are mutually **transverse**, and we write any non-empty intersection as

$$M_I \;=\; M_{i_1} \cap \cdots \cap M_{i_k}. \tag{7}$$

So M_I is the inverse image under π of the codim k face F_I . M_I has codim $2k$, and its **isotropy subgroup** is $T(F_I)$.

The restriction of $\rho_I = \rho_{i_1} \oplus \cdots \oplus \rho_{i_k}$ to M_I is isomorphic to the **normal bundle** ν_I of its embedding in M, for any face F_I.

The cohomology ring of a toric manifold

The bundles ρ_i are important in understanding the **integral cohomology** ring of M.

Let u_i be the first Chern class $c_1(\rho_i)$ in $H^2(M)$; then $H^*(M)$ is generated by the elements $u_1, \ldots, u_m$, modulo two sets of relations.

The first are linear, and arise from the refined form (4) of the dicharacteristic; the second are monomial, and arise from the Stanley-Reisner ideal of P.

The linear relations take the form

$$u_i = -\lambda_{i,n+1}u_{n+1} - \ldots - \lambda_{i,m}u_m \qquad (8)$$

for $1 \leqslant i \leqslant n$. So, $u_{n+1}, \ldots, u_m$ suffice to **generate** $H^*(M)$ multiplicatively.

Fixed points of subcircle actions

Given the action α of T^n on M, it is natural to study its restriction to an arbitrary subcircle $T \subset T^n$. We may decompose the **fixed point set** of T into the union of its components as

$$\text{Fix}(T) = M_{I(1)} \cup \cdots \cup M_{I(s)} \cup \cdots \cup M_{I(d)}, \quad (9)$$

Following (4), we represent T by a primitive column vector $l = l(T)$ in $\mathbb{Z}^n$.

Proposition. The components of Fix(T) are specified by those $I(s)$ for which **none** of the coefficients $\alpha_{i(s)_j}$ is **zero** in any expansion of the form

$$l(T) = \alpha_{i(s)_1}\lambda_{i(s)_1} + \cdots + \alpha_{i(s)_k}\lambda_{i(s)_k}, \quad (10)$$

for $1 \leqslant s \leqslant d$.

Stably complex, special unitary, and level-L structures

On a smooth manifold N of dimention d, a **stably complex structure** is an equivalence class of real $2k$–plane bundle isomorphisms

$$\tau(N) \oplus \mathbb{R}^{2k-d} \simeq \zeta,$$

where ζ is a fixed $GL(k,\mathbb{C})$-bundle over N and k is suitably large.
Two such isomorphisms are equivalent when they agree up to **stabilisation**.

If the first Chern class $c_1(\zeta)$ is **zero**, then the stably complex structure is **special unitary**, (or SU); and if it is **divisible** by a positive integer L, then it is **level-**L.

We identify the geometric data required to induce such structures on a toric manifold.

Note that $\mathbb{C}P^n$ is level-L for any divisor L of $(n+1)$, where $n \geqslant 1$.

Combinatorial data underlying an omnioriented toric manifold

An **omniorientation** of a toric manifold M consists of a choice of **orientation** for M and for **every** normal bundle ν_i.
An interior point of the quotient polytope P admits an open neighborhood U, whose inverse image under the projection π is canonically diffeomorphic to $T^n \times U$ as a subspace of M.

Since T^n is oriented by the standard choice of basis, orientations of M correspond bijectively to orientations of P.
Moreover, the dicharacteristic ℓ determines a complex structure on every ρ_i, so it encodes an orientation for every ν_i.

Every pair (P, Λ) therefore determines a $2n$–dim **omnioriented toric manifold**, where P is the combinatorial type of an oriented finely ordered n–dim simple polytope, and Λ is a matrix of the form (4).

Stably complex structures

Theorem. Any omnioriented toric manifold admits a **canonical** stably complex structure, which is **invariant** under the T^n-action.

