A FUNCTION Log ON SPACE OF N-DIMENSIONAL CYCLES AND PARSHIN-KATO THEORY*

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1. One dimensional case

Consider a complex algebraic curve X and two nonzero meromorphic functions f, g on X. With each point $a \in X$ one can associate the so-called the Weil symbol $[f, g]_a$.

Let u be a local parameter about the point a, $u(a) = 0$, and let $f = c_1 u^{k_1} + \dots$ and $g = c_2 u^{k_2} + \dots$ be the leading terms of the Laurent expansion of the functions f and g at the point a.

The Weil symbol $[f, g]_a$ is the number $(-1)^{k_1 k_2} c_1^{k_2} c_2^{-k_1}$.

For all the points on the curve X except for zeros and poles of f and g, the Weil symbol is equal to one.

Weil's theorem.

$$\prod_{a \in X} [f, g]_a = 1.$$

Example (Weil's theorem and Vieta formula). Consider the Riemann sphere as the curve X, the coordinate function x as the function f, and a polynomial $P = a_n x^n + \dots + a_k x^k$ of degree n as the function g. According to Weil's theorem we get

$$\prod x(a) = [x, P]_0^{-1} [x, P]_\infty^{-1} = (-1)^{-k} a_k^{-1} (-1)^n a_n = (-1)^{n-k} a_n / a_k$$

where the product is taken over all nonzero roots a of the polynomial P. This formula coincides with the Vieta formula.

1.1. Toric Surfaces and Weil's Theorem [5]. Consider a a compact (maybe singular) toric surface M. Let D be a zero-dimensional positive divisor located on the union of one-dimensional orbits. Does an algebraic curve on the surface M exist that does not pass through zero-dimensional orbits and intersects one-dimensional orbits at the given divisor D?

Let us fix the orientation in the plane of one-parameter subgroups of $\mathbb{C}^{*2}$. Thereby we fix the parameterization $\pi_j : \mathbb{C}^* \to M_j$ on each one-dimensional orbit M_j in M. Consider a map $\pi : \bigcup_j M_j \to \mathbb{C}^*$ which restriction on M_j equals to π_j^{-1}.

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Theorem. *If $D = \sum k_i a_i$ is the divisor of intersection of a curve not passing through zero-dimensional orbits of M with the union of one-dimensional orbits of M then*

$$\prod(-\pi(a))^{k_i} = 1$$

Sketch of the proof. Each curve in $(C^*)^2$ is given by an equation $P = 0$, where P is a Laurent polynomial. Let Δ be its Newton polygons. On each side $\mathbf{n}_j$ of Δ a polynomial $P_{\mathbf{n}_j}$ in one variable is written. According to Vieta formula the product of all nonzero roots of all polynomials $P_{\mathbf{n}_j}$ is equal to 1.

The Weil's theorem follows from the previous theorem (see [5]).

FIG 1. NEWTON POLYGON FOR A LAURANT POLYNOMIAL $P(x, y)$ SUCH THAT $P(f, g) \equiv 0$

1.2. Topological Proof of the Weil's theorem. Consider a complex algebraic curve X and two nonzero meromorphic functions f and g on X. Let A be the union of the supports of the principle divisors (f) and (g) and let U be $X \setminus A$.

With the couple $\mathbf{f}, \mathbf{g}$ one can associate a certain cohomology class $[\mathbf{f}, \mathbf{g}] \in \mathbf{H^1}(\mathbf{U}, \mathbf{C}^*)$ which is:

1) skew symmetric $[f, g] = [g, f]^{-1}$,

2) multiplicative $[f_1 f_2, g] = [f_1, g][f_2, g]$

3) its value $[f, g, \gamma]$ at a loop γ running around a point $a \in A$ is equal to the Weil symbol: $[f, g, \gamma] = [f, g]_a$.

We will define the class $[f, g, \gamma]$ using so-called **Beilinson Integral**.

Notations:

$\gamma : I \to U, I = [0, 1]$ — a loop in U, $\gamma(0) = \gamma(1)$;

$\ln f$ — a continuous branch of the function $\ln f(t)$ on the interval $0 < t < 1$;

$\deg_\gamma(f)$ — a mapping degree of a map $\dfrac{f}{|f|} : \gamma \to S^1$.

