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Quantum method of the spectral curve

Contents

Introduction

The main focus of these lectures is of direct relevance to the two most important directions of development of geometry and topology of the 20th century, the applications of the theory of Integrable systems and applications of the ideas of quantum physics. The most visible result of the first direction is the solution of the Schottky problem [1], based on the hypothesis of S.P. Novikov. The challenge of characterizing Jacobians among other principally-polarized abelian varieties has been resolved in terms of non-linear equations: a θ-functional expression satisfies the KP equation if and only if the corresponding Abelian variaty is the Jacobian of an algebraic curve. Development of this activity was the proof of the Welters conjecture [2] on Jacobian matrices in terms of triple secant of Kummer's manifolds. The second major layer of results associated with the applications of the quantum field theory in a challenge of constructing topological invariants. Jones-Witten invariants or more broadly - quantum topological field theory generalizes the traditional invariants: Alexander polynomial and Jones polynomial. The invariants in this case are constructed as correlation functions for some quantum field theory [3]. The theory of the Donaldson invariants [4] and its development by Seiberg and Witten is another important example of applications of the quantum physics in topology.

This work is devoted to constructing quantum analogues of algebraic-geometric methods applicable in solving classical integrable systems. These methods are based on the spectral curve concept and the Abel transform. In addition to applications in the topology, explicit description of solutions for quantum integrable systems is directly linked to such problems as calculation the cohomology of the θ-divisor for Abelian varieties [6], calculation of cohomology and characteristic classes of moduli spaces of stable holomorphic bundles [7] and some generalizations [8].

In these lectures we propose a quantum analog of the spectral curve method for rational and elliptic Gaudin model [9]. These cases correspond to the genus 0 and 1 base curve in the Hitchin classification. The material is related to topological invariants of the quantum field theory type, as well as closely connected with the geometric properties of moduli spaces partially in task of the description of the spectrum for quantum systems. The results concern methodological approach based on the concept of the quantum spectral curve. They crystallize in the explicit discrete group symmetry construction on the corresponding spectrum systems.

Classic Integrable systems.

Interaction between the theory of Integrable systems and algebraic geometry appeared quite early. Pioneer work, linking these areas of mathematics, was due to k. Jacoby [12] where the problem of geodesics on ellipsoid was solved in terms of the Abel transfrom for some algebraic curve. The extent of this observation was conceived in the 1970s by S.P. Novikov school [10, 13]. Later the universal geometric description of the phase space of a wide class of finite-dimensional Integrable systems in terms of the cotangent bundle to some moduli space of holomorphic bundles on an algebraic curve was given in the work of N. Hitchin [14].

The algebraic point of view at Integrable systems evolved parallelly was based on principles of Hamiltonian Dynamics and Poisson geometry. Significant progress in the classical theory of Integrable systems was related with the invention of the inverse scattering problem in 60-ies of the last century [16]. It turned out that the Lax representation is extremely effective description of dynamical systems [17]. This language relates Hamiltonian flows with the related Lie algebra action. This point of view allows to introduce the notion of the spectral curve and use methods of algebraic geometry to construct explicit solutions [18], to solve dynamical systems in algebraic terms by the projection method [19] or by a little more general construction of the Sato grassmanian and the corresponding τ-function [20].

Further, we use the term "'the spectral curve method"' for the method of solving dynamical systems having Lax representation in terms of Abel transform for the curve defined by the characteristic polynomial of the Lax operator.

The first part of the work is dedicated to construction of a generalization for the Hitchin type systems in case of the base curves with singularities and fixed points. The main example of the proposed here quantization technic, the Gaudin model, is a particular case of the Hitchin system of the generalized nature.

Quantization.

Examples of quantum integrable models discussed here have an independent physical meaning as spin chain quantum-mechanical systems describing one-dimensional magnets.

However, the main focus of these lectures is to study the structural role of Integrable systems including the quantum level, where their role as Symmetries of more complex objects is also evident. In particular, spin chains that describe very one-dimensional physical systems are associated with 2D problems of statistical physics [9]. The main method of quantum systems called quantum inverse scattering method (QISM) was established in the 70s of the 20th century by the school of L. D. Faddeev [21]. In many aspects this method relies on the classical method of the inverse problem, in particular with regard to the Hamiltonian description. Using QISM there were constructed several examples of quantum integrable systems: quantum nonlinear Schroedinger equation, the Heisenberg magnet and the sine-Gordon model (this is equivalent to the massive Tirring model). The asymptotic correlation functions for these model were found in [47]. Many of the results regarding QISM was aware of earlier framework of the Bethe ansatz method discovered in 1931 [22].

QISM was much generalized by the theory of quantum groups imposed by Drinfeld [23]. The language of Hopf algebras is exclusively convenient for working with algebraic structures of the theory of quantum integrable systems specifically for the generalization of the ring of invariant polynomials on the group. One can consider QISM as the quantum analog of the algebraic part of the integrable systems theory. At the same time, the role of spectral curve and methods of algebraic geometry was out of the QISM paradigm. The second part of the lectures concerns the quantum spectral curve method, whose central object is the quantum characteristic polynomial for the quantum Lax operator. We propose a construction for the $\mathfrak{sl}_n$ Hitchin-type systems for the base curve of genus 0 and 1 with marked points. The elliptic spin Kalogero-Moser system is a particular case of the considered family. The quantum characteristic polynomial is a generating function for quantum Hamiltonians. The construction is based on the methods of the theory of quantum groups, in particular the theory of Yangians and the Felder dynamic elliptic quantum algebra.

As noted above QISM has not provided substantial progress in solving quantum systems on the finite scale level. Despite the fact that separated variables were found for some models the analogue of the Abel transform as the transition from the divisor space to the Jacobian in the quantum case has not been found. In part 3 a family of the geometric symmetries on the set of the quantum system solutions is constructed significantly using the quantum characteristic polynomial of the model. The alternating formulation of the Bethe system is used to construct this family, the formulation in terms of a family of special Fuchsian systems with restricted monodromy representation. In turn, these differential operators are scalar analogue

of the quantum characteristic polynomial. This permits to realize quantum symmetries in terms of very known Schlesinger transformation in theory of isomonodromic deformations [24] and apply known solutions of differential equations of the Painleve type to describe variations of the spectrum of quantum systems changing the inhomogeneity parameters. In a sense to build a family of symmetries of the spectrum is an analog of the Abel transform.

Quantum method of the spectral curve and other areas of modern mathematics.
Study of quantum characteristic polynomial for the Gaudin models has systematized and made much more efficient methods of solving quantum integrable systems. The constructed discrete symmetries of the spectrum systems provide a generalized angle operators, meaning that you can build eigenvectors of the model recurrently. The significance of the results in geometry and topology is the possibility to apply this technique to field theoretic models arising in topological quantum field theories and theories of fields used in constructing the Donaldson and Seiberg-Witten invariants. In addition, the results in the problem of solving quantum systems have direct application in the description of cohomologies of moduli spaces of holomorphic bundles, analogues of the Laumon spaces, as well as affine Jacobians.

The method have got issue in numerous relations and application in other areas of modern mathematics and mathematical physics. In the representation theory of Lie algebras the results are related to the effectivization of the multiplicity formula. Applications of this type occur thanks to special limits of the Gaudin commutative subalgebras which are interpreted as subalgebras of central elements in $U(\mathfrak{sl}_n)^{\otimes N}$ [25]. Another result of this technique is an explicit descriptions of the centre of the universal enveloping algebra of the affine algebra on the critical level for $\mathfrak{sl}_n$. It is also worth noting the importance of quantum spectral curve method in geometrical Lenglends program over $\mathbb{C}$ [26] in the booming field of Noncommutative Geometry, mathematical physics and the condensed matter. Some of the applications are presented in section 4.

1   The classic method of the spectral curve

1.1   Lax representation

This topic describes the classic method of the spectral curve for finite-dimensional Integrable systems. The explanation begins with Lax representation [17], which have led to the formation of the inverse problem method in the theory of Integrable systems. It turns out that the very wide class of Integrable systems is of the Lax type

$$\dot{L}(z) = [M(z), L(z)] \tag{1.1}$$

where $M(z), L(z)$ are matrix-valued functions of the formal variable z, those matrix elements are in turn the functions on the phase space of a model. In other words, the phase space of a system may be embedded into some space of matrix-valued functions where the dynamics is described by the Lax equation (1.1).

Locally, this property is fulfilled for all integrable systems due to the existence of local action-angle variables ([28], 2 4 Example 1). Globally the Lax expression is known for: harmonic oscillator, integrable tops, the Newman model, the problem of geodesics on ellipsoid, the open and periodic Toda chain, the Calogero-Moser system for all types of root systems, the Gaudin model, nonlinear hierarchies: KdV, KP, Toda, as well as their famous matrix generalizations. The Lax representation demonstrates that the Hamiltonian vector field $\dot{L} = \{h, L\}$ can be expressed in terms of the lie algebra structure on the space of matrices. This property is at the heart of many of algebraic analytic techniques, in particular of the r-matrix approach and the decomposition problem [29].

The Lax representation means that the characteristic polynomial of the Lax operator is preserved by the dynamics. The spectral curve is defined by the equation

$$det(L(z) - \lambda) = 0. \tag{1.2}$$

It turns out that the solution of equations that allow the Lax representation simplifies using the so-called linear problem

$$L(z)\Psi(z) = \lambda\Psi(z). \tag{1.3}$$

The Lax equation is equivalent to the compatibility condition of following equations:

$$
\begin{aligned}
\lambda\Psi(z) &= L(z)\Psi(z), \\
\dot{\Psi}(z) &= M(z)\Psi(z).
\end{aligned}
$$

If now we interpret the auxiliary linear problem as a way of specifying a line bundle on a spectral curve, the system can be solved by means of linear coordinates on the moduli space of line bundles on the spectral curve identified with an associate Jacobian.

Further a Hitchin scheme and some of its generalization sets out pretending to the classification description in the theory of finite-dimensional Integrable systems. In this section we also determine the Gaudin model and give details of the classical method of the spectral curve for the system and separated variables technique.

1.2 The Hitchin description

Let Σ_0 be an algebraic curve and $\mathcal{M} = \mathcal{M}_{r,d}(\Sigma_0)$ be the moduli space of holomorphic stable bundles over Σ_0 of rank r and the determinant bundle d [30]. Let us consider the canonical holomorphic simplectic form on the cotangent bundle to the moduli space $T^*\mathcal{M}$

The deformation theory [31] allows to explicitly describe fibers of the cotangent bundle. A tangent vector to the moduli space at E corresponding to the infinitesimal deformation in terms of the Cech cocycle can be realized by an element of $H^1(\text{End}(E))$, in turn the cotangent vector at E to the moduli space $\mathcal{M}$ through the Serre duality is an element of the cohomology space $\Phi \in H^0(\text{End}(E) \otimes \mathcal{K})$; here $\mathcal{K}$ denotes the canonical class of Σ_0. In this description the following family of functions can be defined on $T^*\mathcal{M}$

$$h_i : T^*\mathcal{M} \to H^0(\mathcal{K}^{\otimes i}); \qquad h_i(E, \Phi) = \frac{1}{i} tr\Phi^i. \tag{1.4}$$

The direct sum of the collection of mappings h_i

$$h : T^*\mathcal{M} \longrightarrow \oplus_{i=1}^{r} H^0(\mathcal{K}^{\otimes i})$$

is called the Hitchin map [14] and defines a Lagrangian fibration of the phase space of the integrable system.

1.2.1 Spectral curve

The spectral curve method implies an explicit method of solution in terms of some geometric objects on a certain algebraic curve. Consider the (nonlinear) bundle map

$$char(\Phi) : \mathcal{K} \to \mathcal{K}^{\otimes r}, \tag{1.5}$$

defined by the expression

$$char(\Phi)(\mu) = det(\Phi - \mu * Id) \tag{1.6}$$

where μ defines a fiber point of $\mathcal{K}$, and the expression Id - the tautological section of $\text{End}(E)$. The spectral curve is defined as the preimage of the zero section of $\mathcal{K}^{\otimes r}$. The preimage defines an algebraic curve Σ in the projectivization of the total space of $\mathcal{K}$.

1.2.2 Line bundle

Solution for the Hitchin type system can be constructed in terms of the following line bundle. Consider the projection map π corresponding to the canonical bundles $\mathcal{K}$

$$\pi : \mathcal{K} \to \Sigma_0$$

and the inverse images map

$$\pi^* E \overset{\Phi - \tilde{\mu} * Id}{\longrightarrow} \pi^*(E \otimes \mathcal{K}),$$

where $\tilde{\mu}$ is the tautological section $\pi^* \mathcal{K}$. Let us also consider the quotient $\mathcal{F}$, corresponding to the inclusion

$$0 \longrightarrow \pi^* E \overset{\Phi - \mu * Id}{\longrightarrow} \pi^*(E \otimes \mathcal{K}) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{1.7}$$

The support of $\mathcal{F}$ coincides with the spectral curve Σ defined below by the reason that the solution of the eigen problem exists only for eigen values of a linear operator. Let us restrict the exact sequence (1.7) to Σ

$$0 \longrightarrow \mathcal{L} \longrightarrow \pi^* E|_\Sigma \overset{\Phi - \mu * Id}{\longrightarrow} \pi^*(E \otimes \mathcal{K})|_\Sigma \longrightarrow \mathcal{F}|_\Sigma \longrightarrow 0.$$

It turns out that $\mathcal{L}$ specifies a line bundle on the spectral curve associated with eigenvectors of the Lax operator.

Let us define the Abel transform as follows: let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a basis in $H_1(\Sigma_0, \mathbb{Z})$ with the intersection indexes $(a_i, b_j) = \delta_{ij}$, $\{\omega_i\}$ be the basis of holomorphic differentials in $H^0(\mathcal{K})$ normalized by the condition $\oint_{a_i} \omega_j = \delta_{ij}$, and let $B_{ij} = \oint_{b_i} \omega_j$ be the matrix of b-periods. Then we define the lattice Λ in $\mathbb{C}^g$ generated by the $\mathbb{Z}^g$ and the lattice generated by the columns of the matrix B. Fixing a point $P_0 \in \Sigma$ one can define the Abel transform by the formula

$$A : \Sigma \to Jac_\Sigma = \mathbb{C}^g/\Lambda; \qquad A(P) = \begin{pmatrix} \int_{P_0}^P \omega_1 \\ \vdots \\ \int_{P_0}^P \omega_g \end{pmatrix}. \tag{1.8}$$

This definition does not depend on the integrating path due to the factorization and generalizes to the map from the space of divisor classes to the moduli space of line bundles.

Theorem 1.1 ([14])**.** *The linear coordinates on the Jacobian $Jac(\Sigma)$ applied to the image of the Abel transform $A(\mathcal{L})$ are the "'angle"' variables for the Hitchin system.*

1.3 The Hitchin system on singular curves

1.3.1 Generalizations

The Hitchin construction can be generalized to the case of singular curves and curves with fixed points [32], [33]. This generalization permits to give explicit parametrization to the wide class of integrable systems preserving the geometric analogy with the intrinsic ingredients of the original Hitchin system.

- Fixed points: It can be considered the moduli space of holomorphic bundles on an algebraic curve with additional structures, namely with trivializations at fixed points. This moduli space can be obtained as the quotient of the space of gluing functions by the trivialization change group with the condition of preserving trivializations at fixed points. Let us denote this moduli space by $\mathcal{M}_{r,d}(z_1, \ldots, z_k)$. The tangent vector to the space $\mathcal{M}_{r,d}(z_1, \ldots, z_k)$ at the point E in an element of the space

$$T_E \mathcal{M}_{r,d}(z_1, \ldots, z_k) \simeq H^1(\mathrm{End}(E) \otimes \mathcal{O}(-\sum_{i=1}^k z_i)).$$

  The cotangent vector can be identified with the following element

$$\Phi \in H^0(\mathrm{End}(E) \otimes \mathcal{K} \otimes \mathcal{O}(\sum_{i=1}^k z_i)).$$

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- Singular points: The moduli space of bundles can be considered on curves with singularities of the types: double point, cusp or the so-called scheme double point. In this situation the consistent Hitchin system formalism can be established. This results in constructing a large class of interesting integrable systems. The description of the dualizing sheaf and the moduli space of bundles in this case turns out to be more explicit then in the case of nonsingular curve of the same algebraic genus.

1.3.2 Scheme points

Let us describe in detail the Hitchin formalism on curves with double scheme points.

The singularities class Let us consider the curve Σ^{proj} obtaining by gluing 2 subscheme $A(\epsilon), B(\epsilon)$ of $\mathbb{C}P^1$ (i.e. a curve obtained by adding a point ∞ to the affine curve $\Sigma^{aff} = Spec\{f \in \mathbb{C}[z] : f(A(\epsilon)) = f(B(\epsilon))\}$, where $\epsilon^N = 0$). Calculating the algebraic genus $(dim H^1(\mathcal{O}))$ we obtain:

- Nilpotent elements: $A(\epsilon) = \epsilon, B(\epsilon) = 0, g = N - 1$.

- Roots of unity: $A(\epsilon) = \epsilon, B(\epsilon) = \alpha\epsilon$, where $\alpha^k = 1$. $g = N - 1 - [(N-1)/k]$.

- Different points:

$$A(\epsilon) = a_0 + a_1\epsilon + ... + a_{N-1}\epsilon^{N-1}, \quad B(\epsilon) = b_0 + b_1\epsilon + ... + b_{N-1}\epsilon^{N-1},$$

supposing $a_0 \neq b_0$. $g = N$.

Holomorphic bundles The most convenient way to describes the moduli space of holomorphic bundles for singular curves is an algebraic language, due to the duality between a bundle and the sheaf of its sections, which is a sheaf of locally-free and thus projective modules over the structure sheaf of an algebraic curve.

The geometrical characterisation of a projective module in the affine chart without ∞ of the normalized curve is made in terms of the submodule M_Λ of rank r in the trivial module of vector-functions $s(z)$ on $\mathbb{C}$ satisfying the condition:
$$s(A(\epsilon)) = \Lambda(\epsilon)s(B(\epsilon)),$$
where $\Lambda(\epsilon) = \sum_{i=0,...,N-1} \Lambda_i\epsilon^i$ is a matrix-valued polynomial. The projectivity condition of this module M_Λ is expressed as follows:

- Nilpotent elements: $A(\epsilon) = \epsilon, B(\epsilon) = 0$, condition: $\Lambda_0 = Id$.

- Roots of unity: $A(\epsilon) = \epsilon, B(\epsilon) = \alpha\epsilon$, where $\alpha^k = 1$,
  condition:
  $$\Lambda(\epsilon)\Lambda(\alpha\epsilon)...\Lambda(\alpha^{k-1}\epsilon) = Id.$$

- Different points:

$$A(\epsilon) = a_0 + a_1\epsilon + ... + a_{N-1}\epsilon^{N-1}, B(\epsilon) = b_0 + b_1\epsilon + ... + b_{N-1}\epsilon^{N-1},$$

condition: Λ_0 is invertible.

The open sell of the moduli space of holomorphic bundles for Σ^{proj} is the quotient space of the space of $\Lambda(\epsilon)$ in general position satisfying the condition above with respect to the adjoint action of GL_r.

