Almost homogeneous toric varieties

Ivan Arzhantsev

Moscow State University

based on a joint work with Jürgen Hausen

"On embeddings of homogeneous spaces with small boundary"

1. Automorphisms of toric varieties

M.Demazure (1970) – smooth complete toric varieties (roots of the fan);

D.Cox (1995) – simplicial complete toric varieties (graded automorphisms of the homogeneous coordinate ring);

D.Bühler (1996) – complete toric varieties (a generalization of Cox's results);

W.Bruns - J.Gubeladze (1999) – projective toric varieties in any characteristic (graded automorphisms of semigroup rings, polytopal linear groups).

Let G be a connected simply connected semisimple algebraic group over $\mathbb{C}$, e.g., $G = SL(n), Sp(2n), Spin(n), \ldots$.

Problem 1. Describe all toric varieties X such that $G \to Aut(X)$ and $G : X$ with an open orbit Gx (1-condition).

Problem 2. Solve Problem 1 under the assumption $codim_X(X \setminus Gx) \geqslant 2$ (2-condition).

Examples. $SL(n+1) : \mathbb{P}^n$, $SL(n+1) : \mathbb{P}^n \times \mathbb{P}^n$ $(n > 1)$.

2. Cox's construction

For a given toric variety X with $Cl(X) \cong \mathbb{Z}^r$ there exist an open $U \subset V = \mathbb{C}^N$ and $T \subset \mathbb{S} = (\mathbb{C}^\times)^N : V$ such that $\exists$ a good quotient $\pi : U \to U//T \cong X$ and $(\mathbb{S}/T) : X$

[A good quotient of a T-invariant open subset $U \subset V$ is an affine T-invariant morphism $\pi : U \to Z$ such that the pullback map $\pi^* : \mathcal{O}_Z \to \pi_*(\mathcal{O}_U)^T$ to the sheaf of invariants is an isomorphism]

Moreover, $\exists$ an open $W \subset U$ with $codim_V(V \setminus W) \geqslant 2$ and $\pi^{-1}(\pi(w))$ is a T-orbit for any $w \in W$ (in particular, one may assume that $\pi^{-1}(X^{reg}) \subset W$)

$W = U$ (i.e., π is a geometric quotient) iff X is simplicial

(V, T, U) is the Cox realization of X

Lemma. If V is a T-module, $U \subset V$ admits $\pi : U \to U//T$, and $\exists\ W \subset U$: $codim_V(V \setminus W) \geqslant 2$, $\pi^{-1}(\pi(w))$ is a T-orbit for any $w \in W$, then (V, T, U) is the Cox realization of $X := U//T$.

Claim 1) $G : X \Rightarrow (G \times T) : V$;

2) 1-condition $\Leftrightarrow$ $(G\times T) : V$ with an open orbit (and linearly, [H.Kraft-V.Popov'85]); in this case V is a prehomogeneous vector space;

3) 2-condition $\Leftrightarrow$ $G : V$ with an open orbit.

Example. $SL(n+1) : \mathbb{P}^n \times \mathbb{P}^n \Rightarrow (SL(n+1) \times (\mathbb{C}^\times)^2) : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ with an open orbit;
$SL(n+1) : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ with an open orbit $\Leftrightarrow$ $n > 1$.

Idea: Given a prehomogeneous G-module $V = V_1 \oplus \cdots \oplus V_r$, let $\mathbb{T} = (\mathbb{C}^\times)^r : V$, $(t_1, \dots, t_r) * (v_1, \dots, v_r) = (t_1 v_1, \dots, t_r v_r)$, and $T \subset \mathbb{T}$ be a subtorus. We shall obtain all X as $U//T$ for a $(G \times T)$-invariant open $U \subset V$, and give a combinatorial description of such U.