Beilinson Integral against the loop $\gamma : I \to U$ for the couple of functions f, g a an element $I_\gamma(f, g)$ of the group $\mathbb{C}/\mathbb{Z}$ defined by a formula

$$I_\gamma(f, g) = \left(\frac{1}{2\pi i}\right)^2 \int_\gamma (\ln f) \frac{dg}{g} - \frac{1}{2\pi i} \deg_\gamma(f) \ln g(c).$$

Beilinson integral is:

1) invariant under orientation preserving reparameterization of the loop γ,
2) skew symmetric, i.e., $I_\gamma(f,g) = -I_\gamma(g,f)$,
3) additive, i.e., $I_\gamma(f_1 f_2, g) = I_\gamma(f_1, g) + I_\gamma(f_2, g)$,
4) if $f = g$ then $I_\gamma(f,g) = \frac{1}{2}\deg_\gamma(f)$.

One can defined Beilinson integral $I_\gamma(f,g)$ against a linear combination $\gamma = \sum k_i \gamma_i$ of loops γ_i with integer coefficients k_i.

Theorem. *Beilinson integral $I_\gamma(f,g)$ depends only on a homology class of the cycle $\gamma = \sum k_i\gamma_i$ in U and defines an element in $H^1(U, \mathbb{C}/\mathbb{Z})$.*

Corollary. *A number $[f,g,\gamma] = \exp 2\pi i I_\gamma(f,g)$ depends only on a homology class of the cycle $\gamma = \sum k_i\gamma_i$ and defines an element in $H^1(U, \mathbb{C}^*$.*

Cohomology class $[f,g,\gamma]$ is:
1) skew symmetric, i.e., $[f,g,\gamma] = [f,g,\gamma]^{-1}$,
2) multiplicative, i.e., $[f_1 f_2, g, \gamma] = [f_1, g, \gamma][f_2, g, \gamma]$,

Theorem. *Consider a small ball B_a centered at a point $a \in X$. Let γ be its boundary $\gamma = \partial B$. Then $[f,g,\gamma]$ equals to the Weil symbol $[f,g]_a$.*

Topological proof of the Weil theorem: Let A be the union of the supports of the principle divisors (f) and (g) and let U be $X \setminus A$. Let B be the union of small balls B_a centered at the points $a \in A$ and let $\gamma = \sum_{a \in A} \partial B_a$ be its boundary. Then $\gamma = 0$ in $H_1(U, \mathbb{Z})$ and $[f,g,\gamma] = \prod_{a\in A}[f,g]_a = 1$.

2. Product of roots of a system of equations with general Newton polyhedra [4]

Consider a system of equations

$$P_1 = \cdots = P_n = 0 \tag{1}$$

in $(\mathbb{C}^*)^n$ where $P_1, \ldots, P_n$ are Laurent polynomials. Let $\Delta_1, \ldots, \Delta_n$ be Newton polyhedra of $P_1, \ldots, P_n$.

Problem: Compute the product in the group $(\mathbb{C}^*)^n$ of roots of the system (1) assuming that the collection of Newton polyhedra $\Delta_1, \ldots, \Delta_n$ is generic enough.

2.1. Developed sets of polyhedra. Let $\Delta_1, \ldots, \Delta_n$ be convex polyhedra in $\mathbb{R}^n$ and let Δ be their Minkowski sum. Each face Γ of the polyhedron Δ can be uniquely represented as a sum

$$\Gamma = \Gamma_1 + \cdots + \Gamma_n$$

A collection of n polyhedra $\Delta_1, \ldots, \Delta_n$ is **developed** if for each face Γ of the polyhedron Δ among the components of its decomposition there is at least one vertex.

For a developed collection of polyhedra $\Delta_1, \ldots, \Delta_n$ a map

$$f : \partial\Delta \to \partial\mathbb{R}^n_+$$

of the boundary of $\Delta = \sum \Delta_i$ into the boundary of the positive octant is called **characteristic** if the component f_i of the map $f = (f_1, \ldots, f_n)$ vanishes on and only on the faces Γ for which the i-th component of the decomposition is a point (a vertex of the polyhedron Δ_i). The preimage of the origin under the characteristic map is precisely the set of all the vertices of the polyhedron Δ.

Definition. The **combinatorial coefficient** C_A of a vertex A of the polyhedron Δ is the local degree of the germ of the map

$$f\colon (\partial\Delta, A) \to (\partial\mathbb{R}^n_+, 0)$$

of the restriction of a characteristic map on the boundary of the polyhedron Δ.

