Equivariant intersection cohomology of toric varieties and applications

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GIT construction of toric varieties

Apply $\mathrm{Hom}(\,-\,,\mathbb{C}^*)$ to an exact sequence

$$0 \to \mathbb{Z}^d \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^{n-d} \to 0$$

to get exact sequence of tori:

$$0 \to T^{n-d} \to T^n \to T^d \to 0.$$

A choice $\beta \in \mathbb{Z}^n$ gives rise to a toric variety

$$X := \mathbb{C}^n /\!/_\beta T^{n-d}.$$

The residual torus $T = T^d$ acts with finitely many orbits.

X is projective over an affine base.

The moment polyhedron

Let $v_1, \ldots, v_n \in \mathbb{Z}^d$ be the rows of matrix B.
The polyhedron

$$P = \{x \in \mathbb{R}^d \,|\, \langle x, v_j \rangle \geq -\beta_j, j = 1, \ldots, n\}$$

is the image of the moment map

$$\mu: X \to \mathbb{R}^d = \mathrm{Lie}(T^d_{\mathbb{R}})^*.$$

The geometry of X can be read off from P.
Let $[P] =$ poset of faces of P.
μ gives a bijection $F \mapsto O_F$

$$[P] \longleftrightarrow \{T - \mathrm{orbits\,of}\, X\}$$

- X is an orbifold $\iff$ P is simple
- X is complete $\iff$ P is bounded
- $\beta = 0 \implies P$ is a cone $\implies$ X is affine

Equivariant cohomology

Assume P is simple, $\mathrm{int}(P) \neq \emptyset$ (so $\dim X_P = d$).

$\rightrightarrows$ We take all cohomology with $\mathbb{R}$ coefficients.
There is a canonical identification

$$A_F := H_T^\bullet(O_F) = \mathrm{Sym}(\mathbb{R}^d / \mathrm{lin.span}\, F).$$

Restricting to orbits gives an identification of elements of $H_T^\bullet(X)$ with tuples $(\alpha_F)_{F \in [P]}$, $\alpha_F \in A_F$, so that if $F \subset F'$, then $\alpha_F \mapsto \alpha_{F'}$ under the quotient $A_F \to A_{F'}$.

This in turn is identified with the "face ring" of the dual simplicial complex: if $F_1, \dots, F_k$ are the facets of P, then

$$H_T^\bullet(X) = \mathbb{R}[e_1, \dots, e_k] / I,$$

$I = \langle e_{i_1} \cdots e_{i_r} \mid F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset \rangle$.

Here e_i comes from parallel translation of the facet F_i:

Formality

All equivariant cohomology groups are modules over

$$A := H_T^\bullet(\mathrm{pt}) = \mathrm{Sym}(\mathbb{R}^d).$$

The map $H_T^\bullet(\mathrm{pt}) \to H_T^\bullet(X)$ sends $\alpha \in A$ to (α_F), where $\alpha_F =$ image of α under the quotient $S \to A_F$.

For any graded A-module, M, put $\overline{M} := M \otimes_A \mathbb{R}$.

The toric variety X is equivariantly formal:

- $H_T^\bullet(X)$ is a free A-module
- $H^\bullet(X) = \overline{H_T^\bullet(X)}$ canonically, as rings

h and g-polynomials

Let $f_i = $ #faces of P of codimension i
[Warning: this is the opposite of the usual convention]

Define:

$$h(P,t) = \sum_{i=0}^{d} h_i(P)t^i = \sum_{i=0}^{d} f_i t^i (1-t)^{d-i}$$

Then $h_i(P) = \dim_{\mathbb{R}} H^{2i}(X_P)$.

If P is a bounded simple polytope, define

$$g(P,t) = \sum_{i=0}^{[d/2]} g_i(P)t^i, \; g_i = h_i - h_{i-1}$$

(starting with $g_0 = 1$).

The g-theorem

Theorem. *[Stanley, McMullen, Billera-Lee]*
There exists a simple polytope (bounded polyhedron) with face numbers (f_i) if and only if

1. $h_i = h_{d-i}$ *for* $0 \leqslant i \leqslant d$
2. $1 = h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{[d/2]}$
3. $(g_0, \ldots, g_{[d/2]})$ *is an "M-sequence": equivalently, there is a graded algebra* $H = \oplus H_k$*, generated by* H_1*, with* $g_k = \dim H_k$

Necessity of these conditions follows (when P is rational) from facts about the cohomology of X_P:

Poincaré duality $\Rightarrow$ 1. (also has purely combinatorial proof)

Hard Lefschetz $\Rightarrow$ 2,3: if $\lambda \in H^2(X_P)$ is ample, the ring $H^\bullet(X_P)/\langle\lambda\rangle$ has Betti numbers (g_k).