3. Quotients of torus actions

T is a torus, $\Xi(T)$ - the character lattice, $\Xi(T)_{\mathbb{Q}} = \Xi(T) \otimes_{\mathbb{Z}} \mathbb{Q}$

$T : V = \bigoplus_{i=1}^{r} V_{\chi_i}$, where $\chi_1, \dots, \chi_r \in \Xi(T)$, and
$V_\chi = \{v \in V \mid t * v = \chi(t)v\}$

open T-invariant $U \subset V$ is a good T-set if it admits a good quotient
$\pi : U \to U//T$

$W \subset U$ is saturated if $\exists\ Y \subset U//T$ such that $W = \pi^{-1}(Y)$

$U \subset V$ is maximal if it is a good T-set maximal with respect to open saturated inclusions

$$\Omega(V) = \{Cone(\chi_{i_1}, \ldots, \chi_{i_p}) \subset \Xi(T)_{\mathbb{Q}} \ \mid \ \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, r\} \ \}$$

a collection of cones $\Psi \subset \Omega(V)$ is connected if $\forall \tau_1, \tau_2 \in \Psi : \ \tau_1^0 \cap \tau_2^0 \neq \emptyset$

Ψ is maximal if it is connected and is not a proper subcollection of a connected collection

$$v = v_{\chi_{i_1}} + \cdots + v_{\chi_{i_p}} \in V, \ v_{\chi_j} \neq 0 \Rightarrow \omega(v) = Cone(\chi_{i_1}, \ldots, \chi_{i_p})$$

$$\Psi \Rightarrow U(\Psi) = \{v \in V \mid \exists \ \omega_0 \in \Psi : \omega_0 \preceq \omega(v)\} \subset V$$

$$U \subset V \Rightarrow \Psi(U) = \{\omega(v) \mid v \in U, \ Tv \text{ is closed in } U\}$$

Theorem $\{$ maximal $U \subseteq V\} \Leftrightarrow \{$ maximal $\Psi \subset \Omega(V)\}$

[A. Białynicki-Birula - J. Święcicka (1998)]

Which maximal Ψ defines quasiprojective $U(\Psi)//T$?

$$\xi \in \Xi(T) \Rightarrow \tau = \tau(\xi) = \bigcap_{\xi \in \omega(v)} \omega(v) \Rightarrow \Psi(\tau) = \{\omega \in \Omega(V) \mid \tau^0 \subset \omega^0\}$$

the cones $\tau(\xi)$ form a fan (GIT-fan) with support $Cone(\chi_1, \ldots, \chi_r)$

Claim. $U(\Psi)//T$ is quasiprojective iff $\Psi = \Psi(\tau)$ for some $\tau = \tau(\xi)$

$U(\Psi(\tau))//T$ is projective $\Leftrightarrow Cone(\chi_1, \ldots, \chi_r)$ is strictly convex and all χ_i are non-zero

Ψ is interior if $Cone(\chi_1, \ldots, \chi_r) \in \Psi \Leftrightarrow$ generic fibers of π are T-orbits

$U(\Psi(\xi))//T$ is affine $\Leftarrow Cone(\chi_1, \ldots, \chi_r) = \Xi(T)_{\mathbb{Q}}$ and $\xi = 0$

a toric variety X is 2-complete if $X \subset X'$ with $codim_{X'}(X' \setminus X) \geqslant 2$ implies $X = X'$. Examples: complete, affine

X is 2-complete iff the fan of X cannot be enlarged without adding new rays

for an interior maximal collection Ψ the variety $U(\Psi)//T$ is 2-complete and for a 2-complete variety X the subset U in the Cox realization is maximal in V

4. A description of 2-complete toric G-varieties

Ingradients:

- prehomogeneous $(G \times \mathbb{T})$-module $V = V_1 \oplus \cdots \oplus V_r$, $\dim V_i \geqslant 2$;
- a subtorus $T \subset \mathbb{T} \Leftrightarrow$ a primitive sublattice $S_T \subset \mathbb{Z}^r = \Xi(\mathbb{T})$;
- the standard basis $e_1, \ldots, e_r$ of $\mathbb{Z}^r$ and the projection $\phi : \mathbb{Z}^r \to \mathbb{Z}^r / S_T \cong \Xi(T)$;
- $\Omega = \{Cone(\phi(e_{i_1}), \ldots, \phi(e_{i_p})) \mid \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, r\}\}$;
- an interior maximal collection $\Psi \subset \Omega$