FIG 2. COMBINATORIAL COEFFICIENT IN TWO DIMENSIONAL CASE EQUALS TO $+1, 0$ OR -1.

2.2. Parshin Symbols. Consider $n+1$ monomials $c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}$ in n complex variables, where $c_i \in \mathbb{C}^*$, $\mathbf{x} = x_1, \dots, x_n$, $\mathbf{k}_i \in (\mathbb{Z})^n, \mathbf{k}_i = k_{i,1}, \dots, k_{i,n}$, $c_i\mathbf{x}^{\mathbf{k}_i} = c_i x_1^{k_1} \cdot \dots \cdot x_n^{k_n}$.

Parshin Symbol of the sequens of monomials by definition is

$$[c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}] =$$

$$= (-1)^{D(\mathbf{k}_1,\dots,\mathbf{k}_{n+1})} c_1^{-\det(\mathbf{k}_2,\dots,\mathbf{k}_{n+1})} \dots c_{n+1}^{(-1)^{n+1}\det(\mathbf{k}_1,\dots,\mathbf{k}_n)} =$$

$$= (-1)^{D(\mathbf{k}_1,\dots,\mathbf{k}_{n+1})} \exp\left(-\det \begin{pmatrix} \ln c_1 & k_{1,1} & \dots & k_{1,n} \\ \vdots & \vdots & & \vdots \\ \ln c_{n+1} & k_{n+1,1} & \dots & k_{n+1,n} \end{pmatrix} \right),$$

where $D(\mathbf{k}_1, \dots, \mathbf{k}_{n+1}) \in \mathbb{Z}/2\mathbb{Z}$ is a special function. The function D depends only on $\pi(\mathbf{k}_1)$,..., $\pi(\mathbf{k}_{n+1}) \in (\mathbb{Z}/2\mathbb{Z})^n$ where $\pi : (\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^n$ is a natural projection. By definition D is the only nonzero multilinear function of $n+1$ vectors in the n-dimensional space over the field $\mathbb{Z}/2\mathbb{Z}$ that is invariant under all linear transformations, and is zero whenever the rank of $n+1$ vectors is less than n.

(About function D see [6].)

Example (Parshin symbol and Weil symbol). Consider two monomials $c_1x^{k_1}$, $c_2x^{k_2}$. Parshin symbol $[c_1x^{k_1}, c_2x^{k_2}]$ equals to $(-1)^{k_1k_2}c_1^{-k_2}c_1^{k_1}$. So it is equal to the Weil symbol of functions $c_1x^{k_1}, c_2x^{k_2}$ at the origin $x = 0$.

Parshin symbol is
1) skew symmetric, for example
$[c_1\mathbf{x}^{\mathbf{k}_1}, c_2\mathbf{x}^{\mathbf{k}_2}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}] = [c_2\mathbf{x}^{\mathbf{k}_2}, c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}]^{-1}$,
2) multiplicative, i.e.
$[a_1b_1\mathbf{x}^{\mathbf{l}_1+\mathbf{m}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}] = [a_1\mathbf{x}^{\mathbf{l}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}][b_1\mathbf{x}^{\mathbf{m}_1}, \dots, c_{n+1}\mathbf{x}^{\mathbf{k}_{n+1}}]$.

2.3. The Value of a Character on the Product of Roots of the System. Let $\chi_{\mathbf{k}} : (\mathbb{C}^*)^n \to \mathbb{C}^*$ be a character corresponding to a point $\mathbf{k} \in \mathbb{Z}^n$.

Definition. For every vertex A of the polyhedron $\Delta = \Delta_1 + \cdots + \Delta_n$, where Δ_i is a Newton polyhedron of Laurent polynomial P_i, we associate a number $[P_1, \dots, P_n, \chi_{\mathbf{k}}]_A \in \mathbb{C}^*$ in the following way: Let $A = A_1 + \cdots + A_n$ be the decomposition of the vertex $A \in \Delta$, $A_i \in \Delta_i$. Assume that at the vertex A_i polynomial P_i corresponds to a monomial $c_i\mathbf{x}^{\mathbf{k}_i}$. Then by definition the number $[P_1, \dots, P_n, \chi_{\mathbf{k}}]_A$ is the Parshin symbol $[c_1\mathbf{x}^{\mathbf{k}_1}, \dots, c_n\mathbf{x}^{\mathbf{k}_n}, \chi_{\mathbf{k}}]$.