Note: a simple polyhedron can be deformed to be rational without changing its combinatorial type.

McMullen found a non-toric proof of Hard Lefschetz, proving necessity without a deformation argument.

Intersection cohomology

If P is not simple, X_P has worse than orbifold singularities, and the (equivariant) intersection cohomology $IH^\bullet(X_P)$, $IH_T^\bullet(X_P)$ is a better invariant: satisfies

- Poincaré duality (for X_P compact)
- Hard Lefschetz
- no ring structure, but it's module over $H^\bullet(X)$, $H_T^\bullet(X)$ respectively
- parity vanishing and purity, which implies
- Formality: $IH_T^\bullet(X_P)$ is a free A-module, and

$$IH^\bullet(X_P) = \overline{IH_T^\bullet(X_P)}$$

Stanley's "toric" g and h-polynomials, give the IH generalizations of the formulas for simple polyhedra: in particular $h_k(P) = \dim IH^{2k}(X_P)$.

If P is a bounded polytope, $g(P,\, t)$ is obtained from $h(P, t)$ exactly as in the simple case, and

$$h(P,t) = \sum_{F \in [P]} t^{\operatorname{codim} F}(1-t)^{\dim F} g(P/F, t^{-1}),$$

where P/F is the polytope which gives the "transverse" behavior of P at F.

If there are exactly $\operatorname{codim} F$ facets of P containing F, then P/F is a simplex, and $g(P/F, t) = 1$, so when P is simple this agrees with the old formula.

If P is a cone over a bounded polytope Q, then

$$h_k(P) = g_k(Q) = \dim IH^{2k}(X_P).$$

Combinatorial IH for toric varieties

(Bressler-Lunts,Barthel-Brasselet-Fieseler-Kaup,Karu)

Let $[P]$ = lattice of faces of P, topologized by

$$\overline{\{F\}} = \{E \in [P] \mid E \subset F\}.$$

The "structure sheaf" on $[P]$ is $U \mapsto \mathcal{A}(U)$, where

$$\mathcal{A}(U) = \{(\alpha_F)_{F \in U} \mid \alpha_E \mapsto \alpha_F \text{ if } E \subset F\}$$

A *minimal extension sheaf* on $[P]$ is a sheaf $\mathcal{L}$ of $\mathcal{A}$-modules satisfying:

1. Every stalk $\mathcal{L}_F$ is a free A_F-module
2. $\mathcal{L}$ is flabby: sections on an open set U extend to all of $[P]$
3. The "generic" stalk $\mathcal{L}_P$ is $A_P = A$.
4. $\mathcal{L}$ is minimal with respect to 1–3.

If P is simple, then $\mathcal{L} = \mathcal{A}$.
If P is rational, then $\mathcal{L}([P]) = IH_T^\bullet(X_P)$ *canonically*.
More generally $\mathcal{L}(U) = IH_T^\bullet(X_U)$,
where $X_U = \bigcup_{F \in U} O_F$.

For P a bounded polytope, not necessarily rational, Karu showed that Hard Lefschetz still holds for $\mathcal{L}([P])$. This implies that its graded rank is the toric h-polynomial of P, even though no toric variety exists!

Karu's proof also implies that $\mathcal{L}$ is rigid (has only scalar automorphisms).

More generally, build a minimal extension sheaf $\mathcal{L}^F$ supported on $\overline{F}$, starting with $(\mathcal{L}^F)_F = A_F$. Direct sums of shifts of these sheaves are called “pure”; they are exactly the sheaves which are flabby and pointwise free, with no minimality restriction.

Using complexes of pure sheaves we can model T-equivariant (mixed) sheaves on X [B—,Lunts]: there is an equivalence of categories

$$D^b(\mathcal{A} - \mathrm{mod}) \cong K^b(\mathrm{Pure}(\mathcal{A} - \mathrm{mod})),$$

and there is a functor $D^b(\mathcal{A} - \mathrm{mod}) \to D^b_T(X)$ to the topological equivariant derived category, which acts like a forgetful functor from (complexes of) graded modules to (complexes of) ungraded modules.