Dualizing sheaf and global section In smooth situation the canonical class $\mathcal{K}$ is determined by the line bundle of the highest order forms on complex analytical variety M. To reconstruct this object in the singular case we axiomatize the Serre duality condition

$$H^n(\mathcal{F}) \times H^{m-n}(\mathcal{F}^* \otimes \mathcal{K}) \to \mathbb{C}$$

for a coherent sheaf $\mathcal{F}$. In the present case the dualizing sheaf can be defined by its global sections. The global sections of the dualizing sheaf on Σ^{proj} can be described in terms of meromorphic differentials on $\mathbb{C}$ of the form

$$\omega_\phi = Res_\epsilon \left(\frac{\phi(\epsilon)dz}{z - A(\epsilon)} - \frac{\phi(\epsilon)dz}{z - B(\epsilon)} \right), \tag{1.9}$$

for an element $\phi(\epsilon) = \sum_{i=0,\ldots,N-1} \phi_i \frac{1}{\epsilon^{i+1}}$. In this expression fractions should be understood as geometric progression:

$$\frac{1}{z - A(\epsilon)} = \frac{1}{z - a_0 - a_1\epsilon - a_2\epsilon^2 - \ldots} = \frac{1}{(z - a_0)(1 - \frac{a_1\epsilon + a_2\epsilon^2 + \ldots}{z - a_0})}$$

$$= \frac{1}{(z - a_0)}(1 + \frac{a_1\epsilon + a_2\epsilon^2 + \ldots}{z - a_0} + (\frac{a_1\epsilon + a_2\epsilon^2 + \ldots}{z - a_0})^2 + \ldots).$$

The symbol Res_ϵ means the coefficient at $\frac{1}{\epsilon}$. It turns out that for an arbitrary $\phi(\epsilon)$ the expression above gives a holomorphic differential on the singular curve Σ^{proj}, and in addition any differential is obtained in this way. Let us describe the Serre pairing for the structure sheaf. Let us consider the covering consisting of two opens $U_0 = \Sigma^{aff}$ and U_∞ - an open disk centered at ∞. The intersection $U_0 \cap U_\infty$ can be identified with the punctured disk $U_\infty^\bullet$ also centered at ∞. Let $s \in \mathcal{O}_{U_\infty^\bullet}$ - be a representative of $H^1(\mathcal{O})$. The pairing is determined by the formula:

$$< \omega_\phi, s > = \oint_{\delta U_0 \cap U_\infty} \omega_\phi s. \tag{1.10}$$

It is easy to see that the pairing is correctly defined on classes of cohomology.

The endomorphisms of the module M_Λ are described by polynomial matrix-valued functions $\Phi(z)$ satisfying the condition

$$\Phi(A(\epsilon)) = \Lambda(\epsilon)\Phi(B(\epsilon))\Lambda(\epsilon)^{-1}.$$

The action of $\Phi(z)$ on a section $s(z)$ is given by the formula : $s(z) \mapsto \Phi(z)s(z)$. The space $H^1(\text{End}(M_\Lambda))$ is described as the quotient of the space of matrix-valued polynomial functions by two subspaces:

$$\text{End}_{out} = \{\chi(z) \in Mat_n[z] | \chi(z) = const\}$$

and

$$\text{End}_{in} = \{\chi(z) \in Mat_n[z] | \chi(A(\epsilon))) = \Lambda(\epsilon)\chi(B(\epsilon))\Lambda(\epsilon)^{-1}\}.$$

Elements of $H^1(\text{End}(M_\Lambda))$ are treated as tangential vectors to the moduli space of holomorphic bundles at M_Λ. The infinitesimal deformation corresponding to an element $\chi(z)$ is defined by the formula

$$\delta_{\chi(z)}\Lambda(\epsilon) = \chi(A(\epsilon))\Lambda(\epsilon) - \Lambda(\epsilon)\chi(B(\epsilon)). \tag{1.11}$$

Global section $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$ are described by the expressions:

$$\Phi(z) = Res_\epsilon \left(\frac{\Phi(\epsilon)}{z - A(\epsilon)}dz - \frac{\Lambda(\epsilon)^{-1}\Phi(\epsilon)\Lambda(\epsilon)}{z - B(\epsilon)}dz \right), \tag{1.12}$$

where

$$Res_\epsilon(\Lambda(\epsilon)\Phi(\epsilon)\Lambda(\epsilon)^{-1} - \Phi(\epsilon)) = 0$$

and $\Phi(\epsilon) = \sum_i \Phi_i \frac{1}{\epsilon^{i+1}}$ is a polynomial matrix-valued function. This expression also implies a decomposition of the denominator in the geometric progression. It turns out that all global section of $H^0(\text{End}(M_\Lambda) \otimes \mathcal{K})$ are of this form.

Symplectic form on the cotangent bundle to the moduli space of holomorphic bundles can be described in terms of Hamiltonian reduction with respect to the adjoint action of GL_n of the symplectic form on the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$, given by the expression:

$$Res_\epsilon Trd(\Lambda(\epsilon)^{-1}\Phi(\epsilon)) \wedge d\Lambda(\epsilon). \tag{1.13}$$

Integrability The Hitchin system on Σ^{proj} in now defined as a system with phase space which is the Hamiltonian quotient of the space of pairs $\Lambda(\epsilon), \Phi(\epsilon)$. The simplectic form is given by the formula 1.13. The reduction is considered with respect to the adjoint action of the group GL_n. The Lax operator is defined by the formula 2.42. The Hamiltonians are defined by the coefficients of the function $Tr(\Phi(z)^k)$ subject to some basis of holomorphic k-differentials (i.e. sections of $H^0(\mathcal{K}^k)$). Let us remark that $\forall z, w, k, l$ the following commutativity condition is fulfilled: $Tr(\Phi(z)^k)$ and $Tr(\Phi(w)^l)$ commute on the nonreduced space.

The integrability proof realizes the r-matrix technique.

Example 1.1. *Consider a rational curve with a double point $z_1 \leftrightarrow z_2$ (the ring of rational functions on a curve is a subring of rational functions f on $\mathbb{C}P^1$ satisfying the condition $f(z_1) = f(z_2)$) with one marked point z_3. The dualizing sheaf has the global section $dz(\frac{1}{z-z_1} - \frac{1}{z-z_2})$. Consider the moduli space $\mathcal{M}$ of holomorphic bundles E of rank n on Σ_{node} with fixed trivialization at z_3. There is the following isomorphism of linear spaces*

$$T_E\mathcal{M} = H^1(\mathrm{End}(E) \otimes \mathcal{O}(-p)).$$

Let us restrict ourself to the open sell of the moduli space of equivalence classes of matrices Λ with different eigenvalues. The cotangent space is isomorphic to the space of holomorphic sections of $\mathrm{End}^*(E) \otimes \mathcal{K} \otimes \mathcal{O}(p)$. This space can be realized by the space of rational matrix-valued functions on z of the following type

$$\Phi(z) = \left(\frac{\Phi_1}{z - z_1} - \frac{\Phi_2}{z - z_2} + \frac{\Phi_3}{z - z_3} \right) dz,$$

with the following conditions on residues

$$\Phi_1\Lambda = \Lambda\Phi_2 \qquad \Phi_1 - \Phi_2 + \Phi_3 = 0.$$

The phase space of the system is parameterized by elements of $U \in GL(n)$ giving a trivialization at z_3, matrix Λ describing the projective module over $\mathcal{O}(\Sigma_{mode})$, residues of the Higgs field Φ_i. In these coordinates the canonical simplectic form on $\mathcal{T}^*\mathcal{M}$ can be expressed as follows

$$\omega = Tr(d(\Lambda^{-1}\Phi_1) \wedge d\Lambda) + Tr(d(U^{-1}\Phi_3) \wedge dU).$$

After Hamiltonian reduction with respect to the group $GL(n)$ action (the right action on U and the adjoint action on Φ_i, Λ) one obtains the space parameterized by the matrix elements $(\Phi_3)_{ij} = f_{ij}, i \neq j$; eigenvalues e^{2x_i} of the matrix Λ and the diagonal elements of the matrix $(\Phi_1)_{ii} = p_i$ with the following Poisson structure

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{f_{ij}, f_{kl}\} = \delta_{jk}f_{il} - \delta_{il}f_{kj}.$$

The Hamiltonian of the trigonometric spin Calogero-Moser system related with the finite-zone solutions of the matrix generalization for the KP equation [34] can be obtained as the coefficient of $Tr\Phi^2(z)$ at $1/(z-z_1)^2$

$$H = Tr\Phi_1^2 = \sum_{i=1}^n p_i^2 - 4\sum_{i \neq j} \frac{f_{ij}f_{ji}}{\sinh^2(x_i - x_j)}.$$

1.4 The Gaudin model

1.4.1 The Lax operator

The Gaudin model was proposed in [9] (section 13.2.2) as a limit of the XXX Heisenberg magnet. It describes a one-dimensional chain of interacting particles with spin. The Gaudin model can be considered

as a generalization of the Hitchin system for the rational curve $\Sigma = \mathbb{C}P^1$ with N marked points $z_1, \ldots, z_N$. The Higgs field (the Lax operator) can be represented by a rational section $\Phi = L(z)dz$ where

$$L(z) = \sum_{i=1\ldots N} \frac{\Phi_i}{z - z_i}. \tag{1.14}$$

The residues of the Lax operator Φ_i are matrices $n \times n$ those matrix elements lie in $\mathfrak{gl}_n \oplus \ldots \oplus \mathfrak{gl}_n$. $(\Phi_i)_{kl}$ coincides with the kl-th generator of the i-th copy of $\mathfrak{gl}_n$. The generators of the Lie algebra are interpreted as functions on the dual space $\mathfrak{gl}_n^*$. The symmetric algebra $S(\mathfrak{gl}_n)^{\otimes N} \simeq \mathbb{C}[\mathfrak{gl}_n^* \oplus \ldots \oplus \mathfrak{gl}_n^*]$ is equipped with the Poisson structure given by the Kirillov-Kostant bracket:

$$\{(\Phi_i)_{kl}, (\Phi_j)_{mn}\} = \delta_{ij}(\delta_{lm}(\Phi_i)_{kn} - \delta_{nk}(\Phi_i)_{ml}).$$

1.4.2 R-matrix bracket

R- matrix representations of Poisson structures turned out to be a key element of the theory of quantum groups. In some sense the existence of an R-matrix structure is equivalent to integrability. It should be noted that, in theory of quantum groups [35] the important concept is the so-called quasitriangular or braided bialgebra. Let us introduce the notations:

- $\{e_i\}$ - the standard basis in $\mathbb{C}^n$;

- $\{E_{ij}\}$ - the standard basis in $\text{End}(\mathbb{C}^n)$, $(E_{ij}e_k = \delta_k^j e_i)$;

- $e_{ij}^{(s)}$ - generators of the s-th copy $\mathfrak{gl}_n \subset \oplus^N \mathfrak{gl}_n$.

The Lax operator can be represented as

$$L(z) = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}.$$

The Poisson structure can be described in terms of generating functions:

$$\{L(z) \otimes L(u)\} = [R_{12}(z - u), L(z) \otimes 1 + 1 \otimes L(u)] \in \text{End}(\mathbb{C}^n)^{\otimes 2} \otimes S(\mathfrak{gl}_n)^{\otimes N},$$

with the classical Yang R-martix

$$R(z) = \frac{P_{12}}{z}, \qquad P_{12}v_1 \otimes v_2 = v_2 \otimes v_1, \qquad P_{12} = \sum_{ij} E_{ij} \otimes E_{ji}.$$

1.4.3 The integrals

The integrals of motion can be retrieved as the characteristic polynomial coefficients

$$det(L(z) - \lambda) = \sum_{k=0}^{n} I_k(z)\lambda^{n-k}. \tag{1.15}$$

It is often used the alternative basis of symmetric functions of eigenvalues of the Lax operators

$$J_k(z) = TrL^k(z), \qquad k = 1, \ldots, n.$$

Traditional quadratic Hamiltonians can be obtained as follows

$$H_{2,k} = Res_{z=z_k} TrL^2(z) = \sum_{j \neq k} \frac{2Tr\Phi_k\Phi_j}{(z_k - z_j)} = 2\sum_{j \neq k} \frac{\sum_{lm} e_{lm}^{(k)} e_{ml}^{(j)}}{z_k - z_j}.$$

They describe the magnet model that consists of a set of pair interaction particles. It is known that

Proposition 1.2. *The coefficients of the characteristic polynomial of $L(z)$ commute with respect to the Kirillov-Kostant bracket*

$$\{I_k(z), I_m(u)\} = 0.$$

Let us present here the baseline of the proof.

Proof

Let $L_1(z) = L(z) \otimes 1$ and $L_2(u) = 1 \otimes L(u)$.

$$\{J_k(z), J_m(u)\} = Tr_{12}\{L^k(z) \otimes L^m(u)\}$$

$$= Tr_{12} \sum_{ij} L_1^i(z) L_2^j(u) \{L(z) \otimes L(u)\} L_1^{k-i-1}(z) L_2^{m-j-1}(u)$$

$$= Tr_{12} \sum_{ij} L_1^i(z) L_2^j(u) R_{12}(z-u) L_1^{k-i}(z) L_2^{m-j-1}(u) \tag{1.16}$$

$$+ \quad Tr_{12} \sum_{ij} L_1^i(z) L_2^j(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j}(u)$$

$$- \quad Tr_{12} \sum_{ij} L_1^{i+1}(z) L_2^j(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j-1}(u) \tag{1.17}$$

$$- \quad Tr_{12} \sum_{ij} L_1^i(z) L_2^{j+1}(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j-1}(u).$$

In particular,

$$(1.16) + (1.17) = Tr_{12}[\sum_{ij} L_1^i(z) L_2^j(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j-1}(u), L_1(z)].$$

The last expression is zero because it is trace of a commutator.

1.4.4 Algebraic-geometric description

This section describes the basic algebraic-geometrical components of the generalized Hitchin system for curves with marked points. Namely it is constructed a pair $\{\Sigma, \mathcal{L}\}$ - the spectral curve and the line bundle on it, which allows to solve the classical Gaudin model.

Spectral curve The spectral curve of the Gaudin system $\tilde{\Sigma}$ is described by the equation

$$det(L(z) - \lambda) = 0. \tag{1.18}$$

To build a nonsingular compactification of that curve one should consider the total space of the bundle where the Lax operator takes values

$$\Phi(z) = L(z)dz \in H^0(\mathbb{C}P^1, \text{End}(\mathcal{O}^n) \otimes \Lambda) \tag{1.19}$$

where $\Lambda = \mathcal{K}(k) = \mathcal{O}(k-2)$. We define a compactification of Σ by the equation

$$det(\Phi(z) - \lambda) = 0. \tag{1.20}$$

This curve is a subvariety of the rational surface S_{k-2}, obtained by compactification of the total space of the line bundle $\mathcal{O}(k-2)$ over $\mathbb{C}P^1$, or as a projectivisation $P(\mathcal{O}(k-2) \oplus \mathcal{O})$ over the rational curve. This rational surface contains three types of divisors: E_∞ - the infinite divisor, C - the fiber of the bundle and E_0 - the base curve with the following intersections

$$E_0 \cdot E_0 = k - 2,$$
$$E_0 \cdot C = 1,$$
$$C \cdot C = 0,$$
$$E_\infty \cdot C = 1.$$

To determine the genus of the curve Σ we use the adjunction formula. First let us calculate the canonical class of S_{k-2}. It corresponds to the class of divisors

$$\mathcal{K}_{S_{k-2}} = -2E_0 + (k-4)C.$$

Let the class of Σ be equal to $[\Sigma] = n_1 E_0 + n_2 C$. Σ is n-folded covering of $\mathbb{C}P^1$. Hence $[\Sigma] \cdot C = n$ and $n_1 = n$. To calculate n_2 it is sufficient to use the fact that Σ is a spectral curve of a holomorphic section of $\mathrm{End}(\mathcal{O}^n) \otimes \Lambda$ and hence does not intersect E_∞. We obtain

$$[\Sigma] = nE_0.$$

By the adjunction formula we have

$$
\begin{aligned}
2g - 2 &= \mathcal{K}_{S_{k-2}} \cdot [\Sigma] + [\Sigma] \cdot [\Sigma] \\
&= (-2E_0 + (k-4)C) \cdot nE_0 + n^2 E_0 \cdot E_0 \\
&= -2(k-2)n + (k-4)n + (k-2)n^2.
\end{aligned}
$$

This allows to calculate the genus of the spectral curve

$$g(\Sigma) = \frac{(k-2)n(n-1)}{2} - (n-1). \tag{1.21}$$

Line bundle   Let us recall the sequence defining the line bundle on the spectral curve

$$0 \to \mathcal{L} \to \mathcal{O}_\Sigma^n \to \mathcal{O}_S^n((k-2)C + E_\infty)|_\Sigma \to \mathcal{F}_\Sigma \to 0, \tag{1.22}$$

where $\mathcal{L}$ and $\mathcal{F}_\Sigma$ are line bundles. We also obtain the following

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_\Sigma^n) - \chi(\mathcal{O}_S^n((k-2)C + E_\infty)|_\Sigma) + \chi(\mathcal{F}_\Sigma)$$

Let us denote the divisor $(k-2)C + E_\infty \subset S$ as D. Then

$$
\begin{aligned}
\chi(\mathcal{O}_\Sigma^n) &= n(1-g), \\
\chi(\mathcal{O}_S^n(D)|_\Sigma) &= nD \cdot [\Sigma] + n(1-g) \\
&= n^2(k-2) + n(1-g), \\
\chi(\mathcal{F}_\Sigma) &= \chi(\pi^* \mathcal{O}^n(k-2) \otimes \mathcal{O}_S(E_\infty)) - \chi(\mathcal{O}_S^n) \\
&= n\frac{1}{2}(D \cdot D - D \cdot \mathcal{K}_S) = n(k-1). \tag{1.23}
\end{aligned}
$$

Hence $\chi(\mathcal{L}) = -n^2(k-2) + n(k-1)$. Calculating the number of branching points $\nu = 2(g + n - 1) = (k-2)(n^2 - n)$ we obtain

Lemma 1.3.

$$deg(\mathcal{L}) = g + n - 1 - \nu.$$

The dimension of the commutative family   On the affine chart without $\{z_i\}$ and ∞ the spectral curve is given by the equation

$$R(z, \lambda) = 0, \qquad R(z, \lambda) = (-1)^n \lambda^n + \sum_{m=0}^{n-1} \lambda^m R_m(z), \tag{1.24}$$

where

$$R_m(z) = \sum_{i=1}^{k} \sum_{l=1}^{n-m} \frac{R_{m,i}^{(l)}}{(z - z_i)^l}.$$

The number of free coefficients is equal $\sum_{m=0}^{n-1} k(n - m) = k\frac{n(n+1)}{2}$. The central functions (the symmetric polynomials of eigenvalues for corresponding orbits) are the highest coefficients of $R_m(z)$ of the total number $kn - 1$. The Lax operator has double zero at infinity

$$L(z) = \frac{1}{z^2} \sum_i \Phi_i z_i + O(\frac{1}{z^3}).$$

It follows that $R_m(z)$ has zero of order $2(n - m)$ at infinity. This observation, in turn, imposes additional

$$\sum_{m=0}^{n-1} (2(n - m) - 1) = n^2$$

conditions on the values of Hamiltonians. Thus, the dimension of the commutative family is

$$k\frac{n(n+1)}{2} - kn + 1 - n^2 = k\frac{n(n-1)}{2} - n^2 + 1 = g.$$

1.5 Separated variables

For the wide class of integrable systems the separated variables are associated with the divisor of the line bundle $\mathcal{L}$ on the spectral curve. Namely pairs of coordinates of the divisor points are separated. Typically, the divisor is the divisor for the Baker function. A construction of separated variables for some class of integrable systems is given in [36]. In the case of $\mathfrak{sl}_2$-Gaudin model separated variables were known before [37], and can be found even more explicitly.