Theorem [4]. *For the system of equations (1) the value of the character $\chi_{\mathbf{k}}$ on the product $M(P_1, \dots, P_n)$ of roots of the system is computed by*

$$\chi_{\mathbf{k}}(M(P_1, \dots, P_n)) = \prod_{A \in \Delta} ([P_1, \dots, P_n, \chi_{\mathbf{k}}]_A)^{(-1)^n C_A},$$

where the product is conducted over all the vertices A of the polyhedron $\Delta = \Delta_1 + \cdots + \Delta_n$, and C_A is the combinatorial coefficient at the vertex A.

3. Parshin–Kato Theory for Parshin symbols

3.1 Flags, Parameters, Parshin Symbols and Reciprocity Laws [2, 3].

Let $F = (C_0 \subset C_1 \subset \cdots \subset C_{n-1})$ be a flag of the subvarieties of X with $\dim C_k = k$ and C_k are locally irreducible about the point C_0. Definition by induction: a collection of rational functions $u_1, \dots, u_n$ is **a set of parameters about** F if:

1) restrictions of $u_1, \dots, u_{n-1}$ on C_{n-1} are well-defined

2) a set of restrictions of $u_1, \dots, u_{n-1}$ on C_{n-1} is a set of parameters for a flag $F_{n-1} = (C_0, \dots, C_{n-2})$ on C_{n-1}.

3) a principal divisor (u_n) (on the normalization $\tilde{X}$ of X) contains the subvariety C_{n-1} with the coefficient 1.

Using a set of parameters with a rational function f one can associate a **leading monomial** f_F on a flag F. Definition by induction: Let the order of the function f at C_{n-1} be k_n. Then the restriction φ of $fu_n^{-k_n}$ on C_{n-1} is well-defined. Let $\varphi_{F_{n-1}}$ be a leading monomial of φ on flag F_{n-1} with parameters $u_1, \dots, u_{n-1}$. Then by definition $f_F = \varphi_{F_{n-1}}u_n^{k_n}$.

Let $f_1, \dots, f_{n+1}$ be $(n+1)$ rational functions on an n-dimensional algebraic variety X. With a flag F one can associate the **Parshin symbol** $[f_1, \dots, f_{n+1}]_F$: by definition it is equal to the Parshin symbol of a set of leading monomials of the function with respect to a set of parameters:

$$[f_1, \dots, f_{n+1}]_F = [(f_1)_F, \dots, (f_{n+1})_F].$$

One can proof that **Parshin symbol is independent of a set of parameters** and depends only $f_1, \dots, f_{n+1}$ and on the flag F.

Example (Parshin symbol and Weil symbol). Let X be an algebraic curve, let flag F be a point a and let u be a local parameter about the point a. For a couple of rational functions f_1, f_2 Weil symbol $[f_1, f_2]_a$ is coincide with Parshin symbol $[f_1, f_2]_F$.

Fix a collection $f_1, \dots, f_{n+1}$ of rational functions on X and fix a flag $G = C_0, \dots, C_{n-1}$ on X. For each $0 \leq i < n$ denote by $\Psi^i(G)$ set of all flags $\tilde{F} = \tilde{C}_0, \dots, \tilde{C}_{n-1}$ such that $\tilde{C}_j = C_j$ if $j \neq i$ and $\tilde{C}_j$ is any j-dimensional locally irreducable subvariety.

Theorem (Parshin reciprocity laws). . *For each* $0 \leq i < n$

$$\prod_{F \in \Psi^i(G)} [f_1, \dots, f_{n+1}]_F = 1$$

.

3.2. Brilinski–McLaughrin topological version of Parshin symbols theory [1]. Let $f_1, \dots, f_{n+1}$ be $(n+1)$ meromorphic functions on an n-dimensional complex analytic space X. Denote by U the complement in X of the divisors of $f_1, \dots, f_{n+1}$ and the singular locus of X. Brilinski and McLaughlin defined a certain cohomology class $(f_1, \dots, f_{n+1}) \in H^k(U, \mathbb{C}^*)$.

Class $(f_1, \dots, f_{n+1})$ **is skew symmetric in set of functions** $f_1, \dots, f_{n+1}$ **and is multiplicative in each component**.