Proof. Using the theory of **analogous polytopes** we obtained an embedding

$$i_P : P \longrightarrow \mathbb{R}^m_\geqslant$$

which respects facial codimensions and gives a pullback diagram

$$\begin{array}{ccc} \mathscr{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ {\scriptstyle\varrho_P}\Big\downarrow & & \Big\downarrow{\scriptstyle\varrho} \\ P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant \end{array} \tag{11}$$

of identification spaces.
Here $\varrho(z_1, \ldots, z_m)$ is given by $(|z_1|^2, \ldots, |z_m|^2)$, the vertical maps are projections onto the quotients by the T^m-actions, and i_Z is a T^m-equivariant **embedding**.

So, there is a K-**equivariant** decomposition

$$\tau(\mathscr{Z}_P) \oplus \nu(i_Z) \simeq \mathscr{Z}_P \times \mathbb{C}^m,$$

obtained by restricting the tangent bundle $\tau(\mathbb{C}^m)$ to $\mathscr{Z}_P$. Factoring out K yields

$$\tau(M) \oplus (\xi/K) \oplus (\nu(i_Z)/K) \simeq \mathscr{Z}_P \times_K \mathbb{C}^m, \quad (12)$$

where ξ denotes the $(m-n)$–plane bundle of tangents along the fibres of

$$\pi_\ell : \mathscr{Z}_P \longrightarrow M.$$

The right-hand side of (12) is isomorphic to $\bigoplus_{i=1}^{m} \rho_i$ as $GL(m, \mathbb{C})$-bundles.
This is all an example of Szczarba's Theorem.

The embedding $i_{\mathbb{Z}} : \mathscr{Z}_P \longrightarrow C^m \simeq R^{2m}$ is T^m-equivariantly framed, so $\nu(i_{\mathbb{Z}})/K$ is trivial.
The bundle ξ/K canonically isomorphic to the adjoint bundle of the principal K-bundle, which is trivial, because K is an abelian group.

So, (12) reduces to an isomorphism

$$\tau(M) \oplus \mathbb{R}^{2(m-n)} \simeq \rho_1 \oplus \ldots \oplus \rho_m,$$

although different choices of trivialisations may lead to different isomorphisms.

Since M is connected and $GL(2(m-n), \mathbb{R})$ has two connected components, such isomorphisms are equivalent when and only when the induced orientations agree on $\mathbb{R}^{2(m-n)}$.

We choose the orientation which is compatible with those on $\tau(M)$ and $\rho_1 \oplus \ldots \oplus \rho_m$, as given by the omniorientation.

The induced structure is **invariant** under the action of T^n, because i_Z is T^m-equivariant.

SU, and level-L structures

Corollary. The omniorientation induces an SU-**structure** on M precisely when the refined matrix Λ of (4) has every column-sum **equal to** 1;
it induces a **level**-L **structure** when every column-sum is **congruent to** 1 mod L.

Proof. The stably complex structure induced by the omniorientation has first Chern class $\sum_{i=1}^{m} u_i$. It is zero in $H^2(M)$ if and only if

$$\Big(1-\sum_{i=1}^{n} \lambda_{i,n+1}\Big)u_{n+1}+\cdots+\Big(1-\sum_{i=1}^{n} \lambda_{i,m}\Big)u_m \;=\; 0.$$

The same argument shows that it is divisible by L if and only if every column-sum is congruent to 1 mod L.

Complex Cobordism

Complex cobordism functor $U^*(X)$ is a generalized cohomology theory with dual elements (bordism classes) $u \in U_k(X)$ represented by maps $f : M^k \to X$ of closed U-manifolds.
The U-structure on M^k is given by the complex structure in the stable normal bundle of some embedding $M^k \subset \mathbb{R}^{2N+k}$.
The operation of **intersection** of bordism classes is dual to the **product** of cobordism classes.

$U^*(X)$ is a commutative and associative $\mathbb{Z}$-graded ring with a unit $1 \in U^*(pt) = \Omega_U^*$.
By the Milnor and Novikov theorem

$$\Omega_U^* = \mathbb{Z}[a_2, a_4, a_6, \ldots], \quad \deg a_k = -2k.$$

The problem of choosing the appropriate generators for the ring Ω_U^* is very important in the cobordism theory and its applications.