1.5.1 $\mathfrak{sl}_2$-Gaudin model

Let us remind that $\mathfrak{sl}_2$-Gaudin model is obtained from the $\mathfrak{gl}_2$ model (1.14) choosing orbits with $tr = 0$. The Lax operator in this case is:

$$L = \begin{pmatrix} A(z) & B(z) \\ C(z) & -A(z) \end{pmatrix}.$$

We will consider the characteristic polynomial as a function of parameters z, λ and values of the Hamiltonians:

$$det(L(z) - \lambda) = R(z, \lambda, h_1, \dots, h_d).$$

Let us define the variables y_j as zeroes of $C(z)$. For dual variables we take

$$w_j = A(y_j).$$

This set of variables defines the Darboux coordinates of the phase space:

$$\{y_i, w_j\} = \delta_{ij}.$$

Let us consider the generating function $S(I, y)$ of the canonical transformation from the variables y_j, w_j to the "'action-angle"' variables I_j, ϕ_j

$$w_j = \partial_{y_j} S, \qquad \phi_j = \partial_{I_j} S.$$

The point with coordinates (y_j, w_j) is a point of the spectral curve by definition. The fact that "'action"' variables are functions of Hamiltonians allows to separate variables in the problem of finding the canonical transformation S

$$S(I, y_1, \dots, y_d) = \prod_i s(I, y_i),$$

where each factor $s(I, z)$ solves the equation

$$R(z, \partial_z s, h_1, \dots, h_d) = 0.$$

2 The quantization problem

The quantization problem has physical motivation, it is related to the quantum paradigm in modern physics. In mathematical context this problem can be formulated by different manners: in [38] it was considered the problem of deformation of an algebra of functions on a simplectic manifold satisfying the so-called "'correspondence principle"'. The particular case of the deformation quantization for the cotangent bundle to a Lie group used in these lectures was considered in [39]. Further the methods of deformation quantization, $*$-product, the Moyal product and the geometric quantization was generalized to wider class of examples. One of the structure results in this field was the formality theorem by M. Kontsevich [40] which demonstrates the existence of the quantization. Another ensemble of important results in this domain are due to Fedosov [41].

In this work it is proposed radically more strong quantization problem, demanding not only the deformation of an algebra of functions but of a pair: Poisson algebra + Poisson commutative subalgebra, representing an integrable system. Let us call this task the algebraic part of integrable system quantization. Moreover it is stated a problem of constructing quantum analogs of the essential geometric objects from the point of view of algebraic-geometric methods in integrable systems. In general the problem is to find an associative deformation of a Poisson algebra such that the Poisson-commutative subalgebra remains commutative, and moreover the deformation of the spectral curve provides quantum separated variables. The last part of the quantization problem is called "'algebraic-geometric"' quantization.

2.1 The deformation quantization

2.1.1 Correspondence

The traditional scheme of deformation quantization supposes a construction of an associative algebra starting with a Poisson algebra. A Poisson algebra is a commutative algebra $\mathcal{A}_{cl}$ with multiplication denoted by $\cdot$, furnished by an antisymmetric bilinear operation called the Poisson bracket $\{\circ, \circ\}$, such that $\mathcal{A}_{cl}$ is a Lie algebra and both structures are compatible by the Leibniz rule:

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}.$$

A Poisson algebra is an infinitesimal version of an associative algebra. Due to the so-called Drinfeld ε-construction it is not hard to note that the space $\mathcal{A}_{cl}[\varepsilon]/\varepsilon^2$ with multiplication

$$f * g = f \cdot g + \varepsilon \{f, g\}$$

is an associative algebra. The quantization of the Poisson algebra $\mathcal{A}_{cl}$ with the structure defined by operations $(\cdot, \{\circ, \circ\})$ which is called the algebra of classical observables is an associative algebra $\mathcal{A}$ with multiplication $(*)$, satisfying the following conditions:

$$\mathcal{A} \simeq \mathcal{A}_{cl}[[\mathsf{h}]] \text{ as linear spaces.}$$

Moreover if identified the algebra of classical observables and the space of constants in $\mathcal{A}$ the following structure compatibility is required:

$$\begin{aligned} a * b &= a \cdot b + O(\mathsf{h}), \\ a * b - b * a &= \mathsf{h}\{a, b\} + O(\mathsf{h}^2). \end{aligned}$$

The map

$$lim : \mathcal{A} \longrightarrow \mathcal{A}_{cl} : \qquad \mathsf{h} \mapsto 0$$

is called the classical limit.

Example 2.1. *Let us consider the Poisson algebra $S(\mathfrak{gl}_n)$, on the space of symmetric algebra of the Lie algebra $\mathfrak{gl}_n$ defined by the Kirillov-Kostant bracket. This has a canonical quantization, realizing the concept of the deformation quantization: let $U_{\mathsf{h}}(\mathfrak{gl}_n)$ be the deformed universal enveloping algebra*

$$U_{\mathsf{h}}(\mathfrak{gl}_n) = T^*(\mathfrak{gl}_n)[[\mathsf{h}]]/\{x \otimes y - y \otimes x - \mathsf{h}[x, y]\}.$$

The classical limit is defined as the limit $\mathsf{h} \to 0$ which is correctly defined on the family of algebras $U_{\mathsf{h}}(\mathfrak{gl}_n)$. The existence of a limit follows from the common Poincare-Birkhof-Witt basis for this family.

2.1.2 Quantization of an integrable system

An integrable system is a pair: a Poisson algebra $\mathcal{A}_{cl}$ and a Poisson commutative subalgebra $\mathcal{H}_{cl}$ of the dimension $dim(Spec(\mathcal{H}_{cl})) = 1/2 dim(Spec(\mathcal{A}_{cl}))$. An algebraic problem of quantization is the following correspondence

$$\mathcal{H}_{cl} \subset \mathcal{A}_{cl} \Leftrightarrow \mathcal{H} \subset \mathcal{A}$$

satisfying the conditions

- $\mathcal{A} \simeq \mathcal{A}_{cl}[[\hbar]]$ as linear spaces, the map $lim : \mathcal{A} \to \mathcal{A}_{cl}$ is called the classical limit;

- $\mathcal{H}$ is commutative;

- $lim : \mathcal{H} = \mathcal{H}_{cl}$

Remark 2.1. *In the case of quantization for the symmetric algebra of the Lie algebra $\mathfrak{gl}_n$ the correspondence can be simplified. Let us consider $U(\mathfrak{gl}_n)$, which is a filtered algebra (the filtration is given by degree) $\{\mathcal{F}_i\}$. The projection map to the associated graded algebra induces a Poisson structure:*

$$U(\mathfrak{gl}_n) \to Gr(U(\mathfrak{gl}_n)) = \oplus_i \mathcal{F}_i / \mathcal{F}_{i-1} = S(\mathfrak{gl}_n). \tag{2.1}$$

We will associate this map with the classical limit operation. On generators $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$ the induced commutative multiplication and the Poisson bracket are given by the following expressions:

$$a \cdot b = a * b \ mod \ \mathcal{F}_{i+j-1}, \qquad \{a, b\} = a * b - b * a \ mod \ \mathcal{F}_{i+j-2}.$$

2.1.3 The Gaudin model quantization problem

The classical part is defined by the following objects

$$
\begin{aligned}
\mathcal{A}_{cl} &= S(\mathfrak{gl}_n)^{\otimes N} \simeq \mathbb{C}[\mathfrak{gl}_n^* \oplus \ldots \oplus \mathfrak{gl}_n^*], \\
\mathcal{H}_{cl} &- \text{the subalgebra generated by the Gaudin Hamiltonians(1.15)}.
\end{aligned}
$$

The algebraic part of the quantization problem is reduced to constructing a pair with the quantum observables algebra coinciding with the tensor power of the universal envelopping algebra:

$$\mathcal{A} = U(\mathfrak{gl}_n)^{\otimes N},$$

such that the commutative subalgebra $\mathcal{H}$ is a deformation of the subalgebra generated by the classical Gaudin Hamiltonians.

2.2 Quantum spectral curve

2.2.1 Noncommutative determinant

Let us consider a matrix $B = \sum_{ij} E_{ij} \otimes B_{ij}$ those elements are elements of some generally speaking not commutative associative algebra $B_{ij} \in A$. We will use the following definition for the noncommutative determinant in this case

$$det(B) = \frac{1}{n!} \sum_{\tau, \sigma \in \Sigma_n} (-1)^{\tau \sigma} B_{\tau(1),\sigma(1)} \ldots B_{\tau(n),\sigma(n)}.$$

This definition is the same as the classical one for matrices with commuting elements. There is an equivalent definition. Let us introduce the operator A_n of the antisymmetrization in $(\mathbb{C}^n)^{\otimes n}$

$$A_n v_1 \otimes \ldots \otimes v_n = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$

The definition above is equivalent to the following

$$det(B) = Tr_{1...n} A_n B_1 \dots B_n,$$

where B_k denotes an operator in $\mathrm{End}(\mathbb{C}^n)^{\otimes n} \otimes A$ given by the inclusion

$$B_k = \sum_{ij} 1 \otimes \dots \otimes \underbrace{E_{ij}}_{k} \otimes \dots \otimes 1 \otimes B_{ij},$$

the trace is taken on $\mathrm{End}(\mathbb{C}^n)^{\otimes n}$.

2.2.2 Quantum spectral curve

Let us call a quantum Lax operator for the Gaudin system the following expression:

$$L(z) = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}.$$

$L(z)$ is a rational function on a variable z with values in $\mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}$. Let us define a quantum characteristic polynomial of the quantum Lax operator by the formula

$$det(L(z) - \partial_z) = \sum_{k=0}^{n} QI_k(z)\partial_z^{n-k}. \tag{2.2}$$

The following theorem says that this generalization of the classic characteristic polynomial (1.15) allows to construct quantum Hamiltonians.

Theorem 2.1 ([42]). *The coefficients $QI_k(z)$ commute*

$$[QI_k(z), QI_m(u)] = 0$$

and quantize the classical Gaudin Hamiltonians in the following sense

$$lim(QI_k) = I_k.$$

The proof of this fact uses significant results of the theory of quantum groups such as the construction of the Yangian, the Bethe subalgebras and generally fits into the concept of quantum inverse scattering method. The following sections introduce the necessary definitions and provides an outline of the proof of the theorem of quantization of the Gaudin model.

2.2.3 Yangian

This Hopf algebra was constructed in [23] and plays an important role in the problem of description of rational solutions of the Yang-Baxter equation. $Y(\mathfrak{gl}_n)$ first and foremost is an associative algebra generated by the elements $t_{ij}^{(k)}$ (in this section $i = 1, \dots, n;\ j = 1, \dots, n;\ k = 1, \dots, \infty$). Let us introduce the generating function

$$T(u, \mathsf{h}) \in Y(\mathfrak{gl}_n) \otimes \mathrm{End}(\mathbb{C}^n)[[u^{-1}, \mathsf{h}]],$$

which takes the form

$$T(u, \mathsf{h}) = \sum_{i,j} E_{ij} \otimes t_{ij}(u, \mathsf{h}), \qquad t_{ij}(u, \mathsf{h}) = \delta_{ij} + \sum_{k} t_{ij}^{(k)} \mathsf{h}^k u^{-k},$$

where E_{ij} are the matrix unities in $\mathrm{End}(\mathbb{C}^n)$. The relations can be written with the help of the Yang R-matrix

$$R(u) = 1 - \frac{\mathsf{h}}{u} \sum_{i,j} E_{ij} \otimes E_{ji}$$

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and take the form

$$R(z - u, \mathsf{h})T_1(z, \mathsf{h})T_2(u, \mathsf{h}) = T_2(u, \mathsf{h})T_1(z, \mathsf{h})R(z - u, \mathsf{h}). \tag{2.3}$$

Both parts are regarded as elements of

$$\operatorname{End}(\mathbb{C}^n)^{\otimes 2} \otimes Y(\mathfrak{gl}_n)[[z^{-1}, z, u^{-1}, u, \mathsf{h}]],$$

the rational function $\frac{1}{z-u}$ in the R-matrix formula has an expansion

$$\frac{1}{z - u} = \sum_{l=0}^{\infty} \frac{u^l}{z^{l+1}}.$$

We use the following notations

$$T_1(z, \mathsf{h}) = \sum_{i,j} E_{ij} \otimes 1 \otimes t_{ij}(z, \mathsf{h}), \quad T_2(u, \mathsf{h}) = \sum_{i,j} 1 \otimes E_{ij} \otimes t_{ij}(u, \mathsf{h}).$$

The Yangian is a Hopf algebra those comultiplication is given in terms of the generating function by the following formula

$$(id \otimes \Delta)T(z, \mathsf{h}) = T^1(z, \mathsf{h})T^2(z, \mathsf{h}),$$

where we use the notations

$$T^1(z, \mathsf{h}) = \sum_{i,j} E_{ij} \otimes t_{ij}(z, \mathsf{h}) \otimes 1, \quad T^2(z, \mathsf{h}) = \sum_{i,j} E_{ij} \otimes 1 \otimes t_{ij}(z, \mathsf{h}).$$

The evaluation representation   Let us remind the construction of the so-called evaluation homomorphism $\rho : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$. To do this we consider a rational function on u, h with values in $\operatorname{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)$ given by the formula

$$T_{ev}(u, \mathsf{h}) = 1 + \frac{\mathsf{h}}{u} \sum_{i,j} E_{ij} \otimes e_{ij} \stackrel{def}{=\!=} 1 + \frac{\mathsf{h}\Phi}{u}, \tag{2.4}$$

where e_{ij} are the generators of $\mathfrak{gl}_n$. $T_{ev}(u, \mathsf{h})$ satisfy RTT relations (2.3), hence the map $\{t_{ij}^{(1)} \mapsto e_{ij};\ t_{ij}^{(k)} \mapsto 0$ with $k > 1\}$ determines an algebra homomorphism.

Let us consider the tensor product $U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h}, \mathsf{h}^{-1}]]$ and the generating function (2.4) for the evaluation representation to the l-th component of the product $T_{ev}^l(u - z_l, \mathsf{h})$. It turns out that for an arbitrary set of complex numbers $(z_1, \dots, z_N)$, the expression

$$T^\alpha(u, \mathsf{h}) = T_{ev}^1(u - z_1, \mathsf{h})T_{ev}^2(u - z_2, \mathsf{h}) \dots T_{ev}^k(u - z_N, \mathsf{h}), \tag{2.5}$$

which is a rational function on u and h with values in $\operatorname{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}$, determines a homomorphism $\rho_\alpha : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h}, \mathsf{h}^{-1}]]$. More precisely, the following lemma is true.

Lemma 2.2. *The map, defined on the Yangian generators $t_{ij}^{(k)}$ as the ij-th matrix element of the expansion coefficient of $T^\alpha(u, \mathsf{h})\mathsf{h}^{-k}$ at u^{-k} in $u = \infty$ gives an algebra homomorphism*

$$\rho_\alpha : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h}, \mathsf{h}^{-1}]].$$

This lemma follows from the properties of the comultiplication homomorphism and the evaluation homomorphism.

2.2.4 The Bethe subalgebra

This subalgebra is closely related with Quantum inverse scattering Method (QISM) [21, 44, 45], namely its generators are quantum integrals of the Heisenberg XXX model [44, 43]. Here we use the description from [46] (section 2.14): let us consider an $n \times n$-matrix C and $T(u, \mathsf{h})$ - a generating function for the Yangian generators $Y(\mathfrak{gl}_n)$. Let us also use the notation A_n for the antisymmetrization operator in $(\mathbb{C}^n)^{\otimes n}$ and the following elements of $\operatorname{End}(\mathbb{C}^n)^{\otimes n} \otimes Y(\mathfrak{gl}_n)[[u, u^{-1}, \mathsf{h}]]$

$$T_m(u, \mathsf{h}) = \sum_{ij} 1 \otimes \ldots \otimes 1 \otimes \overset{m}{E_{ij}} \otimes 1 \otimes \ldots \otimes 1 \otimes t_{ij}(u, \mathsf{h}).$$

It turns out [46] (section 2.14), that the expressions of the form

$$\tau_k(u, \mathsf{h}) = Tr A_n T_1(u, \mathsf{h}) T_2(u - \mathsf{h}, \mathsf{h}) \ldots T_k(u - \mathsf{h}(k-1), \mathsf{h}) C_{k+1} \ldots C_n \tag{2.6}$$

for $k = 1, \ldots n$, which are called the Bethe generators, constitute a commutative family in $Y(\mathfrak{gl}_n)[[u, u^{-1}, \mathsf{h}]]$ in the following sense:

$$[\tau_i(u, \mathsf{h}), \tau_j(v, \mathsf{h})] = 0.$$

In addition, this family is maximal if the matrix C has simple spectrum. The trace in the formula 2.6 is meant over matrix components $\operatorname{End}(\mathbb{C}^n)^{\otimes n}$, the series expansion of $T_m(u - \mathsf{h}(m-1), \mathsf{h})$ is realized at $u = \infty$, for example

$$\frac{1}{u - \mathsf{h}} = \sum_{m=0}^{\infty} \frac{\mathsf{h}^m}{u^{m+1}},$$

Next we will consider an identity matrix C and images of the Bethe generators with the evaluation homomorphism. For simplicity, we refer to the same letters

$$\tau_k(u, \mathsf{h}) = Tr A_n T_1^{\alpha}(u, \mathsf{h}) T_2^{\alpha}(u - \mathsf{h}, \mathsf{h}) \ldots T_k^{\alpha}(u - \mathsf{h}(k-1), \mathsf{h}) \quad k = 1, \ldots n. \tag{2.7}$$

2.2.5 The commutativity proof

The presence of the comultiplication structure in the theory of quantum groups allows to use the so-called "'fusion"' method to construct non-trivial integrable systems. Literally, the method is as follows: let us consider the image $T(z)$ by the evaluation homomorphism in composition with enough comultiplication operations $\rho_{z_1} \otimes \ldots \otimes \rho_{z_N} \Delta^{N-1}$

$$T^{\aleph}(z) = T_{z_1}^1(z) \ldots T_{z_N}^N(z) \in \operatorname{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}.$$

The image of the Bethe subalgebra raises to some commutative subalgebra which can be described by the generating function:

$$
\begin{aligned}
Q(z, \mathsf{h}) &= Tr A_n (e^{-\mathsf{h}\partial_z} T_1^{\aleph}(z, \mathsf{h}) - 1) \ldots (e^{-\mathsf{h}\partial_z} T_n^{\aleph}(z, \mathsf{h}) - 1) \\
&= \sum_{j=0}^{n} \tau_j(z - \mathsf{h}, \mathsf{h})(-1)^{n-j} C_n^j e^{-j\mathsf{h}\partial_z} \\
&= det(e^{-\mathsf{h}\partial_z} T^{\aleph}(z, \mathsf{h}) - 1).
\end{aligned}
\tag{2.8}
$$

The expression (2.8) can be represented as a series of ∂_z. From the commutativity of the Bethe generators it follows that the coefficients of this series which are rational functions on u with values in $U(\mathfrak{gl}_n)^{\otimes N}[[\mathsf{h}]]$ also commute at different values of the parameter u. Hence the lowest coefficients on h also commute. These are exactly the coefficients of the characteristic polynomial of the Gaudin model. It turns out that the highest coefficient of the expression (2.8) on h has the form

$$det(e^{-\mathsf{h}\partial_z} T^{\aleph}(z, \mathsf{h}) - 1) = \mathsf{h}^n det(L(z) - \partial_z) + O(\mathsf{h}^{n+1})$$

in virtue of the expansion:

$$e^{-\mathsf{h}\partial_z} T^{\aleph}(z) - 1 = \mathsf{h}(L(z) - \partial_z) + O(\mathsf{h}^2).$$

Remark 2.2. *It should be noted that the independence of the quantum Hamiltonians directly follows from the independence of their classic limits, since the algebraic relations on the constructed operators in $U(\mathfrak{gl}_n)^{\otimes N}$ induces a nontrivial relation on their symbols. The maximality follows from the maximality on the classical level.*

2.3 Traditional solution methods

The traditional methods of solving quantum integrable system on finite scale are reduced to the Bethe ansatz method or the method of separated variables which in turn allow to express the condition on the quantum model spectrum in terms of the solutions of some system of algebraic equations or the monodromy properties of some Fuchsian system. Those methods do not suppose any way of solving the substituting problems. However there is quite rich material in solving quantum integrable systems in various limits.