Let $F = (C_0 \subset C_1 \subset \cdots \subset C_{n-1})$ be a flag of the irreducible subspaces of X with $\dim C_k = k$. Brilinski and McLaughlin associated with the flag F a *flag-localized homology class* $\gamma_F \in H_k(U, \mathbb{Z})$*)*.

Theorem (Topological reciprocity laws). *For each* $0 \leq i < n$

$$\sum_{F \in \Psi^i(G)} \gamma_F = 0 \in H_n(U, \mathbb{Z})$$

.

Theorem (Topology and Parshin Symbol). *The Parshin symbol* $[f_1, \dots, f_{n+1}]_F$ *can be obtained by pairing of the cohomology class* $(f_1, \dots, f_{n+1}) \in H^n(U, \mathbb{C}^*)$ *and the flag-localized homology class* $\gamma_F \in H_n(U, \mathbb{Z})$.

4. A Multidimensional Logarithmic function and topology of Parshin symbols [7]

4.1. A Functional Log **on** n**-Dimensional Cycles.** Let $(\mathbb{C}^*)^{n+1}$ be a complex $n+1$ dimensional torus and Let ω be an $(n+1)$-form $\left(\dfrac{1}{2\pi i}\right)^{n+1} \dfrac{dx_1}{x_1} \wedge \dots \wedge \dfrac{dx_{n+1}}{x_{n+1}}$. Denote by E the following subset in $(\mathbb{C}^*)^{n+1}$: $(x_1, \dots, x_{n+1}) \in E \Leftrightarrow \{\exists j, 1 \leq j \leq n+1\} : \{x_j = 1\}$.

Consider an n-dimensional simplicial complex X. Let $\gamma = \sum k_j \Delta_j, k_j \in \mathbb{Z}, \Delta_j \subset X$ be an n dimensional cycle. Consider a smooth map $\mathbf{f} : X \to (\mathbb{C}^*)^{n+1}$, where $\mathbf{f} = f_1, \dots, f_{n+1}$.

Definition of a Log **functional.** For a smooth map $\mathbf{f} : X \to (\mathbb{C}^*)^{n+1}$ and a cycle $\gamma = \sum k_j \Delta_j$ let $\mathrm{Log}(\mathbf{f}, \gamma)$ be an element of the group $\mathbb{C}/\mathbb{Z}$ defined by a formula

$$\mathrm{Log}(\mathbf{f}, \gamma) = \int_\sigma \omega,$$

where σ is an $n+1$-dimensional chain in $(\mathbb{C}^*)^{n+1}$ such that $\partial\sigma - \mathbf{f}(\gamma) \subset E$.

The Log functional is well-defined:

1) Form ω equals to zero on E

2) Let φ_i be the argument of the coordinate function x_i. Then $\omega = d\varphi_1 \wedge \cdots \wedge d\varphi_{n+1} + d\alpha$, where α is a n form on $(\mathbb{C}^*)^{n+1}$

3) if $\partial\sigma_1 - \mathbf{f}(\gamma) \subset E$ and $\partial\sigma_2 - \mathbf{f}(\gamma) \subset E$ then $\partial\sigma_1 - \partial\sigma_2 = \partial\sigma_3$ where σ_3 is a $n+1$-chain in E. Let σ be an $n+1$ cycle $\sigma_1 - \sigma_2 - \sigma_3$. Then $\int_{\sigma_1} \omega - \int_{\sigma_2} \omega = \int_\sigma \omega = \int_\sigma d\varphi_1 \wedge \cdots \wedge d\varphi_{n+1} \in \mathbb{Z}$.

Example (ln **and** Log **functional).** Let X be a point $\{a\}$ and let γ be the point $\{a\}$ with a coefficient 1. Then for a map $f : X \to \mathbb{C}^*$ we have

$$\mathrm{Log}(f, \gamma) = \frac{1}{2\pi i} \ln f(a).$$

Example (Beilinsin integral and Log **functional).** Consider a complex algebraic curve X and two nonzero meromorphic functions f and g on X. Let A be the union of the divisors (f) and (g) and let U be $X \setminus A$. Take a loop γ in U. Then for a map $(f, g) : \gamma \to (\mathbb{C}^*)^2$ we have

$$\mathrm{Log}(f, g, \gamma) = I_\gamma(f, g)$$

Log **functional** is

1) **skew symmetric**, for example
$\mathrm{Log}(f_1, f_2, \ldots, f_{n+1}, \gamma) = -\mathrm{Log}(f_2, f_1, \ldots, f_{n+1}, \gamma)$;

2) **additive**, i.e.,
$\mathrm{Log}(\varphi_1\psi_1, f_2, \ldots, f_{n+1}, \gamma) = \mathrm{Log}(\varphi_1, f_2, \ldots, f_{n+1}, \gamma) + \mathrm{Log}(\psi_1, f_2, \ldots, f_{n+1}, \gamma)$.