Restriction and Kalai's monotonicity

Restricting an IH sheaf to a fixed point set of an attracting action of $\mathbb{C}^*$ yields a sum of IH sheaves. The corresponding fact for combinatorial IH is:

Theorem. *The restriction of the minimal extension sheaf $\mathcal{L}$ to the closure of a face F is pure.*

Applying this to a cone over a polytope P gives:

$$g_k(P) \geqslant \sum_{i+j=k} g_i(F) g_j(P/F).$$

This was originally conjectured by Kalai, and first proved when P is rational by [B—,MacPherson '99], using restriction of IH sheaves on toric varieties.

Corollary. *[Kalai] If $g_k(P)=0$, then $g_{k+1}(P)=0$.*

Question: do the toric g-numbers form an M-sequence?

For fixed point sets of $\mathbb{C}^*$-actions which are neither attracting or repelling, there is a "hyperbolic" localization with mixed supports which preserves purity [B—,'03]. The corresponding combinatorial result is:

Theorem. *Let H be a hyperplane intersecting the bounded polytope P. Then*

$$\sum_{i \geqslant 0} g_i(P) \geqslant \sum_{\substack{i,j \geqslant 0 \\ F \in \mathcal{S}}} g_i(F) g_j(P/F)$$

Where $\mathcal{S}$=maximal elements in $\{F \mid F \subset H\}$.

Again this can be proved without rationality hypotheses, using minimal extension sheaves.

The Koszul resolution

Non-pure sheaves can be studied by resolving them by pure sheaves. Let $\mathcal{M}$ be the "extension by zero" of A_P:

$$\mathcal{M}(U) = \begin{cases} A_P, & U = \{P\} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem. *$\mathcal{M}$ has a unique minimal resolution*

$$0 \to \mathcal{M} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^d \to 0$$

with the $\mathcal{F}^i$ pure. It has only scalar automorphisms.

The multiplicity of $\mathcal{L}^F$ in $\mathcal{F}^i$ is $g_k((P/F)^\vee)$, where $2k = \operatorname{codim} F - i$, and $(P/F)^\vee$ is the dual polytope to P/F.

The multiplicity statement follows from applying an identity of Stanley:

$$\sum_F (-1)^{\dim F} g(F,t) g((P/F)^\vee, t) = 0,$$

to the stalk Euler characteristics of the complex $\mathcal{F}^\bullet$.

Truncating the complex leads to inequalities involving partial sums of terms from Stanley's identity; it is not known if any of them are new.

Koszul Duality

Toric Koszul duality [B—,Lunts '04] relates perverse sheaves on affine toric varieties defined by dual cones. It can be used to lift the statement about the multiplicities of simple sheaves in $\mathcal{F}^\bullet$ to give canonical isomorphisms of vector spaces.

In particular, if P and $P^\vee$ are dual rational convex cones of dimension $d = 2k+1$, then there is a dual pairing

$$IH^{2k}(X_P) \otimes IH^{2k}(X_{P^\vee}) \to \mathbb{R},$$

which is canonical after fixing an orientation of P.

Hypertoric varieties

With the same exact sequence of tori:

$$0 \to T^{n-d} \to T^n \to T^d \to 0$$

we get an action of T^{n-d} on $T^*\mathbb{C}^n = \mathbb{C}^n \times (\mathbb{C}^n)^*$, and a corresponding Hyperkähler quotient

$$Y = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_\beta T^{n-d},$$

of dimension $2d$, depending on $\beta \in \mathbb{Z}^n$.

Its topology is governed by the arrangement $\mathcal{H}$ of hyperplanes $H_i = \{x \in \mathbb{R}^d \mid \langle x, v_j \rangle = -\beta_j\}$.

- Y is an orbifold $\Leftrightarrow \mathcal{H}$ is a simple arrangement
  Y is smooth $\Leftrightarrow \mathcal{H}$ is simple and unimodular
- The toric varieties $\{X_P \mid P$ is a chamber of $\mathcal{H}\}$ are d-dimensional subvarieties of Y.
- Y has a stratification indexed by $[\mathcal{H}]$, the poset of flats of the arrangement $\mathcal{H}$.

Define a sheaf of rings on $[\mathcal{H}]$ exactly as for polyhedra: for $F \in [\mathcal{H}]$, put

$$A_F = \mathrm{Sym}(\mathbb{R}^d/\mathrm{lin.span}\, F),$$

and define

$$\mathcal{A}(U) = \{(\alpha_F)_{F \in U} \mid \alpha_E \mapsto \alpha_F \text{ if } E \subset F\}.$$

If $\mathcal{H}$ is simple, then $H_T^\bullet(Y_\mathcal{H}) = \mathcal{A}([\mathcal{H}])$, which is the face ring of the dual simplicial complex (the "independence complex" of the corresponding matroid).