Further we explain two basic methods in the case of the simplest Gaudin model.

2.3.1 Bethe ansatz

Let us consider the quantum $\mathfrak{sl}_2$ Gaudin model. The Lax operator in this case takes the form

$$L = \left(\begin{array}{cc} A(z) & B(z) \\ C(z) & -A(z) \end{array} \right) = \sum_i \frac{\Phi_i}{z - z_i},$$

where

$$\Phi_i = \left(\begin{array}{cc} h_i/2 & e_i \\ f_i & -h_i/2 \end{array} \right).$$

The quantum characteristic polynomial is a differential operator of the second order with values in the algebra of quantum observables:

$$det(L(z) - \partial_z) = \partial_z^2 - \frac{1}{2} \sum_i \frac{c_i^{(2)}}{(z - z_i)^2} - \sum_i \frac{H_i}{z - z_i}.$$

The Gaudin Hamiltonians are the residues

$$H_i = \sum_{i \neq j} \frac{h_i h_j/2 + e_i f_j + e_j f_i}{z_i - z_j}.$$

The coefficients at the poles of second order are also elements of the commutative subalgebra but of trivial nature - they are central in the quantum algebra.

The Bethe ansatz method was firstly proposed for the Heisenberg model but fits well for a wide class of systems. The Bethe ansatz method for the Gaudin model was realized in [9]. Let us observe the construction. We consider the $\mathfrak{sl}_2$ Gaudin model in fixed representation $V_\lambda = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_N}$ where V_{λ_i} are the finite dimensional irreducible representations of highest weights λ_i.

Lemma 2.3. *The vector*

$$\Omega = \prod_{j=1}^M C(\mu_j)|vac>$$

is the common eigenvector for the ensemble of Gaudin Hamiltonanians if the set of parameters μ_j (called the Bethe roots) satisfies the system of Bethe equations

$$-\frac{1}{2} \sum_i \frac{\lambda_i}{\mu_j - z_i} + \sum_{k \neq j} \frac{1}{\mu_j - \mu_k} = 0, \qquad j = 1, \ldots, M. \tag{2.9}$$

The eigenvalues of H_i on the vector Ω are expressed as follows

$$H_i^\Omega = -\lambda_i \left(\sum_j \frac{1}{z_i - \mu_j} - \frac{1}{2} \sum_{j \neq i} \frac{\lambda_j}{z_i - z_j} \right).$$

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Proof

In this case the quantum characteristic polynomial takes the form:

$$det(L(z) - \partial_z) = \partial_z^2 - A^2(z) - C(z)B(z) + A'(z) = \partial_z^2 - H(z).$$

The following commutation relations on the matrix elements of the Lax operator are also true:

$$[A(z), B(z)] = -B'(z), \qquad [A(z), C(u)] = \frac{1}{z-u}(C(z) - C(u)),$$

$$[A(z), C(z)] = C'(z), \qquad [B(z), C(u)] = \frac{2}{u-z}(A(z) - A(u)).$$

Using this relation and the condition:

$$H(z)|vac> = \left(\frac{1}{4}(\sum_i \frac{\lambda_i}{z - z_i})^2 - \frac{1}{2}\sum_i \frac{\lambda_i}{(z - z_i)^2} \right) |vac> = h_0(z)|vac>$$

we obtain:

$$
\begin{aligned}
H(z)\Omega &= \left(h_0(z) + 2\sum_{j=1}^{M} \frac{1}{\mu_j - z} A(z) + \sum_{j \neq k} \frac{1}{(\mu_j - z)(\mu_k - z)} \right) \Omega \\
&+ 2C(z) \sum_{j=1}^{M} \frac{1}{z - \mu_j} \prod_{l \neq j} C(\mu_l) \left(\sum_{k \neq j} \frac{1}{\mu_k - \mu_j} + A(\mu_j) \right).
\end{aligned}
$$

Let us remark that the Bethe equations can be rephrased in the form:

$$\sum_{k \neq j} \frac{1}{\mu_k - \mu_j} + A(\mu_j) = 0.$$

This proves the lemma.

2.3.2 Quantum separated variables

Let us consider the quantum $\mathfrak{sl}_2$ Gaudin model as in the previous section. An irreducible representation of this type can be realized as the quotient of the Verma module $\mathbb{C}[t_i]/t_i^{\lambda_i+1}$, such that the generators of $\mathfrak{sl}_2$ act as differential operators:

$$h^{(s)} = -2t_s \frac{\partial}{\partial t_s} + \lambda_s, \; e^{(s)} = -t_s \frac{\partial^2}{\partial t_s^2} + \lambda_s \frac{\partial}{\partial t_s}, \; f^{(s)} = t_s.$$

Let us explore the problem in the tensor product of the Verma modules which is realized in this case on the space of polynomials on N variables $\mathbb{C}[t_1, \ldots, t_N]$. Let us introduce the set of variables y_j, defined by the formula:

$$C(z) = C_0 \frac{\prod_j (z - y_j)}{\prod_i (z - z_i)}.$$

They are elements of some algebraic extension of the ring $\mathbb{C}[t_1, \ldots, t_N]$. Let us denote by the same symbols functions and operators of multiplication by those functions.

Let Ω be a common eigenvector for the Gaudin Hamiltonians in $\mathbb{C}[t_1, \ldots, t_N]$

$$H(z)\Omega = h(z)\Omega. \tag{2.10}$$

Considering both parts of 2.10 as rational functions on z and substituting $z = y_j$ from the left we obtain:

$$
\begin{aligned}
H(y_j) &= A^2(y_j) - A'(y_j) \\
&= \frac{1}{4}\sum_{i,k} \frac{1}{(y_j - z_i)(y_j - z_k)} h_i h_k + \frac{1}{2}\sum_k \frac{1}{(y_j - z_k)^2} h_k.
\end{aligned}
\tag{2.11}
$$

Using the definition of the separated variables let us express the partial derivatives:

$$\partial_{y_j} = \sum_k \frac{\partial t_k}{\partial y_j} \partial_{t_k} = \sum_k \frac{t_k}{y_j - z_k} \partial_{t_k}. \tag{2.12}$$

Substituting 2.12 in 2.11 we obtain:

$$\left(-\partial_{y_j} + \frac{1}{2} \sum_k \frac{\lambda_k}{y_j - z_k} \right)^2 \Omega = h(y_j)\Omega.$$

Hence the common eigenfunction for the Gaudin Hamiltonians factorizes, its dependence on y_j is separated:

$$\Omega = \prod_j \omega(y_j).$$

Each of the factors $\omega(z)$ is related to the solution for the Sturm-Liouville equation

$$(\partial_z^2 - h(z))\widetilde{\omega}(z) = 0$$

as follows:

$$\widetilde{\omega}(z) = \prod_i (z - z_i)^{-\lambda_i/2} \omega(z).$$

2.3.3 The monodromy of Fuchsian systems

The results of traditional separation of variables in quantum integrable systems discussed above demonstrate that the spectrum description is closely related with the families of Fuchsian equations obeying special monodromy properties. These properties are quite natural in the Heisenberg approach explained in [47], and correspond to existence of globally defined wave-functions.

In the considered $\mathfrak{sl}_2$ Gaudin model it was obtained that if Ω is a common Bethe eigenvector with values H_i^Ω then the equation

$$\left(\partial^2 - \frac{1}{4} \sum_i \frac{\lambda_i(\lambda_i + 2)}{(z - z_i)^2} - \sum_i \frac{H_i^\Omega}{z - z_i} \right) \Psi(z) = 0 \tag{2.13}$$

has a solution of the form

$$\Psi(z) = \prod_i (z - z_i)^{-\lambda_i/2} \prod_j (z - \mu_j),$$

where the set of parameters μ_j satisfy the system of Bethe equations.

This observation was generalized in [48]. Let us consider the quantum characteristic polynomial:

$$det(L(z) - \partial_z) = \partial_z^2 - \sum_i \frac{C_i^{(2)}}{(z - z_i)^2} - \sum_i \frac{H_i}{z - z_i}.$$

Let $\mathcal{H}$ be the algebra generated by the coefficients of the quantum characteristic polynomial. A character χ of the algebra $\mathcal{H}$ is called admissible if it takes values $\chi(C_i^{(2)}) = \frac{1}{4}(\lambda_i + 2)\lambda_i$ on central elements.

Theorem 2.4 ([48]). *There is a one-to-one correspondence between the set of "'admissible"' characters χ for those the differential equation*

$$\chi(det(L(z) - \partial_z))\Psi(z) = 0$$

has monodromy ± 1, and the set of common eigenvectors of the Gaudin model in the representation V_λ.

In contrast with the traditional Bethe ansatz and separation of variables methods this spectrum characterization can be generalized to the $\mathfrak{sl}_n$ case.

2.4 Elliptic case

It turns out that the elements of the algebraic-geometric part of the quantization problem can the constructed also in the case of the Elliptic Gaudin model: the quantum spectral curve and the quantum separated variables. Let us remark that the elliptic Gaudin model can be obtained in generalized Hitchin system framework. This corresponds to the moduli space of holomorphic semistable bundles with the trivial determinant bundle over an elliptic curve with a set of marked points. A modified algebraic structure is applicable for this problem, namely the dynamical $\mathfrak{gl}_n$ RLL equation corresponding to the "'elliptic quantum group"' $E_{\tau,\hbar}(\mathfrak{gl}_n)$, defined in [50].

The commutativity in this case is meant modulo the Cartan subalgebra. To obtain an integrable system one should restrict the constructed family to the zero weight subspace with respect to the diagonal action of the Lie algebra.

2.4.1 The notations

Let us define the so-called odd Riemann θ-functions on an elliptic curve. Let $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ be the parameter of elliptic curve $\mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ - is the periods lattice. The odd θ-function $\theta(u) = -\theta(-u)$ is defined by the relations

$$\theta(u+1) = -\theta(u), \qquad\qquad \theta(u+\tau) = -e^{-2\pi i u - \pi i \tau}\theta(u), \qquad\qquad \theta'(0) = 1. \qquad (2.14)$$

Let us also introduce some matrix notations. Let

$$T = \sum_j t_j \cdot a_{1,j} \otimes \ldots \otimes a_{N,j}$$

be a tensor over an algebra $\mathfrak{R}$, where $t_j \in \mathfrak{R}$ and $a_{i,j}$ are elements of the space $\operatorname{End}\mathbb{C}^n$. Then the notation $T^{(k_1,\ldots,k_N)}$ corresponds to the following element of $\mathfrak{R} \otimes (\operatorname{End}C^n)^{\otimes M}$ for numbers $M \geqslant N$:

$$T^{(k_1,\ldots,k_N)} = \sum_j t_j \cdot 1 \otimes \ldots \otimes a_{1,j} \otimes \ldots \otimes a_{N,j} \otimes \ldots \otimes 1.$$

Here each element $a_{i,j}$ is placed in the k_i-th tensor component, the numbers k_i are pairwise different and the following condition fulfills $1 \leqslant k_i \leqslant M$.

We need also the notation: let $F(\lambda) = F(\lambda_1, \ldots, \lambda_n)$ be a function on n parameters λ_k, taking values in an algebra $\mathfrak{R}$: i.e. $F \colon \mathbb{C}^n \to \mathfrak{R}$. In this case we define special shifts

$$
\begin{aligned}
F(\lambda + P) &= F(\lambda_1 + P_1, \ldots, \lambda_n + P_n) \\
&= \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1+\ldots+i_n} F(\lambda_1,\ldots,\lambda_n)}{\partial\lambda_1^{i_1} \cdots \partial\lambda_n^{i_n}} P_1^{i_1} \cdots P_n^{i_n}
\end{aligned}
\qquad (2.15)
$$

for some set $P = (P_1, \ldots, P_n)$, $P_k \in \mathfrak{R}$. We do not discuss here the convergency questions, in our context all such expressions will be well defined.

2.4.2 Felder algebra

Let us introduce the notion of the elliptic L-operator, corresponding to the Felder R-matrix.

We use the notations $\{e_i\}$, $\{E_{ij}\}$ from the section 1.4.2. Let $\mathfrak{h}$ be a commutative algebra of dimension n. In [50] it was constructed an element of $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$, meromorphly depending on the parameter u and n dynamical parameters $\lambda_1, \ldots, \lambda_n$:

$$R(u;\lambda) = R(u;\lambda_1,\ldots,\lambda_n) = \frac{\theta(u+\hbar)}{\theta(u)} \sum_{i=1}^n E_{ii} \otimes E_{ii} +$$

$$+ \sum_{i \neq j} \Big(\frac{\theta(\lambda_{ij}+\hbar)}{\theta(\lambda_{ij})} E_{ii} \otimes E_{jj} + \frac{\theta(u-\lambda_{ij})\theta(\hbar)}{\theta(u)\theta(-\lambda_{ij})} E_{ij} \otimes E_{ji} \Big), \quad (2.16)$$

where $\lambda_{ij} = \lambda_i - \lambda_j$. This element is called the dynamical Felder R-matrix. This satisfy the dynamical Yang-Baxter equation

$$R^{(12)}(u_1 - u_2; \lambda)R^{(13)}(u_1 - u_3; \lambda + \hbar E^{(2)})R^{(23)}(u_2 - u_3; \lambda) =$$
$$= R^{(23)}(u_2 - u_3; \lambda + \hbar E^{(1)})R^{(13)}(u_1 - u_3; \lambda)R^{(12)}(u_1 - u_2; \lambda + \hbar E^{(3)}),$$

and the additional conditions

$$R^{(21)}(-u; \lambda)R^{(12)}(u; \lambda) = \frac{\theta(u + \hbar)\theta(u - \hbar)}{\theta(u)^2},$$
$$(E_{ii}^{(1)} + E_{ii}^{(2)})R(u; \lambda) = R(u; \lambda)(E_{ii}^{(1)} + E_{ii}^{(2)}),$$
$$(\widehat{D}_\lambda^{(1)} + \widehat{D}_\lambda^{(2)})R(u; \lambda) = R(u; \lambda)(\widehat{D}_\lambda^{(1)} + \widehat{D}_\lambda^{(2)}),$$

where

$$\widehat{D}_\lambda = \sum_{k=1}^n E_{kk} \frac{\partial}{\partial \lambda_k}, \quad \widehat{D}_\lambda^{(i)} = \sum_{k=1}^n E_{kk}^{(i)} \frac{\partial}{\partial \lambda_k}.$$

We should mention that the expression λ in formulas above denotes a vector $\lambda_1, \ldots, \lambda_n$, and the expression $\lambda + \hbar E^{(s)}$ implies a shift of the type 2.15 with the parameters values $P_i = \hbar E_{ii}^{(s)}$.

Let $\mathfrak{R}$ be a $\mathbb{C}[[\hbar]]$-algebra, $L(u; \lambda)$ an invertible $n \times n$ matrix over $\mathfrak{R}$ depending on the spectral parameter u and n dynamical parameters $\lambda_1, \ldots, \lambda_n$. Let $h_1, \ldots, h_n$ be a set of pairwise commuting elements of $\mathfrak{R}$. $L(u; \lambda)$ is called an elliptic dynamical L-operator corresponding to the set of Cartan elements h_k if $L(u; \lambda)$ satisfies the dynamical RLL relation

$$R^{(12)}(u - v; \lambda)L^{(1)}(u; \lambda + \hbar E^{(2)})L^{(2)}(v; \lambda)$$
$$= L^{(2)}(v; \lambda + \hbar E^{(1)})L^{(1)}(u; \lambda)R^{(12)}(u - v; \lambda + \hbar h), \tag{2.17}$$

and a condition of the form

$$(E_{ii} + h_i)L(u; \lambda) = L(u; \lambda)(E_{ii} + h_i).$$

Let us introduce an equivalent but more symmetric form of RLL relations. For an L-operator we define an expression:

$$L_D(u) = e^{-\hbar \widehat{D}_\lambda} L(u; \lambda). \tag{2.18}$$

The equation (2.17) can be rewritten in new notations as follows:

$$R^{(12)}(u - v; \lambda)L_D^{(1)}(u)L_D^{(2)}(v) = L_D^{(2)}(v)L_D^{(1)}(u)R^{(12)}(u - v; \lambda + \hbar h). \tag{2.19}$$

The next lemma plays the role analogous to the fusion method in the rational case, namely this describes a method of elliptic L-operators construction.

Lemma 2.5. *If $L_1(u; \lambda) \in \text{End}(\mathbb{C}^n) \otimes \mathfrak{R}_1$ and $L_2(u; \lambda) \in \text{End}(\mathbb{C}^n) \otimes \mathfrak{R}_2$ are two elliptic dynamical L-operators with respect to two sets of Cartan elements: $h^1 = (h_1^1, \ldots, h_n^1)$ and $h^2 = (h_1^2, \ldots, h_n^2)$, then the product $L_2(u; \lambda)L_1(u; \lambda + \hbar h^2) \in \text{End}(\mathbb{C}^n) \otimes \mathfrak{R}_1 \otimes \mathfrak{R}_2$ is also an elliptic dynamical L-operator with respect to the set $h = h^1 + h^2 = (h_1^1 + h_1^2, \ldots, h_n^1 + h_n^2)$. Hence, if $L_1(u; \lambda), \ldots, L_m(u; \lambda)$ are elliptic dynamical L-operators with the sets of Cartan elements $h^1, \ldots, h^m$, then the matrix*

$$\overleftarrow{\prod_{m \geqslant j \geqslant 1}} L_j\big(u; \lambda + \hbar \sum_{l=j+1}^m h^l\big) \tag{2.20}$$

is also an elliptic dynamical L-operator with the following set of Cartan elements $h = \sum_{i=1}^m h^i$.