Let M be a smooth real manifold and let $\mathbf{f}$ be a smooth map $\mathbf{f} : M \to (\mathbb{C}^*)^{n+1}$. With each n-dimensional cycle on can associate an element $\mathrm{Log}(\mathbf{f}, \gamma)$ in the group $\mathbb{C}/\mathbb{Z}$.

Theorem. *If the form $\mathbf{f}^*\omega$ is identically equal to zero on M, then a functional $\gamma \to \mathrm{Log}(\mathbf{f}, \gamma)$ is a cohomology class in $H^n(M, \mathbb{C}/\mathbb{Z})$.*

Main Example. Let X be an n-dimensional complex algebraic variety, $f_1, \ldots, f_{n+1}$ — rational functions on X and $M = X \setminus \cup(f_j)$. Consider a map $\mathbf{f} : M \to (\mathbb{C}^*)^{n+1}$, $\mathbf{f} = f_1, \ldots, f_{n+1}$. Then $\mathbf{f}^*\omega \equiv 0$. So a functional $\gamma \to \mathrm{Log}(\mathbf{f}, \gamma)$ is a cohomology class in $H^n(M, \mathbb{C}/\mathbb{Z})$.

4.2. Multiplicative version. Using exponential function one can define a multiplicative version of Log functional.

Functional $\exp 2\pi i \,\mathrm{Log}(\mathbf{f}, \gamma)$ takes value in $\mathbb{C}^*$. It is:

1) **skew symmetric**, for example
$\exp 2\pi i \,\mathrm{Log}(f_1, f_2, \ldots, f_{n+1}, \gamma) = (\exp 2\pi i \,\mathrm{Log}(f_2, f_1, \ldots, f_{n+1}, \gamma))^{(-1)}$;

2) **multiplicative**, i.e.

$$\exp 2\pi i \,\mathrm{Log}(\varphi_1\psi_1, f_2, \ldots, f_{n+1}, \gamma)$$
$$= \exp 2\pi i \,\mathrm{Log}(\varphi_1, f_2, \ldots, f_{n+1}, \gamma) \exp 2\pi i \,\mathrm{Log}(\psi_1, f_2, \ldots, f_{n+1}, \gamma).$$

Theorem. *If the form $\mathbf{f}^*\omega$ is identically equal to zero on M, then*

$$\gamma \to \exp 2\pi i \operatorname{Log}(\mathbf{f}, \gamma)$$

is a cohomology class in $H^n(M, \mathbb{C}^*)$.

Theorem. *Let $X \subset (\mathbb{C}^*)^n$ be a real n-dimensional torus $|y_1| = \cdots = |y_n| = 1$ and let γ be the fundamental cycle of the manifold X. Consider $n+1$ monomials $c_1\mathbf{y}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{y}^{\mathbf{k}_{n+1}}$. Let $\mathbf{f} : X \to (\mathbb{C}^*)^{n+1}$ be a restriction of the map $(c_1\mathbf{y}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{y}^{\mathbf{k}_{n+1}}) : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n+1}$ on X. Then*

$$\exp(2\pi i \operatorname{Log}(\mathbf{f}, \gamma)) = [c_1\mathbf{y}^{\mathbf{k}_1}, \dots, c_{n+1}\mathbf{y}^{\mathbf{k}_{n+1}}].$$

Theorem. *Let $f_1, \dots, f_{n+1}$ be $(n+1)$ meromorphic functions on an n-dimensional complex analytic space X. Let F be a flag on X and let $\gamma_F \in H_k(U, \mathbb{Z}))$ be the flag-localized homology class. Then*

$$\exp(2\pi i \operatorname{Log}(f_1, \dots, f_{n+1}, \gamma_F)) = [f_1, \dots, f_{n+1}]_F.$$

.

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