V is G-prehomogeneous $\Leftrightarrow X = U(\Psi)//T$ satisfies 2-condition

V is $(G \times T)$-prehomogeneous $\Leftrightarrow X = U(\Psi)//T$ satisfies 1-condition

Interior maximal collections $\Leftrightarrow$ Bunches of cones in the divisor class group (F. Berchtold - J. Hausen'2004)
$\Rightarrow$ Fans (via a linear Gale transformation)

5. Examples

1) $V = V_1 \Rightarrow$ a) $T = \mathbb{T} = \mathbb{C}^\times$, $\phi(e_1) = e_1$, $\Psi = \{\mathbb{Q}_+\}$,

$$U(\Psi) = V \setminus \{0\},\ X = \mathbb{P}(V);$$

b) $T = \{e\}$, $\phi(e_1) = 0$, $\Psi = \{0\}$, $U(\Psi) = X = V$

2) $V = V_1 \oplus V_2 \ \Rightarrow$ a) $T = \mathbb{T} = (\mathbb{C}^\times)^2$, $X = \mathbb{P}(V_1) \times \mathbb{P}(V_2)$;

b) $T = \{e\}$, $X = V$;

c) $\dim T = 1$, here we consider only some particular cases:

(1) $S_T = \langle(1,-1)\rangle$, $t(v_1,v_2) = (tv_1, tv_2) \Rightarrow \phi(e_1) = \phi(e_2) = 1$, $\Psi = \{\mathbb{Q}_+\}$, $U(\Psi) = V \setminus \{0\}$, $X = \mathbb{P}(V)$;

(2) $S_T = \langle(0,1)\rangle$ $t(v_1,v_2) = (tv_1, v_2) \Rightarrow \phi(e_1) = 1$, $\phi(e_2) = 0$, $\Psi = \{\mathbb{Q}_+\}$, $U(\Psi) = \{(v_1,v_2) \mid v_1 \neq 0\}$, $X = \mathbb{P}(V_1) \times V_2$;

(3) $S_T = (1,1)$ $t(v_1,v_2) = (tv_1, t^{-1}v_2) \Rightarrow \phi(e_1) = 1$, $\phi(e_2) = -1 \Rightarrow$

(3.1) $\Psi = \{0, \mathbb{Q}\}$, $U(\Psi) = V$, $X = V//T$ and may be realized as the cone $C = \{v_1 \otimes v_2\}$ of decomposible tensors in $V_1 \otimes V_2$;

(3.2) $\Psi = \{\mathbb{Q}_+, \mathbb{Q}\}$, $U(\Psi) = \{(v_1,v_2) \mid v_1 \neq 0\}$, X is a "small blow-up" of C at the vertex with the exceptional fiber $\mathbb{P}(V_1)$

6. Prehomogeneous vector spaces

1) G is simple and V is G-prehomogeneous– E.B. Vinberg (1960)

- $G = SL(m)$, $\quad V = (\mathbb{C}^m)^r,\ r < m, \quad V^*$;
- $G = SL(2m+1)$, $\quad V = \bigwedge^2 \mathbb{C}^{2m+1}, \quad V^*$;
- $G = SL(2m+1)$, $\quad V = \bigwedge^2 \mathbb{C}^{2m+1} \oplus \bigwedge^2 \mathbb{C}^{2m+1}, \quad V^*$;
- $G = SL(2m+1)$, $\quad V = \bigwedge^2 \mathbb{C}^{2m+1} \oplus (\mathbb{C}^{2m+1})^*, \quad V^*$;
- $G = Sp(2m)$, $\quad V = \mathbb{C}^{2m}$;
- $G = Spin(10)$, $\quad V = \mathbb{C}^{16}$

2) V is irreducible and $(G \times \mathbb{T})$-prehomogeneous – M. Sato, T. Kimura (1977)

3) G contains $\leqslant 3$ simple factors and V is $(G \times \mathbb{T})$-prehomogeneous – T. Kimura, K. Ueda, T. Yoshigaki (1983,...)