Remark 2.3. *The arrow in the product notation above denotes the order of multipliers with growing indexes: for example, the expression $\overleftarrow{\prod}_{3 \geqslant i \geqslant 1} A_i$ means $A_3 A_2 A_1$.*

The main example of elliptic dynamical L-operator is given by the Felder R-matrix: $L(u) = R(u - v; \lambda)$. In this case the second space $\mathrm{End}(\mathbb{C}^n)$ takes the role of the algebra $\mathfrak{R}$. Here v is a complex number and the Cartan elements coincide with the diagonal matrices $h_k = E_{kk}^{(2)}$. Lemma 2.5 allows to generalize this example: let $v_1, \ldots, v_m$ be a set of complex numbers, then the matrix

$$\mathbb{R}^{(0)}(u; \{v_j\}; \lambda) = \overleftarrow{\prod_{m \geqslant j \geqslant 1}} R^{(0j)}\Big(u - v_j; \lambda + \hbar \sum_{l=j+1}^{m} E^{(l)}\Big) \tag{2.21}$$

is a dynamical elliptic L-operator with the Cartan elements $h_k = \sum\limits_{l=1}^{m} E_{kk}^{(l)}$.

A more general class of dynamical elliptic L-operators is related with the so-called *small elliptic quantum group* $e_{\tau,\hbar}(\mathfrak{gl}_n)$ constructed in [51]. This represents a $\mathbb{C}[[\hbar]]((\lambda_1, \ldots, \lambda_n))$-algebra generated by $\tilde{t}_{ij}$ and h_k with relations

$$\tilde{t}_{ij} h_k = (h_k - \delta_{ik} + \delta_{jk}) \tilde{t}_{ij},$$
$$t_{ij} \lambda_k - (\lambda_k - \hbar \delta_{ik}) t_{ij} = 0,$$
$$t_{ij} t_{ik} - t_{ik} t_{ij} = 0, \tag{2.22}$$
$$t_{ik} t_{jk} - \frac{\theta(\lambda_{ij}^{\{1\}} + \hbar)}{\theta(\lambda_{ij}^{\{1\}} - \hbar)} t_{jk} t_{ik} = 0, \qquad i \neq j,$$
$$\frac{\theta(\lambda_{jl}^{\{2\}} + \hbar)}{\theta(\lambda_{jl}^{\{2\}})} t_{ij} t_{kl} - \frac{\theta(\lambda_{ik}^{\{1\}} + \hbar)}{\theta(\lambda_{ik}^{\{1\}})} t_{kl} t_{ij} - \frac{\theta(\lambda_{ik}^{\{1\}} + \lambda_{jl}^{\{2\}})\theta(\hbar)}{\theta(\lambda_{ik}^{\{1\}})\theta(\lambda_{jl}^{\{2\}})} t_{il} t_{kj} = 0,$$

with $i \neq k$, $j \neq l$, where $t_{ij} = \delta_{ij} + \hbar \tilde{t}_{ij}$, $\lambda_{ij}^{\{1\}} = \lambda_i - \lambda_j$, $\lambda_{ij}^{\{2\}} = \lambda_i - \lambda_j - \hbar h_i + \hbar h_j$, it is also supposed that $h_1, \ldots, h_k, \lambda_1, \ldots, \lambda_k$ commute. One constructs a generating function for these generators $T(-u)$

$$T_{ij}(-u) = \theta(-u + \lambda_{ij} - \hbar h_i) t_{ji}. \tag{2.23}$$

Representing this matrix in the form

$$T(-u) = \theta(-u) e^{-\hbar \sum_{k=0}^{n}(h_k + E_{kk})\partial_{\lambda_k}} L_0(u; \lambda) e^{\hbar \sum_{k=0}^{n} h_k \partial_{\lambda_k}} \tag{2.24}$$

we obtain a dynamical elliptic L-operator $L_0(u; \lambda)$ for the algebra $\mathfrak{T} = e_{\tau,\hbar}(\mathfrak{gl}_n)[[\partial_\lambda]]$ with the Cartan elements $h = (h_1, \ldots, h_n)$, where $\mathbb{C}[[\partial_\lambda]] = \mathbb{C}[[\partial_{\lambda_1}, \ldots, \partial_{\lambda_n}]]$. The elements $\partial_{\lambda_k} = \frac{\partial}{\partial \lambda_k}$ commute with h_i and do not commute with $\tilde{t}_{ij}$.

2.4.3 Commutative algebra

Let us consider a dynamical elliptic L-operator $L(u; \lambda)$ with a set of Cartan elements h_k. This function takes values in the algebra $\mathrm{End}\, \mathbb{C}^n \otimes \mathfrak{R}$.

Let us introduce the operators

$$\mathbb{L}^{[m,N]}(\{u_i\}; \lambda) = e^{-\hbar \widehat{D}_\lambda^{(m+1)}} L^{(m+1)}(u_{m+1}; \lambda) \cdots e^{-\hbar \widehat{D}_\lambda^{(N)}} L^{(N)}(u_N; \lambda), \tag{2.25}$$

where $m < N$. Let us consider a particular case with the parameters values $u_i = u + \hbar(i - 1)$,

$$\mathbb{L}^{[a,b]}(u; \lambda) = \mathbb{L}^{[a,b]}(\{u_i = u + \hbar(i - a - 1)\}; \lambda)$$

for $a < b$. Let $\mathbb{A}_n = \mathbb{C}((\lambda_1, \ldots, \lambda_n))$ be the completed function space. The operators $\widehat{D}_\lambda$ act on the space $\mathbb{A}_n \otimes \mathbb{C}^n$, in turn the operators $\mathbb{L}^{[a,b]}(u; \lambda)$ act from $\mathbb{A}_n \otimes (\mathbb{C}^n)^{\otimes(b-a)}$ to the space $\mathbb{A}_n \otimes (\mathbb{C}^n)^{\otimes(b-a)} \otimes \mathfrak{R}$: fixing u we obtain $\mathbb{L}^{[a,b]}(u; \lambda) \in \mathrm{End}(\mathbb{C}^n)^{\otimes(b-a)} \otimes \mathfrak{A}_n$, where $\mathfrak{A}_n = \mathbb{A}_n[e^{\pm \hbar \partial_\lambda}] \otimes \mathfrak{R}$. Let us consider a subalgebra $\mathfrak{h} \subset \mathfrak{R} \subset \mathfrak{A}_n$ generated by the elements h_k and its normalizer $\mathfrak{A}_n$:

$$\mathfrak{N}_n = \mathfrak{N}_{\mathfrak{A}_n}(\mathfrak{h}) = \{x \in \mathfrak{A}_n \mid \mathfrak{h}x \subset \mathfrak{A}_n \mathfrak{h}\}. \tag{2.26}$$

Let us remark that $\mathfrak{A}_n \mathfrak{h}$ is a two-sided ideal in $\mathfrak{N}_n$. In [49] the following statement is proved

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Theorem 2.6. *Let us define $\mathfrak{A}_n$-valued functions*

$$t_m(u) = tr\big(A^{[0,m]}\mathbb{L}^{[0,m]}(u;\lambda)\big), \tag{2.27}$$

where we suppose the trace operation over m spaces $\mathbb{C}^n$. These expressions commute with the Cartan elements h_k:

$$h_k t_m(u) = t_m(u) h_k. \tag{2.28}$$

Hence they are elements of the subalgebra $\mathfrak{N}_n$. Moreover these generators commute modulo the ideal $\mathfrak{A}_n\mathfrak{h} \subset \mathfrak{N}_n$:

$$t_m(u)t_s(v) = t_s(v)t_m(u) \qquad\qquad \mathrm{mod}\ \mathfrak{A}_n\mathfrak{h}. \tag{2.29}$$

2.4.4 Characteristic polynomial

As in the rational case the generators $t_m(u)$ can be organized into a generating function called the quantum characteristic polynomial. This generating function is constructed as a "'determinant"' of the corresponding L-operator.

Proposition 2.7. *Let us consider the matrix $M = e^{-\hbar\widehat{D}_\lambda}L(u;\lambda)e^{\hbar\frac{\partial}{\partial u}}$. Then the determinant of $1 - M$ generate the family $t_m(u)$ in the following sense:*

$$P(u, e^{\hbar\partial_u}) = det(1 - e^{-\hbar\widehat{D}_\lambda}L(u;\lambda)e^{\hbar\frac{\partial}{\partial u}}) = \sum_{m=0}^{n}(-1)^m t_m(u)e^{m\hbar\frac{\partial}{\partial u}}, \tag{2.30}$$

where $t_0(u) = 1$. This property induces the commutativity of the quantum characteristic polynomial with elements h_k, and the pairwise commutativity modulo $\mathfrak{A}_n\mathfrak{h}$ of the generating functions:

$$[P(u, e^{\hbar\partial_u}), h_k] = 0, \qquad\qquad [P(u, e^{\hbar\partial_u}), P(v, e^{\hbar\partial_v})] = 0 \quad \mathrm{mod}\ \mathfrak{A}_n\mathfrak{h}. \tag{2.31}$$

2.4.5 The limit and the Gaudin model

Let us consider degenerated elliptic dynamical RLL relations at $\hbar \to 0$. This limit describes the elliptic quantum Gaudin model. To do this we use a shift of the L-operator. The limit of the generating function for the generic family gives the generating function for the Hamiltonians of the elliptic Gaudin model. The result obtained generalizes the works [52],[53].

Let $L(u;\lambda)$ be a dynamical elliptic L-operator of the form

$$L(u;\lambda) = 1 + \hbar\Lambda(u;\lambda) + o(\hbar), \tag{2.32}$$

those matrix elements are elements of the algebra $\mathfrak{R}_0 = \mathfrak{R}/\hbar\mathfrak{R}$. The matrix $\Lambda(u;\lambda)$ is called a classical dynamical elliptic L-operator. This satisfy the rLL-relations

$$[\Lambda^{(1)}(u;\lambda) - \widehat{D}_\lambda^{(1)}, \Lambda^{(2)}(v;\lambda) - \widehat{D}_\lambda^{(2)}] - \sum_{k=1}^{n}h_k\frac{\partial}{\partial\lambda}r(u-v;\lambda) =$$

$$= [\Lambda^{(1)}(u;\lambda) + \Lambda^{(2)}(v;\lambda), r(u-v;\lambda)] \quad (2.33)$$

with the classical dynamical elliptic r-matrix

$$\begin{aligned}
r(u;\lambda) &= \frac{\theta'(u)}{\theta(u)}\sum_{i=1}^{n}E_{ii}\otimes E_{ii} \\
&+ \sum_{i\neq j}\Big(\frac{\theta'(\lambda_{ij})}{\theta(\lambda_{ij})}E_{ii}\otimes E_{jj} + \frac{\theta(u-\lambda_{ij})}{\theta(u)\theta(-\lambda_{ij})}E_{ij}\otimes E_{ji}\Big).
\end{aligned} \tag{2.34}$$

The matrix (2.34) is related with the Felder R-matrix (2.16) by the formula

$$R(u;\lambda) = 1 + \hbar r(u;\lambda) + o(\hbar). \tag{2.35}$$

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Theorem 2.8. *Let $\mathcal{A}_n = \mathfrak{R}_0 \otimes \mathbb{A}_n[\partial_\lambda]$ $\mathcal{N}_n = \mathfrak{N}_{\mathcal{A}_n}(\mathfrak{h}) = \{x \in \mathcal{A}_n \mid \mathfrak{h}x \subset \mathcal{A}_n\mathfrak{h}\}$, where $\mathbb{A}_n = \mathbb{C}((\lambda_1,\ldots,\lambda_n))$. Let us define a set of $\mathcal{N}_n$-valued functions $s_m(u)$ by the formula*

$$Q(u, \partial_u) = det\Big(\frac{\partial}{\partial u} - \widehat{D}_\lambda + \Lambda(u; \lambda)\Big) = \sum_{m=0}^{n} s_m(u)\Big(\frac{\partial}{\partial u}\Big)^{n-m}, \tag{2.36}$$

where $s_0(u) = 1$. They commute with the Cartan elements h_k:

$$h_k s_m(u) = s_m(u) h_k \tag{2.37}$$

and moreover pairwise commute modulo $\mathcal{A}_n\mathfrak{h}$:

$$s_m(u)s_l(v) = s_l(v)s_m(u) \mod \mathcal{A}_n\mathfrak{h}. \tag{2.38}$$

The values of the functions $s_1(u)$, $s_2(u)$, ..., $s_n(u)$ generate a commutative subalgebra in $\mathcal{N}_n$ on the level $h_k = 0$. This means that the images of these elements with respect to the canonical homomorphism $\mathcal{N}_n \to \mathcal{N}_n/\mathcal{A}_n\mathfrak{h}$ pairwise commute.

The quantum elliptic Gaudin model is defined with the help of the Lax operator

$$\Lambda_{ij}(u; \lambda) = e_{ji}(u; \lambda), \qquad \Lambda_{ii}(u; \lambda) = e_{ii}(u; \lambda) + \sum_{k \neq i} \frac{\theta'(\lambda_{ik})}{\theta(\lambda_{ik})} h_k, \tag{2.39}$$

those coefficients are expressed by the formulas:

$$e_{ii}(u) = \frac{\theta'(u-z)}{\theta(u-z)} e_{ii} = \sum_{m \geqslant 0} \frac{(-1)^m}{m!} \Big(\frac{\theta'(u)}{\theta(u)}\Big)^{(m)} e_{ii} z^m, \tag{2.40}$$

$$e_{ij}(u; \lambda) = \frac{\theta(u-z+\lambda_{ij})}{\theta(u-z)\theta(\lambda_{ij})} e_{ij} = \sum_{m \geqslant 0} \frac{(-1)^m}{m!} \Big(\frac{\theta(u+\lambda_{ij})}{\theta(u)\theta(\lambda_{ij})}\Big)^{(m)} e_{ij} z^m. \tag{2.41}$$

An analog of the evaluation representation is the homomorphism to the small elliptic quantum group defined by the generating function (2.24). Let us consider an expansion on the parameter $\hbar$ of the dynamical L-operator corresponding to the tensor power of the small elliptic group. It turns out that the coefficient at $\hbar$ of this expansion coincides with the elliptic Gaudin model L-operator.

2.4.6 The explicit form of the $\mathfrak{sl}_2$ elliptic Gaudin model

The L-operator of the elliptic $\mathfrak{sl}_2$ Gaudin model considered in [52, 53, 54] has the form

$$\Lambda(u; \lambda) = \begin{pmatrix} h(u)/2 & f_\lambda(u) \\ e_\lambda(u) & -h(u)/2 \end{pmatrix}, \tag{2.42}$$

where $\lambda = \lambda_{12} = \lambda_1 - \lambda_2$ and the currents are expressed by the formulas

$$h(u) = e_{11}(u) - e_{22}(u) = \sum_{s=1}^{N} \frac{\theta'(u-v_s)}{\theta(u-v_s)}(e_{11}^{(s)} - e_{22}^{(s)}),$$

$$e_\lambda(u) = e_{12}(u; \lambda) = \sum_{s=1}^{N} \frac{\theta(u-v_s+\lambda)}{\theta(u-v_s)\theta(\lambda)} e_{12}^{(s)},$$

$$f_\lambda(u) = e_{21}(u; \lambda) = \sum_{s=1}^{N} \frac{\theta(u-v_s-\lambda)}{\theta(u-v_s)\theta(-\lambda)} e_{21}^{(s)}.$$

The fact that the L-operator depends only on the difference $\lambda = \lambda_1 - \lambda_2$ allows to restrict the generating function of the commutative subalgebra $Q(u, \partial_u)$ to the space $\mathbb{A} = \{a \in \mathbb{A}_2 \mid (\partial_{\lambda_1} + \partial_{\lambda_1})a = 0\} \subset \mathbb{A}_2$ coinciding with $\mathbb{C}((\lambda_{12}))$. Let $\mathcal{A} = \mathfrak{R}_0 \otimes \mathbb{A}[\partial_\lambda]$ then the values of $s_m(u)$ are elements of $\mathcal{N} = \mathfrak{N}_\mathcal{A}(\mathfrak{h}) = \{x \in$

$\mathcal{A} \mid hx \in \mathcal{A}h\}$. In virtue of the representation $\rho\colon h_1 + h_2 \to 0$ the operator $\widehat{D}_\lambda$ has the form $H\partial_\lambda$, where $H = E_{11} - E_{22}$.

Let us find the quantum characteristic polynomial in this case:

$$Q(u, \partial_u) = det\Big(\frac{\partial}{\partial u} - \widehat{D}_\lambda + \widetilde{\Lambda}(u; \lambda) - \frac{\theta'(\lambda)}{\theta(\lambda)}\frac{h}{2}\Big) =$$

$$= det\begin{pmatrix} \frac{\partial}{\partial u} - \partial_\lambda + h^+(u)/2 - \frac{\theta'(\lambda)}{\theta(\lambda)}h/2 & f_\lambda^+(u) \\ e_\lambda^+(u) & \frac{\partial}{\partial u} + \partial_\lambda - h^+(u)/2 - \frac{\theta'(\lambda)}{\theta(\lambda)}h/2 \end{pmatrix}$$

$$= \Big(\frac{\partial}{\partial u}\Big)^2 - \frac{\theta'(\lambda)}{\theta(\lambda)}h\frac{\partial}{\partial u} - S_\lambda(u), \tag{2.43}$$

where $h = h_1 - h_2$. $S_\lambda(u)$ is an $\mathcal{N}$-valued function

$$S_\lambda(u) = \big(\partial_\lambda - h(u)/2\big)^2 + \partial_u h(u)/2 + e_\lambda(u)f_\lambda(u) \qquad \text{mod } \mathcal{A}h.$$

The commutativity condition can be formulated in terms of this generating function as follows:

$$[S_\lambda(u), S_\lambda(v)] = 0 \mod \mathcal{A}h.$$

Using the commutation relations

$$[e_\lambda^+(u), f_\lambda^+(u)] = -\frac{\partial}{\partial u}h^+(u) + \big(\frac{\theta'(\lambda)}{\theta(\lambda)}\big)'h$$

one can simplify this generating function:

$$S_\lambda(u) = \big(\partial_\lambda - h(u)/2\big)^2 + \big(e_\lambda(u)f_\lambda(u) + f_\lambda(u)e_\lambda(u)\big)/2 \qquad \text{mod } \mathcal{A}h. \tag{2.44}$$

3   Solution for quantum integrable systems

As was mentioned above the traditional methods of solving quantum integrable systems on the finite scale in some cases allow to solve the Hamiltonian diagonalization problem in terms of solutions of a system of algebraic equations (the Bethe system). However, the system of equations itself, in cases where it can be deduced, turns out to be quite complicated and hypothetically prevents algebraic solutions. In this section we use an equivalent formulation for quantum eigenproblem in terms of Fuchsian systems with special monodromy representation. In turn the construction of relevant Fuchsian systems uses the quantum characteristic polynomial of a model. This observation also emphasizes the quantum characteristic polynomial among others generating function for the commutative subalgebra.