Formality holds, so we get a presentation for cohomology: $H^\bullet(Y_\mathcal{H}) = \overline{\mathcal{A}([\mathcal{H}])}$, originally due to Konno and Hausel-Sturmfels.

Thus the Betti numbers of $Y_\mathcal{H}$ for $\mathcal{H}$ simple are the h-numbers of the independence complex.

Equivariant IH of hypertoric varieties

(work in progress with N. Proudfoot)

Take a nonsimple arrangement $\mathcal{H}$.

Define a minimal extension sheaf in exactly the same way as before: $\mathcal{L}$ is a sheaf of $\mathcal{A}$-modules on $[\mathcal{H}]$ satisfying:

1. Every stalk $\mathcal{L}_F$ is a free A_F-module
2. $\mathcal{L}$ is flabby: sections on an open set U extend to all of $[P]$
3. The "generic" stalk $\mathcal{L}_P$ is $A_P = A$.
4. $\mathcal{L}$ is minimal with respect to 1–3.

Theorem. *The sheaf $\mathcal{L}$ exists, and is unique up to a unique isomorphism. The global sections are canonically identified with $IH^\bullet_T(Y_\mathcal{H})$.*

In fact, the sheaf $\mathcal{L}$ makes sense, and has the expected Betti numbers, even if the arrangement $\mathcal{L}$ is not rational.

A ring structure

The IH Betti numbers of $Y_{\mathcal{H}}$ are the h-numbers of the *broken circuit complex,* a subcomplex of the independence complex [Proudfoot-Webster '04]. It depends on a choice of ordering of the vectors v_i, but its h-numbers do not.

Proudfoot and Speyer described a ring $R_{\mathcal{H}}$, not involving choices, which degenerates to the face ring of the broken circuit complex for any choice of ordering.

Theorem. *There is a canonical identification*

$$R_{\mathcal{H}} = \mathcal{L}([\mathcal{H}]),$$

and thus a canonical ring structure on $IH^{\bullet}_{T}(Y_{\mathcal{H}})$.

The proof goes by showing that there are natural ring homomorphisms making $F \mapsto R_{\mathcal{H}_F}$ into a minimal extension sheaf.

When $\mathcal{H}$ is unimodular, we have conditions which uniquely characterizing this ring structure in terms of the geometry of $Y_{\mathcal{H}}$.

Gale duality and IH

Let $\mathcal{H}$ be a central arrangement of n hyperplanes in $\mathbb{R}^d$, defined over $\mathbb{Q}$.

The Gale dual arrangement $\mathcal{H}^\vee$ is the arrangment of n hyperplanes in $\mathbb{R}^{n-d}$ defined by taking the dual of the defining exact sequence:

$$0 \to \mathbb{Z}^d \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^{n-d} \to 0.$$

Let $\tilde{\mathcal{H}}$ be an arrangement with the same defining matrices as $\mathcal{H}$, but with β chosen so that $\tilde{\mathcal{H}}$ is simple. There is a natural map $Y_{\tilde{\mathcal{H}}} \to Y_{\mathcal{H}}$, which is a semi-small (orbifold) resolution of singularities.

Theorem. *There is a canonical dual pairing*

$$H^{2d}(Y_{\tilde{\mathcal{H}}}) \otimes IH^{2d}(Y_{\mathcal{H}^\vee}, Y^+) \to \mathbb{R},$$

where $Y^+ = \{x \in Y_{\mathcal{H}^\vee} \mid \lim_{t\to 0} \rho(t)x = 0\}$*, for any cocharacter* $\rho\colon \mathbb{C}^* \to T^\vee$ *with* $(Y_{\mathcal{H}^\vee})^{\rho(\mathbb{C}^*)} = (Y_{\mathcal{H}^\vee})^T$.

Remarks:
$IH^\bullet(Y_{\mathcal{H}^\vee}, Y^+)$ is halfway between closed and compact supports. It is nonzero only in degree $2d$.

All these terms can be defined purely in terms of the arrangement, and the theorem remains true even for non-rational arrangements.

The Betti numbers of $Y_{\tilde{\mathcal{H}}}$ and the IH Betti numbers of $Y_{\mathcal{H}}$ can both be obtained by specializations of the Tutte polynomial $T_{\mathcal{H}}(x, y)$. The fact that the spaces in the theorem have the same dimension follows from the identity $T_{\mathcal{H}^\vee}(x, y) = T_{\mathcal{H}}(y, x)$.