The standard set of multiplicative generators for Ω_U^* is constructed using the projective spaces $\mathbb{C}P^i$, $i \geqslant 0$, and **Milnor hypersurfaces** $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$, $1 \leqslant i \leqslant j$.
The hypersurface $H_{i,j}$ is defined by

$$H_{i,j} = \Big\{ z \in \mathbb{C}P^i, w \in \mathbb{C}P^j \mid \sum_{q=0}^{i} z_q w_q = 0 \Big\}.$$

Note, that $H_{i,j}$ is **not** a toric manifold if $i > 1$.

Theorem. An alternative set of multiplicative generators for Ω_U^* is toric manifolds $\{B_{i,j}\}$.

Application to complex cobordism

Theorem. Every complex cobordism class is represented by a **disjoint** union of omnioriented toric manifolds, which are suitably oriented products of the $B_{i,j}$.

Our modification of **connected sum** of toric manifolds gives:

Theorem. In dimensions > 2, every complex cobordism class contains a toric manifold, necessarily **connected**, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.

Similar methods combine with Diagram(11) to give:

Theorem. Every complex cobordism class may be represented by the quotient of a free torus action on a real quadratic complete intersection.

Davis and Januszkiewicz's space BP

Let BP be the Borel construction $ET^n \times_{T^n} M^{2n}$. We have the fibration

$$p : BP \xrightarrow{M^{2n}} BT^n$$

and the following results:

$$U^*(BT^n) = \Omega_U[[v_1, \ldots, v_n]];$$

$$U^*(BP) = \Omega_U[[u_1, \ldots, u_m]]/I,$$

where I is Stanley-Reisner ideal of P;

$U^*(BP)$ is a free $U^*(BT^n)$-module with generators $u_{n+1}, \ldots, u_m$, where $u_k = c_1^U(\rho_k)$ and $p^*v_i = u_i + \lambda_{i,n+1}u_{n+1} + \cdots + \lambda_{i,m}u_m$.

The universal index for a toric manifold

$$\mathfrak{I}(M^{2n}) = [p : BP \longrightarrow BT^n] \in U^{-2n}(BT^n).$$

We have

$$\mathfrak{I}(M^{2n}) = p_!(1),$$

where

$$p_! : U^*(BP) \longrightarrow U^*(BT^n)$$

is the Gysin homomorphism and

$$\varepsilon^*\mathfrak{I}(M^{2n}) = \mathfrak{I}(M^{2n})(0) = [M^{2n}] \in \Omega_U^{-2n},$$

where $\varepsilon^* : U^*(BT^n) \to \Omega_U^*$ induced by $\varepsilon : (pt) \to BT^n$.

The expression for Gysin homomorphism in terms of isolated fixed points of T^n-action on M^{2n} gives a universal formula for $\mathfrak{I}(M^{2n})$ (especially for$[M^{2n}]$) in terms of combinatorial data (P, Λ).

Krichever's results

Let F_s be a connected component of the set $\{F_s\}$ of fixed points under the action of S^1 on a stably complex S^1-manifold M.

Suppose that the representation of S^1 in the normal bundle to F_s is given by $\sum\limits_i \eta^{j_{s,i}}$. Then if $c_1(M) = 0$, all the sums $r_s = \sum\limits_i j_{s,i}$ are equal.
The resulting integer is called the **type** of the circle action on the SU-manifold M.

Theorem. If the action of S^1 on any SU-manifold M has nonzero type, then

$$T_{ell_*}([M]) = 0,$$

where T_{ell_*} is Krichever's **generalized elliptic genus**.

The generalized elliptic genus of a toric manifold

Let M^{2n} be a toric SU-manifold and $T \subset T^n$ be an arbitrary subcircle. So

$$l(T) = (z_1, \dots, z_n) \in \mathbb{Z}^n.$$

Proposition. The action of T on M^{2n} has type $\sum_{i=1}^{n} z_i$.

Corollary. The generalized elliptic genus T_{ell_*} of any toric SU-manifold M^{2n} is zero.

Corollary. If M^{2n} is any toric SU-manifold, where $n < 5$, then $[M^{2n}] = 0$.

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