3.1   Monodromic formulation

3.1.1   A scalar and a matrix Fuchsian equations

Consider a Fuchsian system defined by a connection in trivial bundle of rank 2 on the disk with punctures:

$$A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \sum_{i=1}^{k} \frac{A_i}{z - z_i} \tag{3.1}$$

with residues satisfying the conditions:

$$Tr(A_i) = 0; \quad Det(A_i) = -d_i^2; \quad \sum_i A_i = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}. \tag{3.2}$$

The Fuchsian system is written by the equation

$$(\partial_z - A(z))\Psi(z) = 0. \tag{3.3}$$

The components of this system may be represented as follows

$$\begin{aligned} \psi_1' &= a_{11}\psi_1 + a_{12}\psi_2, \\ \psi_2' &= a_{21}\psi_1 + a_{22}\psi_2. \end{aligned}$$

The first vector component satisfies the second order equation

$$\psi_1'' = (a_{12}'/a_{12})\psi_1' + u\psi_1,$$

where

$$u = a_{11}' + a_{11}^2 - a_{11}(a_{12}'/a_{12}) + a_{12}a_{21}.$$

With the following variable change: $\Phi = \psi_1/\chi$, where $\chi = \sqrt{a_{12}}$, we obtain the equation

$$\Phi'' + U\Phi = 0,$$

those potential is defined by the formula

$$U = \chi''/\chi - (a_{12}'/a_{12})\chi'/\chi - u. \tag{3.4}$$

Introducing the expression for χ to U we obtain:

$$U = \frac{1}{2}\left(\frac{a_{12}'}{a_{12}}\right)' - \frac{1}{4}\left(\frac{a_{12}'}{a_{12}}\right)^2 + a_{11}\frac{a_{12}'}{a_{12}} - a_{11}' - a_{11}^2 - a_{12}a_{21}. \tag{3.5}$$

Let us suppose that $a_{12}(z)$ has no multiple poles

$$a_{12}(z) = c\frac{\prod_{j=1}^{k-2}(z - w_j)}{\prod_{i=1}^{k}(z - z_i)}.$$

We should note that the number of zeros agrees with the normalization (3.2). The expression for the logarithmic derivative can be simplified:

$$\frac{a_{12}'}{a_{12}} = \sum_{j=1}^{k-2}\frac{1}{z - w_j} - \sum_{i=1}^{k}\frac{1}{z - z_i}. \tag{3.6}$$

The potensial U taked the form

$$U = \sum_{j=1}^{k-2}\frac{-3/4}{(z - w_j)^2} + \sum_{i=1}^{k}\frac{1/4 + \det A_i}{(z - z_i)^2} + \sum_{j=1}^{k-2}\frac{H_{w_j}}{z - w_j} + \sum_{i=1}^{k}\frac{H_{z_i}}{z - z_i}, \tag{3.7}$$

in which

$$\begin{aligned} H_{w_j} &= a_{11}(w_j) + \frac{1}{2}\left(\sum_{i\neq j}\frac{1}{w_j - w_i} - \sum_i\frac{1}{w_j - z_i}\right); \\ H_{z_i} &= \left(\frac{1}{2} + a_{11}^i\right)\sum_j\frac{1}{z_i - w_j} - \sum_{j\neq i}\frac{Tr(A_i A_j) + a_{11}^i + a_{11}^j + 1/2}{z_i - z_j}. \end{aligned}$$

Let us remark that the coefficients at $(z - z_i)^{-2}$ take values

$$1/4 + \det A_i = (1/2 - d_i)(1/2 + d_i). \tag{3.8}$$

In what follows we identify these factors with the values of the quadratic Casimir elements of the Lie algebras $\mathfrak{sl}_2$ in the representations of highest weights λ_i ($\lambda_i = 2d_i - 1$ in our case).

3.1.2 Dual equation

As was shown in previous calculations, the matrix form connection leads to the Sturm-Liouville operator with additional poles at points w_j. A consideration of the second vector component of a solution of the matrix equation Ψ_2 leads to another scalar differential operator with poles at points z_i and additional points $\widetilde{w}_j$, determined by the formula

$$a_{21}(z) = \widetilde{c}\frac{\prod_{j=1}^{k-2}(z - \widetilde{w}_j)}{\prod_{i=1}^{k}(z - z_i)}.$$

Let us call the corresponding Sturm-Liouville operator

$$\partial_z^2 - \widetilde{U}$$

the dual $\mathfrak{sl}_2$-oper. In this case, the potential is expressed by the formula

$$\widetilde{U} = \sum_{j=1}^{k-2}\frac{-3/4}{(z - \widetilde{w}_j)^2} + \sum_{i=1}^{k}\frac{1/4 + detA_i}{(z - z_i)^2} + \sum_{j=1}^{k-2}\frac{H_{\widetilde{w}_j}}{z - \widetilde{w}_j} + \sum_{i=1}^{k}\frac{\widetilde{H}_{z_i}}{z - z_i}. \tag{3.9}$$

3.1.3 Backup

In this section we construct an inverse map, namely for a Sturm-Liouville operator that has trivial monodromy we construct a rank 2 connection of the form (3.3) with the monodromy representation in the subgroup $\mathbb{Z}/2\mathbb{Z} \subset GL(2)$ of scalar matrices ± 1.

Let us consider an ansatz for the solution of the matrix linear equation (3.3)

$$(\partial_z - A(z))\Psi = 0$$

of the type

$$\psi_l = \prod_{i=1}^{k}(z - z_i)^{-s_i}\phi_l(z), \qquad l = 1, 2; \tag{3.10}$$

which satisfy

$$\phi_1 = \prod_{j=1}^{M}(z - \gamma_j),$$

$$\phi_2/\phi_1 = \sum_{j=1}^{M}\frac{\alpha_j}{z - \gamma_j}. \tag{3.11}$$

Let us rewrite the system (3.3) taking into account the new parameterization (3.11)

$$\partial_z\psi_1/\psi_1 = a_{11} + a_{12}\phi_2/\phi_1, \tag{3.12}$$

$$(\partial_z\psi_1/\psi_1)(\phi_2/\phi_1) + \partial_z(\phi_2/\phi_1) = a_{21} + a_{22}\phi_2/\phi_1. \tag{3.13}$$

Let us represent these equations more precisely:

$$-\sum_i\frac{s_i}{z - z_i} + \sum_j\frac{1}{z - \gamma_j} = \sum_i\frac{a_{11}^i}{z - z_i} + \sum_i\frac{a_{12}^i}{z - z_i}\sum_j\frac{\alpha_j}{z - \gamma_j}, \tag{3.14}$$

$$\left(-\sum_i\frac{s_i}{z - z_i} + \sum_j\frac{1}{z - \gamma_j}\right)\sum_j\frac{\alpha_j}{z - \gamma_j} - \sum_j\frac{\alpha_j}{(z - \gamma_j)^2} =$$

$$\sum_i\frac{a_{21}^i}{z - z_i} - \sum_i\frac{a_{11}^i}{z - z_i}\sum_j\frac{\alpha_j}{z - \gamma_j}. \tag{3.15}$$

The comparison of residues of both part of (3.14), (3.15) at points $z = z_i$ gives:

$$-s_i \;=\; a_{11}^i + a_{12}^i \sum_j \frac{\alpha_j}{z_i - \gamma_j}, \tag{3.16}$$

$$-\sum_j \frac{\alpha_j}{z_i - \gamma_j} s_i \;=\; a_{21}^i - a_{11}^i \sum_j \frac{\alpha_j}{z_i - \gamma_j}. \tag{3.17}$$

These equations coupled with the condition of zero trace $a_{11}^i + a_{22}^i = 0$ lead to a condition that s_i must be one of the eigenvalues of A_i, in particular, can be adopted as $s_i = d_i$. Let us consider the behavior at the poles $z = \gamma_j$. Let us note that the second order poles of the equation (3.15) at these points cancel. Calculating residues of both sides of the equations (3.14) and (3.15) we obtain

$$1 \;=\; \alpha_j \sum_i \frac{a_{12}^i}{\gamma_j - z_i}, \tag{3.18}$$

$$\alpha_j \left(-\sum_i \frac{s_i}{\gamma_j - z_i} + \sum_{i \neq j} \frac{1}{\gamma_j - \gamma_i} \right) + \sum_{i \neq j} \frac{\alpha_i}{\gamma_j - \gamma_i} \;=\; -\alpha_j \sum_i \frac{a_{11}^i}{\gamma_j - z_i}. \tag{3.19}$$

Let us recall that one of the normalization condition controls the diagonal form of the residue at ∞

$$\sum_{i=1}^k a_{12}^i \;=\; 0, \tag{3.20}$$

$$\sum_{i=1}^k a_{21}^i \;=\; 0. \tag{3.21}$$

We also note that the choice of Sturm-Liouville operator poles captures zeros of the rational function $a_{12}(z)$, which is determined up to a constant:

$$a_{12}(z) = c \frac{\prod_{j=1}^{k-2}(z - w_j)}{\prod_{i=1}^k (z - z_i)}.$$

Then the condition (3.20) will be satisfied automatically. The coefficients a_{12}^i are expressed by the formula

$$a_{12}^i = c \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)}. \tag{3.22}$$

The coefficients a_{11}^i are expressed by the following formula in virtue of (3.16)

$$a_{11}^i = -s_i - c \frac{\prod_{j=1}(z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)} \sum_l \frac{\alpha_l}{z_i - \gamma_l}. \tag{3.23}$$

Let us substitute the expressions for a_{12}^i and a_{11}^i to the equations (3.18), (3.19). Then expressing α_j from the first and substituting to the second we obtain:

$$- \sum_k \frac{2s_k}{\gamma_j - z_k} + \sum_{k \neq j} \frac{1}{\gamma_j - \gamma_k} + \sum_{k,m} \frac{\prod_l (z_k - w_l) \prod_{s \neq k}(\gamma_m - z_s)}{\prod_{l \neq k}(z_k - z_l) \prod_s (\gamma_m - w_s)(\gamma_j - z_k)}$$

$$+ \frac{\prod_i (\gamma_j - w_i)}{\prod_i (\gamma_j - z_i)} \sum_{k \neq j} \frac{\prod_i (\gamma_k - z_i)}{\prod_i (\gamma_k - w_i)(\gamma_j - \gamma_k)} = 0.$$

An equivalent form can be obtained if one divides both sides by $\frac{\prod_i(\gamma_j - z_i)}{\prod_i(\gamma_j - w_i)}$

$$- \sum_k \frac{2s_k}{\gamma_j - z_k} + \sum_{k \neq j} \frac{1}{\gamma_j - \gamma_k} + \sum_{k,m} \frac{\prod_l (z_k - w_l) \prod_{s \neq k}(\gamma_m - z_s)}{\prod_{l \neq k}(z_k - z_l) \prod_s (\gamma_m - w_s)(\gamma_j - z_k)}$$

$$+ \sum_{k \neq j} \frac{\prod_i (\gamma_k - z_i)}{\prod_i (\gamma_k - w_i)(\gamma_j - \gamma_k)} \frac{\prod_m (\gamma_j - w_m)}{\prod_m (\gamma_j - z_m)} = 0.$$

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Let us consider the left-hand side of equality as a rational function $F(\gamma_j)$ and calculate its primitive fractions decomposition at poles z_k, w_k, γ_k and ∞. It turns out that this decomposition will look like:

$$F(\gamma_j) = -\sum_k \frac{2s_k - 1}{\gamma_j - z_k} - \sum_k \frac{1}{\gamma_j - w_k} + 2\sum_{i \neq j} \frac{1}{\gamma_j - \gamma_i}. \tag{3.24}$$

Thus, the equality is equivalent to an equation of the Bethe system. Let us demonstrate, for example, the residue calculating at the point $\gamma_j = w_i$

$$
\begin{aligned}
Res_{\gamma_j = w_i} F(\gamma_j) &= \sum_k \frac{\prod_l (z_k - w_l) \prod_{s \neq k}(w_i - z_s)}{\prod_{l \neq k}(z_k - z_l) \prod_{s \neq i}(w_i - w_s)(w_i - z_k)} \\
&= \frac{\prod_s (w_i - z_s)}{\prod_{s \neq i}(w_i - w_s)} \sum_k \frac{\prod_l (z_k - w_l)}{\prod_{l \neq k}(z_k - z_l)(w_i - z_k)^2}.
\end{aligned} \tag{3.25}
$$

Let us write down the expesion on the right side of the equality

$$(Res_{z=w_i}\Phi(z))^{-1} \sum_k Res_{z=z_k}\Phi(z),$$

where

$$\Phi(z) = \frac{\prod_l(z - w_l)}{\prod_l(z - z_l)(z - w_i)^2},$$

and therefore is -1.

The sufficiency condition was proved in [55].

Theorem 3.1. *If the set of numbers γ_i where $i = 1, \ldots, M$ satisfies the system of Bethe equations (2.9) with the parameters: the set of poles $z_1, \ldots, z_k$ and $w_1, \ldots, w_{k-2}$ and the set of highest weights $2s_1 - 1, \ldots, 2s_k - 1$ and $1, \ldots, 1$ correspondingly, then the vector*

$$\Psi = \prod_{i=1}^k (z - z_i)^{-s_i} \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix}, \tag{3.26}$$

where

$$\phi_1 = \prod_{j=1}^M (z - \gamma_j),$$

$$\phi_2/\phi_1 = \sum_{j=1}^M \frac{\alpha_j}{z - \gamma_j}, \tag{3.27}$$

and the coefficients α_j are given by the expressions

$$\alpha_j = \frac{\prod_i(\gamma_j - z_i)}{\prod_i(\gamma_j - w_i)}, \tag{3.28}$$

solves the matrix linear problem (3.3), where the connection coefficients are given by

$$a_{12}^i = \frac{\prod_j(z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)}, \tag{3.29}$$

and the coefficients a_{11}^i and a_{21}^i are determined form (3.16), (3.17). The conditions of normalization (3.2) are fulfilled.

Proof. Actually, we should prove just that normalization condition (3.21) does not depend on the choice of the parameter c, in particular, it may be taken 1. Indeed, on the basis of (3.16), (3.17) we obtain:

$$a_{21}^i = -\left(2s_i \sum_j \frac{\alpha_j}{z_i - \gamma_j} + \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)} \left(\sum_j \frac{\alpha_j}{z_i - \gamma_j} \right)^2 \right). \tag{3.30}$$

We need to prove that

$$\sum_i 2s_i \sum_j \frac{\alpha_j}{z_i - \gamma_j} + \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)} \left(\sum_j \frac{\alpha_j}{z_i - \gamma_j} \right)^2 = 0. \tag{3.31}$$

The first summand of (3.31) then using the Bethe equations can be converted to the following:

$$\sum_i 2s_i \sum_j \frac{\alpha_j}{z_i - \gamma_j} = \sum_j \alpha_j \sum_i \frac{2s_i}{z_i - \gamma_j}$$

$$= \sum_j \alpha_j \left(-\sum_i \frac{1}{\gamma_j - z_i} + \sum_i \frac{1}{\gamma_j - w_i} - 2\sum_{i \neq j} \frac{1}{\gamma_j - \gamma_i} \right). \tag{3.32}$$

Now we will simplify the second summand (3.31) changing the order

$$\sum_{m \neq l} \alpha_m \alpha_l \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)(z_i - \gamma_m)(z_i - \gamma_l)}$$

$$+ \sum_m (\alpha_m)^2 \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)(z_i - \gamma_m)^2}. \tag{3.33}$$

Considering the second summand (3.33) let us note that

$$\sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)(z_i - \gamma_m)^2} = -\partial_{\gamma_m} \Phi^1(\gamma_m), \tag{3.34}$$

where

$$\Phi^1(\gamma_m) = \sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)(z_i - \gamma_m)} = \frac{\prod_j (\gamma_m - w_j)}{\prod_j (\gamma_m - z_j)}. \tag{3.35}$$

Therefore, the expression (3.34) becomes:

$$-\frac{\prod_j (\gamma_m - w_j)}{\prod_j (\gamma_m - z_j)} \left(\sum_s \frac{1}{\gamma_m - w_s} - \sum_s \frac{1}{\gamma_m - z_s} \right), \tag{3.36}$$

which is reduced with the relevant part of (3.32). Let us consider the first summand (3.33), this also can be simplified:

$$\sum_i \frac{\prod_j (z_i - w_j)}{\prod_{j \neq i}(z_i - z_j)(z_i - \gamma_m)(z_i - \gamma_l)}$$

$$= \frac{\prod_j (\gamma_l - w_j)}{\prod_j (\gamma_l - z_j)(\gamma_m - \gamma_l)} - \frac{\prod_j (\gamma_m - w_j)}{\prod_j (\gamma_m - z_j)(\gamma_m - \gamma_l)}. \tag{3.37}$$

Substituting the expression in (3.33) we finish the proof $\square$

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3.2 Schlesinger transformations

There is a discrete group of transformations that preserve the connection form (3.3) and, moreover, do not change the class of monodromy representation. However, these changes shift characteristic exponents at fixed points by half-integer values. Such transformations are called Schlesinger, Hecke or Backlund transformations depending on the context. They have simple geometric interpretation explained in the beginning of this section.

3.2.1 Action on bundles

Let us consider a curve C, a holomorphic bundle F on it, the corresponding sheaf of sections $\mathcal{F}$, the additional set of data $x \in C$ and a point of the dual space to the fiber $l \in F_x^*$. Then the lower Hecke transform $T_{(x,l)}E$ is defined by the subsheaf $\mathcal{F}' = \{s \in \mathcal{F} : (s(x), l) = 0\}$, which in turn corresponds to a certain holomorphic bundle on the curve C.

The equivalent definition can be defined in terms of gluing functions. Let us consider the action on holomorphic bundles on $\mathbb{C}P^1$. In virtue of the Birkhoff-Grothendieck theorem [56] any holomorphic bundle on $\mathbb{C}P^1$ of rank n is isomorphic to the sum of line bundles $\mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_n)$ for a specific set of integers $(k_1, \ldots, k_n)$ called the type of a bundle and determined up to the symmetric group action. Let us consider the open covering of $\mathbb{C}P^1$ consisting in: U_∞ - a disk around ∞ which does not contain $z = z_i$, $i = 1, \ldots, N$ and the domain $U_0 = \mathbb{C}P^1 \backslash \{\infty\}$. We consider holomorphic rank 2 bundles and parameterize them by gluing function $G(z)$ which is a holomorphically invertible function on $U_0 \cap U_\infty$ with values in $GL(2)$. Let us say that a pair $S_\infty(z) \in \mathcal{O}^{(2)}(U_\infty)$ and $S_0(z) \in \mathcal{O}^{(2)}(U_0)$ defines a global section if $S_0(z) = G(z)S_\infty(z)$.

We describe the transformation on bundles in terms of actions on corresponding gluing functions defined as a multiplication on the left by an element

$$G_s(z) = G_s \left(\begin{array}{cc} z - z_s & 0 \\ 0 & 1 \end{array} \right) G_s^{-1} \tag{3.38}$$

for some constant matrix G_s and some point $z_s \in U_0$

Remark 3.1. *The action on the space of gluing functions can be reduced to the action on the isomorphism classes of holomorphic bundles if one chooses G_s appropriately, if changing a trivialization in U_0 by $T(z)$ we change also the matrix G_s as follows $T(z_s)G_s$. This is obviously referring to the invariant definition above.*

We will investigate the composition of these changes applied at two points.

Lemma 3.2. *A composition of two transformations specified by an expression $G_i(z)G_j^{-1}(z)$, for a generic choice of matrices G_i, G_j preserves the trivial bundle.*

Proof It is sufficient to find a decomposition for $G(z) = G_i(z)G_j^{-1}(z)$ with $G(z) = G_{ij}(z)G_\infty(z)$, where $G_{ij}(z)$, $G_\infty(z)$ are invertible respectively at U_0, U_∞. The thought-consideration for this evidence is the cohomological dimension count in families at a generic point. Indeed, for a particular choice $G_i^{-1}G_j = 1$ we get a trivial bundle which is semistable and hence minimizes the dimension of $H^0(\text{End}(V))$ for V of degree 0. In this context, the trivial bundle is generic in the family of bundles for different G.

Despite the general argument here we propose a proof in spirit of the decomposition lemma in [56]. Let us introduce the notations

$$G_i = \left(\begin{array}{cc} 1 & x_i \\ y_i & 1 \end{array} \right). \tag{3.39}$$

We can decompose the product

$$G(z) = G_i \left(\begin{array}{cc} z & 0 \\ 0 & 1 \end{array} \right) G_i^{-1} G_j \left(\begin{array}{cc} (z-1)^{-1} & 0 \\ 0 & 1 \end{array} \right) G_j^{-1} \tag{3.40}$$

into the alternative product

$$G(z) = G_{ij}(z)G_\infty(z),$$

where $G_{ij}(z)$, $G_\infty(z)$ are holomorphically invertible functions on U_0, U_∞ respectively. The traditional calculations allows to find a decomposition in the form

$$G_\infty(z) = \begin{pmatrix} \frac{z(1-x_jy_i)(1-x_jy_j)-x_j(y_i-2y_j-x_jy_iy_j)}{(1-2x_jy_i+x_jy_j)(1-x_jy_i)(1-x_jy_j)(z-1)} & -\frac{x_j}{(1-x_jy_i)(1-x_jy_j)(z-1)} \\ \frac{y_i-2y_j+x_jy_iy_j}{(1-x_jy_i)(1-2x_jy_i+x_jy_j)} & \frac{1}{1-x_jy_i} \end{pmatrix}.$$

3.2.2 The action on connections

The action of Hecke transformations on classes of holomorphic bundles can be extended to the space of pairs: (bundle, connection) when certain conditions are satisfied. Let us describe in detail the induced action. A connection is a sheave map satisfying Leibniz rule with respect to the action of the structure sheaf:

$$\Delta : \mathcal{F} \to \mathcal{F} \otimes \Omega^1.$$

Hecke transformations can be defined on the space of connections preserving the space $Ann_l = \{v \in \mathcal{F}_x :< l, v >= 0\}$

$$\Delta_x : Ann_l \to Ann_l \otimes \Omega^1_x.$$

In our case we consider the composition of pairs of Hecke transformations localized at z_i, z_j, preserving the trivial rank 2 bundle.

As is mentioned above, the action can be defined by using the gluing functions language. Let us consider the trivial bundle specified by the gluing function 1. Hecke transformation change the bundle structure, the global section is defined by the pair S_0, S_∞, such that $S_0 = GS_\infty$, where $G = G_{ij}G_\infty$. One can define the action on connections as follows: let $\partial_z - A$ is a connection in the trivial bundle, determined by this expression on both opens, the transformed object is the pair of connection forms:

$$\partial_z - A \qquad \text{over} \qquad U_\infty,$$
$$G(\partial_z - A)G^{-1} \qquad \text{over} \qquad U_0.$$

After the basis change in U_∞ of the type $\widetilde{S_\infty} = G_\infty S_\infty$ we obtain the connection of the form

$$\partial_z - A \;\to\; G_\infty(\partial_z - A)G_\infty^{-1}. \tag{3.41}$$

The trivialization change in U_0 of the kind $\widetilde{S_0} = G_{ij}^{-1}S_0$ gives the following

$$G(\partial_z - A)G^{-1} \;\to\; G_{ij}^{-1}G(\partial_z - A)G^{-1}G_{ij} = G_\infty(\partial_z - A)G_\infty^{-1}. \tag{3.42}$$

Therefore, the transformed connection is of the same type as the initial one. The analytic properties at ∞ are preserved in virtue of the fact that G_∞ is holomorphically invertible in U_∞.

Using the results of the previous sections we calculate explicitly the Hecke action. To preserve the normalization condition $A(z)$ at ∞ it is necessary to consider transformations of the kind

$$\begin{aligned} \tilde{G}(z) &= G_\infty^{-1}(\infty)G_\infty(z) \\ &= \frac{1}{z-1}\left(z - \begin{pmatrix} \frac{x_1(y_0-2y_1+x_1y_0y_1)}{(1-x_1y_0)(1-x_1y_1)} & \frac{x_1(1-2x_1y_0+x_1y_1)}{(1-x_1y_0)(1-x_1y_1)} \\ \frac{y_0-2y_1+x_1y_0y_1}{(1-x_1y_0)(1-x_1y_1)} & \frac{1-2x_1y_0+x_1y_1}{(1-x_1y_0)(1-x_1y_1)} \end{pmatrix}\right). \end{aligned} \tag{3.43}$$

Then one just needs to apply the gauge transformation $\tilde{G}(z)$ to the connection

$$A \mapsto \tilde{G}(z)A\tilde{G}^{-1}(z) + \partial_z\tilde{G}(z)\tilde{G}^{-1}(z).$$

The complete family of Hecke transformations in the case of 3 points associated with the analysis of the Painleve VI equation was described in [57].

Remark 3.2. *The choice of the highest weights* 1 *in moving poles w_i is not obligatory, but in some ways, the most general. One can consider a potential of the form*

$$U = \sum_{j=1}^{m} \frac{-1/4(\eta_j + 2)\eta_j}{(z - w_j)^2} + \sum_{i=1}^{k} \frac{1/4 + det A_i}{(z - z_i)^2} + \sum_{j=1}^{k-2} \frac{H_{w_j}}{z - w_j} + \sum_{i=1}^{k} \frac{H_{z_i}}{z - z_i} \tag{3.44}$$

with the higher values of weights. It can be implemented if one requires that $a_{12}(z)$ have zeroes w_j with multiplicities η_j satisfying the condition $\sum_{j=1}^{m} \eta_j = k - 2$.

The local analysis at poles shows that the eigenvalues of residues A_i transform due to the 4 following rules depending on the choice of the low and upper Hecke transformations subspaces:

$$(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) \longmapsto (\ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots),$$
$$(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) \longmapsto (\ldots, \lambda_i + 1, \ldots, \lambda_j + 1, \ldots),$$
$$(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) \longmapsto (\ldots, \lambda_i - 1, \ldots, \lambda_j - 1, \ldots),$$
$$(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) \longmapsto (\ldots, \lambda_i - 1, \ldots, \lambda_j + 1, \ldots).$$

The result obtained allows to treat recurrent relations on the space of solutions for the Bethe equation system. The most interesting in the program of explicit solving of quantum systems is the set of transformations lowering the highest weight values in both points. The consecutive application of these transformations could reduce the highest weight to zero, which corresponds to the trivial representation of the quantum algebra and hence the trivial quantum problem.

3.3 Elliptic case

The elliptic $\mathfrak{sl}_2$ Gaudin model is provided by a similar technique of quantum model solution including the quantum spectral curve, quantum separated variables and Hecke symmetries on the spectrum.

3.3.1 Separated variables

Let us recall the traditional method of separation of variables for this system [53], [59]. As in the rational case we consider the $\mathfrak{sl}_2$ Gaudin model with fixed representation $V = V_1 \otimes \ldots \otimes V_k$ of the quantum algebra $U(\mathfrak{sl}_2)^{\otimes k}$, where V_i is the finite dimensional irreducible representation of the highest weight Λ_i. V_i can be realized as the quotient of the Verma module $\mathbb{C}[t_i]/t_i^{\Lambda_i+1}$, such that the generators of $\mathfrak{sl}_2$ act by differential operators:

$$h^{(s)} = -2t_s \frac{\partial}{\partial t_s} + \Lambda_s, \qquad e^{(s)} = -t_s \frac{\partial^2}{\partial t_s^2} + \Lambda_s \frac{\partial}{\partial t_s}, \qquad f^{(s)} = t_s.$$

Let us start with the study of quantum problem on the tensor product of Verma modules $W = \mathbb{C}[t_1, \ldots, t_k]$. We introduce the variables $C, \{y_j\}$ defined by:

$$\sum_{s=1}^{k} \frac{\theta(u - u_s - \lambda)}{\theta(u - u_s)\theta(-\lambda)} t_s = C \prod_{s=1}^{k} \frac{\theta(u - y_s)}{\theta(u - u_s)}.$$

Let us now represent the elliptic Gaudin model eigenvector as a function of introduced variables:

$$S_\lambda(u)\Omega(C, y_1, \ldots, y_k) = s_\lambda(u)\Omega(C, y_1, \ldots, y_k). \tag{3.45}$$

In this formula $s_\lambda(u)$ is a scalar functions on u of the form

$$s_\lambda(u) = \sum c_i \wp(u - u_i) + \sum d_i \frac{\theta'(u - u_i)}{\theta(u - u_i)}; \qquad c_i = \Lambda_i^2/4 + \Lambda_i/2. \tag{3.46}$$

Introducing further $u = y_j$ by the left in formula (3.45) we obtain:

$$\left(\frac{\partial}{\partial y_j} - \frac{1}{2} \sum_{s=1}^{k} \frac{\theta'(y_j - u_s)}{\theta(y_j - u_s)} \Lambda_s \right)^2 \Omega(C, y_1, \ldots, y_k) = s_\lambda(y_j)\Omega(C, y_1, \ldots, y_k).$$

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This equation induces a factorization of an eigenvector:

$$\Omega(C, y_1, \ldots, y_k) = C^a \prod_j \omega(y_j),$$

moreover it may be argued that the expression $w(u) = \prod_{s=1}^k \theta(u - u_s)^{-\Lambda_s/2} \omega(u)$ associated with the component of the eigenvector satisfies the equation:

$$\left(\partial_u^2 - s_\lambda(u)\right) w(u) = 0 \tag{3.47}$$

Therefore, each equation 3.47 of the form 3.46, having solution $s_\lambda(u)$ with half-integer exponents at $\{u_1, \ldots, u_k\}$ and meromorphic outsides these points corresponds to an eigenvector for the elliptic Gaudin Hamiltonians in representation V, obtained projecting the vector Ω.

Hypothesis 3.3. *There is a one-to-one mapping between this kind of differential operators and the eigenvectors of the model in the representation V.*

Through the following sections we will consider only such eigenvectors for the Gaudin model that correspond to elliptic Sturm-Liouville operators with the described analytic properties.

3.3.2 Bethe ansatz

The traditional Bethe ansatz method in the elliptic case [59] can be obtained considering the following particular solution with simple zeroes

$$\psi(u) = \prod_i \theta^{-\Lambda_i/2}(u - u_i) \prod_j \theta(u - \gamma_j) \tag{3.48}$$

for the elliptic Sturm-Liouville equation

$$\left(\partial_u^2 - \sum_i c_i \wp(u - u_i) - \sum_i d_i \frac{\theta'(u - u_i)}{\theta(u - u_i)}\right) \psi(u) = 0. \tag{3.49}$$

This condition is equivalent to the following system of equations:

$$
\begin{aligned}
c_i &= \Lambda_i^2/4 + \Lambda_i/2, \\
d_i &= \Lambda_i \left(\sum_j \frac{\theta'(u_i - \gamma_j)}{\theta(u_i - \gamma_j)} - \sum_{j \neq i} \frac{\Lambda_j \theta'(u_i - u_j)}{2\theta(u_i - u_j)}\right), \\
0 &= \sum_i \Lambda_i/2 \frac{\theta'(\gamma_j - u_i)}{\theta(\gamma_j - u_i)} - \sum_{i \neq j} \frac{\theta'(\gamma_j - \gamma_i)}{\theta(\gamma_j - \gamma_i)},
\end{aligned}
\tag{3.50}
$$

the latter is called the elliptic Bethe system.

3.3.3 Matrix form of the Bethe equations

In this section we find a matrix Fuchsian system equivalent to the elliptic Sturm-Liouville equation (3.49),

$$(\partial_u - A(u)) \Psi(u) = 0, \tag{3.51}$$

where

$$\Psi(u) = \begin{pmatrix} \psi_1(u) \\ \psi_2(u) \end{pmatrix},$$

$$A(u) = \begin{pmatrix} a_{11}(u) & a_{12}(u) \\ a_{21}(u) & a_{22}(u) \end{pmatrix} = \begin{pmatrix} \sum a_{11}^i \frac{\theta'(u - z_i)}{\theta(u - z_i)} & \sum a_{12}^i \frac{\theta(u - z_i - \lambda)}{\theta(u - z_i)\theta(-\lambda)} \\ \sum a_{21}^i \frac{\theta(u - z_i + \lambda)}{\theta(u - z_i)\theta(\lambda)} & \sum a_{11}^i \frac{\theta'(u - z_i)}{\theta(u - z_i)}. \end{pmatrix}$$

The equivalence relation of the matrix and scalar systems is the following: the function $w = \psi_1/\sqrt{a_{12}}$ solves the equation $w'' - Uw = 0$, of the same form as 3.47, with the potential whose highest term is given by the formula:

$$U(u) = -\sum(1/4 + det(A_i))\wp(u - z_i) + \sum 3/4\wp(u - w_i) + \ldots$$

Here the points w_j are defined by the condition

$$a_{12}(u) = c\frac{\prod \theta(u - w_i)}{\prod \theta(u - z_i)}.$$

In turn A_i are defined as residues of $A(u)$ at z_i.

Remark 3.3. *Note that the sets of poles of the Sturm-Liouville operator and of the matrix problem do not match, the first one is compiled from two subsets*

$$\{u_1, \ldots, u_{2l}\} = \{z_1, \ldots, z_l, w_1, \ldots, w_l\}.$$

It turns out that method of solution construction for the matrix problem from a solution for the Sturm-Liouville equation is also explicit. Let us consider a scalar problem that corresponds to the set of marked points $\{z_1, \ldots, z_l, w_1, \ldots, w_l\}$, the set of highest weights $2s_1 - 1, \ldots, 2s_k - 1, 1, \ldots, 1\}$ and the set of Bethe roots $\{\gamma_1, \ldots, \gamma_\rho\}$. Then the 2-vector function Ψ with components:

$$
\begin{aligned}
\psi_1 &= \prod_{i=1}^{k} \theta(u - u_i)^{s_i} \prod_{j=1}^{\rho} \theta(u - \gamma_j) \\
\psi_2 &= \sum_{j=1}^{\rho} \frac{\alpha_j \theta(u - \gamma_j + \lambda)}{\theta(u - \gamma_j)} \psi_1,
\end{aligned}
\tag{3.52}
$$

those coefficients α_j are given by the formula

$$\alpha_j = \frac{\prod_i \theta(\gamma_j - w_i)}{\prod_i \theta(\gamma_j - u_i)},$$

satisfy the matrix equation 3.51.

An explicit calculation shows that the equation (3.51) for Ψ given by the expression (3.52) is equivalent to the following system of equations:

$$
\begin{aligned}
det \begin{pmatrix} a_{11}^i - s_i & a_{12}^i \\ a_{21}^i & a_{22}^i - s_i \end{pmatrix} &= 0, \\
-\sum_k (2s_k - 1)\frac{\theta'(\gamma_j - u_k)}{\theta(\gamma_j - u_k)} - \sum_k \frac{\theta'(\gamma_j - w_k)}{\theta(\gamma_j - w_k)} + 2\sum_{i \neq j} \frac{\theta'(\gamma_j - \gamma_i)}{\theta(\gamma_j - \gamma_i)} &= 0, \\
\frac{\prod_i \theta(\gamma_j - w_i)}{\prod_i \theta(\gamma_j - u_i)} &= \alpha_j.
\end{aligned}
$$

The system of equations means that exponents are eigenvalues of the residues of the connection and the set of γ_j satisfy the elliptic Bethe system (3.50) corresponding to the set of marked points

$$\{u_1, \ldots, u_k, w_1, \ldots, w_k\}$$

and the set of highest weights $\{2s_1 - 1, \ldots, 2s_k - 1, 1, \ldots, 1\}$.

3.3.4 Hecke transformations

Let us describe in more details how the Hecke transformations are calculated over an elliptic curve. The most suitable way of parameterization of holomorphic bundles for an elliptic curve Σ implies the lift of a bundle to the universal covering $\mathbb{C}$ ([58] (2,6)). The monodromy group $\mathbb{Z}^2$ acts by homomorphisms on the sheaf of sections $\pi^* \mathcal{E}$ corresponding to the bundle E. In case of the degree 0 line bundle the only way to define the multiplier set up to equivalence is the set of quasiperiodic factors of the expression:

$$f(z) = \frac{\theta(z - \lambda)}{\theta(z)}$$

for $\lambda \in \Sigma$. Let us denote the corresponding line bundle by $\mathcal{O}_\lambda$. The Hecke transform at a point w supposes considering the subsheaf of $\mathcal{O}_\lambda$ taking values 0 at w. This sheaf is isomorphic to the sheaf of sections of some line bundle of degree 1

$$s(z) \mapsto \frac{s(z)}{\theta(z - w)}.$$

This map is an isomorphism due to the property that $\theta(z)$ has a unique zero at $z = 0$. The Hecke transformations on connections on the rank 2 bundle $\mathcal{O}_{\lambda/2} \oplus \mathcal{O}_{-\lambda/2}$ construct connections on a bundle $\mathcal{O}_{\mu/2} \oplus \mathcal{O}_{-\mu/2}$ as follows. Let the residues of the connection have the decomposition:

$$A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ a_{21}^{(i)} & -a_{11}^{(i)} \end{pmatrix} = G_i \begin{pmatrix} d_i & 0 \\ 0 & -d_i \end{pmatrix} G_i^{-1}$$

where G_i are constant matrices. Then the connection is transformed by the gauge transformation with the group element

$$G_{ij}(z) = \tilde{G}_i \begin{pmatrix} 1 & 0 \\ 0 & \theta(z - z_i) \end{pmatrix} \tilde{G}_i^{-1} G_j \begin{pmatrix} \theta^{-1}(z - z_j) & 0 \\ 0 & 1 \end{pmatrix} G_j^{-1},$$

where

$$\tilde{G}_i = G_j \begin{pmatrix} \theta^{-1}(z_i - z_j) & 0 \\ 0 & 1 \end{pmatrix} G_j^{-1} G_i.$$

As well as in the rational case we consider a pair of Hecke transformations at various points u_i, u_j with different signs $T_{ij} = T_{(u_i, l_i)}^{-1} T_{(u_j, l_j)}$ acting on rank 2 bundles with trivial determinant. Depending on the choice of subspaces of upper and lower transformations we get the following action of T_{ij} on a variety of highest weights of the Gaudin model

$$
\begin{aligned}
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) &\longmapsto (\ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots), \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) &\longmapsto (\ldots, \lambda_i + 1, \ldots, \lambda_j + 1, \ldots), \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) &\longmapsto (\ldots, \lambda_i - 1, \ldots, \lambda_j - 1, \ldots), \\
(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) &\longmapsto (\ldots, \lambda_i - 1, \ldots, \lambda_j + 1, \ldots).
\end{aligned}
$$

As in the rational case it is of particular interest the family of transformations that lower the weights of all representations hence simplifying the diagonalization problem.

4   Applications

This section is devoted to two main applications of the quantum spectral curve method. The first application is related with the geometric Langlands correspondence and mainly consists in an effective description of the center $U_{crit}(\widehat{\mathfrak{gl}_n})$ which in turn plays a key role in the Beilinson-Drinfeld quantization of the Hitchin system. Let us note that this problem is closely related to the representation theory of affine Lie algebras.

4.1   Geometric Langlands correspondence

4.1.1   The center of $U(\widehat{\mathfrak{gl}_n})$ on the critical level

We introduce the following notation $U_{crit}(\widehat{\mathfrak{gl}_n})$ for the local completion $U(\widehat{\mathfrak{gl}_n})/\{C-crit\}$, where C is a central element and $crit = -h^\vee = -n$ is the critical level inverse to the dual Coxeter number of the Lie algebra $\mathfrak{sl}_n$. It was proved in [61] that $U_{crit}(\widehat{\mathfrak{gl}_n})$ has a center isomorphic to the polynomial ring of the Cartan algebra as a linear space. Despite the geometric description of the center there was absent an explicit construction for the generators of this commutative algebra. For this purpose we use the Adler-Costant-Symes scheme [60]. This approach has an important place in the theory of integrable systems: can be exploited to construct a wide family of commutative algebras, allows to make relation of integrable systems with decomposition problems and provide an algebraic interpretation for the Lax representation, r-matrix structures. The AKS scheme can be generalized to the quantum level and takes important role in description, solution and classification of quantum integrable models. The most simple case is that of finite dimensional Lie algebra allowing a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ into the sum of two Lie subalgebras. To each choice of normal ordering one can attach an isomorphism of linear spaces

$$\phi : U(\mathfrak{g}) \to U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-).$$

Let us introduce a notation $\mathfrak{g}_-^{op}$ for the inverse Lie algebra structure to the space $\mathfrak{g}_-$ defining by the formula $-\{\circ, \circ\}$. Let us denote the Lie algebra $\mathfrak{g}_+ \oplus \mathfrak{g}_-^{op}$ by the symbol $\mathfrak{g}_r$. The corresponding enveloping algebras can be identified as linear spaces with the help of the Poincare-Birkhoff-Witt basis:

$$U(\mathfrak{g}_-^{op}) \simeq U(\mathfrak{g}_-).$$

Lemma 4.1. *The center of $\mathfrak{z}(U(\mathfrak{g}))$ is mapped to a commutative subalgebra in $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-^{op})$ by ϕ.*

Proof Let is denote the commutator in $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-^{op})$ as follows $[*,*]_R$. Let c_1, c_2 be two central elements in $U(\mathfrak{g})$ represented as follows

$$c_i = \sum_j x_j^{(i)} y_j^{(i)} \qquad x_j^{(i)} \in U(\mathfrak{g}_+), \ y_j^{(i)} \in U(\mathfrak{g}_-).$$

The result of calculating the modified commutator is as follows

$$
\begin{aligned}
[\phi(c_1), \phi(c_2)]_R &= [\sum_j x_j^{(1)} y_j^{(1)}, \sum_k x_k^{(2)} y_k^{(2)}]_R \\
&= \sum_{j,k} [x_j^{(1)}, x_k^{(2)}]_R y_j^{(1)} y_k^{(2)} + x_j^{(1)} x_k^{(2)} [y_j^{(1)}, y_k^{(2)}]_R.
\end{aligned}
$$

In virtue of the definition above we have

$$[x_j^{(1)}, x_k^{(2)}]_R = [x_j^{(1)}, x_k^{(2)}] \qquad [y_j^{(1)}, y_k^{(2)}]_R = -[y_j^{(1)}, y_k^{(2)}]$$

$$[\phi(c_1), \phi(c_2)]_R = \sum_k [c_1, x_k^{(2)}] y_k^{(2)} - \sum_j x_j^{(1)} [y_j^{(1)}, c_2]$$

The last expression is zero due to the centrality of elements c_1, c_2 $\square$

Remark 4.1.   *In what follows we will be interested in applying this scheme for $U_{crit}(\widehat{\mathfrak{gl}_n})$. To use the result of the AKS lemma in the infinite dimensional case one should choose an appropriate completion of an algebra. In our case we use the completion corresponding to the bigrading $deg(gt^k) = (k,0)$, $deg(gt^{-k}) = (0,k)$ for $k \geq 0$. One needs to prove that the considering central elements belong to this completion $U_{crit}^\cdot(\widehat{\mathfrak{gl}_n})$. This is a matter of fact due to the classical limit argument. In what follows we omit the completion in notation $U_{crit}(\widehat{\mathfrak{gl}_n})$, $U(\mathfrak{g}_r)$ and the tensor products for the sake of the simplicity.*

One considers also the linear space map

$$\varsigma : U(\mathfrak{g}) \to U(\mathfrak{g}_+) \oplus U(\mathfrak{g})\mathfrak{g}_-$$

defining by the direct sum decomposition for the Lie algebra. Let us denote by φ the projector to the first subspace $U(\mathfrak{g}_+)$.

Lemma 4.2. *The image of $\mathfrak{z}(U(\mathfrak{g}))$ with respect to φ is a commutative subalgebra of $U(\mathfrak{g}_+)$.*
Proof Let $c_1, c_2 \in Z$.

$$[c_1 - \varphi(c_1), c_2 - \varphi(c_2)] = [\varphi(c_1), \varphi(c_2)]$$

The r.h.s. belongs to $U(\mathfrak{g}_+)$; the l.h.s in an element of $U(\mathfrak{g})\mathfrak{g}_-$; this takes issue in vanishing of both sides $\square$

We will identify $U_{crit}(\widehat{\mathfrak{gl}_n})$ with the loop algebra as linear spaces. Let us list several important facts about the loop algebra.

Proposition 4.3. *Let us consider $\mathfrak{g} = \mathfrak{gl}_n[t, t^{-1}] = \mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n[t]$ those generators $e_{ij}^{(k)} = e_{ij}t^k$ can be represented by the generating series*

$$L_{full}(z) = \sum_{s=-\infty,\infty} \Phi_s z^{-s-1} \tag{4.1}$$

where

$$\Phi_s = \sum_{ij} E_{ij} \otimes e_{ij}^{(s)}.$$

Here as above e_{ij} are generators of the Lie algebra $\mathfrak{gl}_n$, and E_{ij} are matrix unities in Mat_n. The Lie algebra structure on $\mathfrak{g}_r$ can be described as the following commutation relations

$$\{L_{full}(z) \otimes L_{full}(u)\} = [\frac{P}{z-u}, L_{full}(z) \otimes 1 + 1 \otimes L_{full}(u)]. \tag{4.2}$$

Let us remark that these relations are the same as for the Gaudin Lax operator (2.42).

The center of $(U_{crit}(\widehat{\mathfrak{gl}_n}))$ and a commutative subalgebra in $U(t\mathfrak{gl}_n[t])$ Let us also introduce the "'positive"' Lax operator:

$$L(z) = \sum_{k>0} \Phi_k z^{-k-1},$$

which satisfies the following R-matrix relations:

$$\{L(z) \otimes L(u)\} = [\frac{P}{z-u}, L(z) \otimes 1 + 1 \otimes L(u)]. \tag{4.3}$$

Theorem 4.4. *The commutative subalgebra in $U(t\mathfrak{gl}_n[t])$ defined by the set of coefficients of the quantum characteristic polynomial $det(L(z) - \partial_z)$ coincide with the image of $\mathfrak{z}(U_{crit}(\widehat{\mathfrak{gl}_n}))$ by the projection $\varphi : U_{crit}(\widehat{\mathfrak{gl}_n}) \to U(t\mathfrak{gl}_n[t])$.*
Proof The proof is based on the results of [62] where it was proved that the centralizer of the set of quadratic Gaudin Hamiltonians H_2^i in $U(t\mathfrak{gl}_n[t])$ coincide with the projection of $U(\widehat{\mathfrak{gl}_n})$ on the critical level.

Remark 4.2. *This particular property, namely the fact that the quadratic generators determine the complete commutative subalgebra is known also in the theory of Fomenko-Mishenko subalgebras [63] and in the theory of the Calogero-Moser system [64].*

Following the proposed logic and using the fact that the subalgebra defined by the coefficients of $det(L(z) - \partial_z)$ commute with H_2^i, one can show that this subalgebra is a subalgebra of the algebra obtained from the center. In order to prove their coincidence it is sufficient ot consider the classical limit $\square$

Remark 4.3. *The analogous strategy is applicable in the case of projection to $U(\mathfrak{gl}_n[t])$. One needs to take into account that both algebras are invariant with respect with the $GL(n)$ action.*

4.1.2 Explicit description of the center of $U_{crit}(\widehat{\mathfrak{gl}_n})$)

Theorem 4.5. *The center of $U_{crit}(\widehat{\mathfrak{gl}_n})$ is isomorphic to a subalgebra in $U(\mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n^{op}[t])$ defined by the coefficients of th quantum characteristic polynomial $det(L_{full}(z) - \partial_z)$. The isomorphism is induced by the mapping*

$$I : U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n^{op}[t]) \to U_{crit}(\widehat{\mathfrak{gl}_n}), \qquad I : h_1 \otimes h_2 \to h_1 h_2 \tag{4.4}$$

Proof follows the same lines as those of [62]. Let us firstly show that the algebra generated by the coefficients of the characteristic polynomial of the Lax operator $L_{full}(z)$ coincide with the centralizer of its quadratic elements. Further using the Sugawara formula for the quadratic center generators we prove that their image in $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n[t])$ coincide with the quadratic elements of the quantum characteristic polynomial. For proving the first statement we consider a special limit of the commutative family.

Using the commutation relations 4.2, 4.3 and the traditional r-matrix calculations we show that $TrL_{full}^m(z)$ are central in the symmetric algebra $S(\mathfrak{gl}_n[t, t^{-1}])$ and moreover $TrL_{full}^m(z)$ generate the commutative Poisson subalgebra in $S(\mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n^{op}[t])$.

Let us consider the family of automorphisms of the algebra $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n^{op}[t])$ defined in terms of the Lax operator as follows: let K is a generic diagonal $n \times n$ matrix. The Lax operator

$$L_{full}^{\hbar}(z) = L_{full}(z) + \hbar K$$

also satisfies the r-matrix reletions (4.2). This automorphism family is parameterized by the parameter $\hbar$.

Let us consider the family of commutative subalgebras

$$M^{\hbar} \subset U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n^{op}[t])$$

defined by the generating function $det(L_{full}^{\hbar}(z) - \partial_z)$. $M^{\hbar}$ centralizes the set of quadratic generators $QI_2(L_{full}^{\hbar}(z))$. $QI_k(z, \hbar)$ has the following leading term in expansion on $\hbar$

$$QI_k(z, \hbar) = \hbar^k TrA_n K_1 K_2 \ldots K_k + O(\hbar^{k-1}).$$

Changing the basis

$$QI_k(z, \hbar) \mapsto \widetilde{QI_k}(z, \hbar) = (QI_k(z, \hbar) - \hbar^k TrA_n K_1 K_2 \ldots K_k)\hbar^{-k+1}$$

and considering the limit $\hbar \to \infty$

$$\widetilde{QI_k}(z, \hbar) \to Tr(L_{full}(z)K^{k-1})$$

we obtain that these expressions generate the Cartan subalgebra

$$\mathfrak{H} = \mathfrak{H}_- \otimes \mathfrak{H}_+ = U(\mathfrak{h}[t^{-1}]) \otimes U(t\mathfrak{h}[t]).$$

Let us demonstrate that this subalgebra coincide with the centralizer of its quadratic generators

$$H_2^{\infty}(z) = lim_{\hbar \to \infty} \widetilde{QI}_2(z, \hbar) = \sum_{i=-\infty,\infty} Tr(\Phi_i K)z^{-i-1}.$$

Obviously $\mathfrak{H} \subset Z(H_2^{\infty}(z))$. Let us introduce the notations $(k_1, \ldots, k_n)$ for the diagonal elements of K. Let us also denote by $h_i \in \mathfrak{H}$ the sum of the form

$$h_i = \sum_{s=1}^{n} (\Phi_i)_{ss} k_s,$$

then $H_2^{\infty}(z) = \sum_{i=-\infty,\infty} h_i z^{-i-1}$. The centralizer elements should commute with h_1 and h_{-1}. Let $\sum_{i=-\infty}^{\infty} x_i y_i$ be the infinite series such that $x_i \in U(\mathfrak{g}[t^{-1}])$, $y_i \in U(t\mathfrak{g}[t])$. We also suppose that this series is an element of the considered completion, i.e. such that it contains only finite number of elements of

each bigrading. The operators $[h_1, *]$ and $[h_{-1}, *]$ are homogeneous of bigrading $(0, 1)$ and $(1, 0)$. Hence the centralizer description question is reduced to the analogous question in the polynomial algebra. The answer is given by the formulas

$$Z(h_1) = U(\mathfrak{gl}_n[t^{-1}]) \otimes \mathfrak{H}_+, \qquad Z(h_{-1}) = \mathfrak{H}_- \otimes U(t\mathfrak{gl}_n^{op}[t]).$$

An intersection of these subspaces in a completed sense coincides with the Cartan subalgebra $\mathfrak{H}$.

Summarizing we obtain that in the generic point $\hbar$ of the family the commutative subalgebra $M^\hbar$ belongs to the centralizer of the set QI_2 and in the limit $\hbar \to \infty$ generates the centralizer. From the arguments analogous to those of [62] at the generic point $M^\hbar$ should coincide with the centralizer of the quadratic generators. To finish the proof let us remind the Sugawara formula $U_{crit}(\widehat{\mathfrak{gl}_n})$

$$c_2(z) =: Tr(L_{full}^2(z)):$$

This uses the normal ordering symbol $::$ for currents in $\mathfrak{sl}_2$. These elements project to $QI_2(z)$ up to a central elements in $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n[t])$ ∎

4.1.3 The Beilinson-Drinfeld scheme

In [65] it was proposed a universal construction for the Hitchin system quantization. Let Σ be the connected smooth projective curve over $\mathbb{C}$ of genus $g > 1$, G - a semisimple Lie group, $\mathfrak{g}$ - the corresponding Lie algebra, Bun_G - the moduli stack of principal G-bundles on Σ. Let us also define the Langlands dual group ^LG as a group determined by the dual root data, namely such that its root lattice coincides with the dual lattice for G.

The main result of [65] can be reduced to the following:

- There exists a commutative ring of differential operators on $\mathfrak{z}(\Sigma, G)$, acting on sections of the canonical bundle K_{Bun_G} such that the symbol map produces the commutative subalgebra of classical Hitchin Hamiltonians on T^*Bun_G.

- The spectrum of the ring $\mathfrak{z}(\Sigma, G)$ is canonically isomorphic to the moduli space of $^L\mathfrak{g}$-opers (for the $G = SL_2$ case an $^L\mathfrak{g}$-oper is just the Sturm-Liouville operator on S; in general case this is a flat connection in a principal ^LG bundle with a parabolic structure).

- To each $^L\mathfrak{g}$-per one can correspond a D-module on Bun_G by fixing eigenvalues of the Hitchin Hamiltonians. This D-module is an eigensheaf for the Hecke action defined naturally on the moduli stack of bundles. Moreover the eigenvalue in this case coincide with the corresponding $^L\mathfrak{g}$-oper.

The basement of this construction is the natural action of the center of $U_{crit}(\widehat{\mathfrak{g}})$ on the loop group of the corresponding Lie group. This action can induce an action by differential operators on $Bun_G(\Sigma)$ in virtue of one of the realizations of the moduli stack of principal bundles

$$Bun_G(\Sigma) \simeq G(F) \backslash G(\mathcal{A}_F)/G(\mathcal{O}_F) \simeq G_{in} \backslash G[[z, z^{-1}]/G_{out}$$

here G_{in} and G_{out} denote the subgroups of function converging in U_{in} and U_{out}, where U_{in} and U_{out} determine a covering of Σ of the type: U_{in} is an open disk centered in P with the local parameter z, $U_{out} = \Sigma \backslash P$. The middle part of the equality represents the so-called adelic realization of the moduli stack of principal G-bundles for an algebraic group G. The construction uses the adel group $G(\mathcal{A}_F)$ for the field F of rational functions on Σ, the group of entire adels $G(\mathcal{O}_F)$ and the group of principal adels $G(F)$. This realization is convenient for describing the geometric complex analogy of the arithmetic Langlands correspondence and the quantum Gaudin model.

4.1.4 Correspondence

Historically the Langlands hypothesis generalize the field-class theory [66, 67], one of those principal results is the following statement in the case of a number field. Namely let F be a number field (this means a finite

extension of $\mathbb{Q}$), $\bar{F}$ - its maximal algebraic extension, F^{ab} - its maximal abelian extension. The Galois group of an extension $F \subset F'$ is

$$Gal(F', F) \;\;=\;\; \{\sigma \in Aut(F') : \sigma(x) = x \;\; \forall x \in F\}.$$

The abelian reciprocity law
There exists a group isomorphism

$$Gal(F^{ab}, F) \simeq \text{ The group of connectivity components of } \; F^{\times} \backslash \mathcal{A}_F^{\times}$$

where $\mathcal{A}_F^{\times}$ is the idel group of the ring F, $F^{\times}$ is the group of invertible elements of F. The topology of of the completion product is considered.

The Langlands hypothesis is formulated as an n-dimensional (non-commutative) generalization of the abelian reciprocity law. Namely it is assumed the isomorphism between the cathegory of the Galois group representations of the maximal algebraic extension of a ring and the category of automorphic representations for the corresponding idel group. By an automorphic representation we mean a $GL_n(\mathcal{A}_F)$ - representation realized on the space of functions on

$$GL_n(F) \backslash GL_n(\mathcal{A}_F),$$

meet some additional conditions [68, 69]. The right part is traditionally called automorphic for the following reason. For $n = 2$ these representations are related with the theory of modular functions. It should be reminded that modular functions are functions on the upper-half Siegel plain matching the condition

$$f((az + b)/(cz + d)) = \chi(a)(cz + d)^k f(z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In particular, the modular functions can be represented as functions on the following quotient space

$$SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \simeq K \backslash GL_2(\mathcal{A}_{\mathbb{Q}})/GL_2(\mathbb{Q})$$

The Langlands program covers the following types of fields F:

- A number field.

- Field of functions on an algebraic curve over the finite field F_q (In this case, the hypothesis was proven in [70]).

- Field of functions on an algebraic curve over $\mathbb{C}$. This is called the geometrical case over $\mathbb{C}$. The following papers are on the subject [71].

The correspondence over $\mathbb{C}$:
In this case on the Galois side one considers classed of representations of the fundamental group or classes of flat connections in a holomorphic bundle of rank n. The automorphic side deals with the Hitchin D-module on

$$GL(F) \backslash GL(\mathcal{A}_F)/GL(\mathcal{O}_F) \simeq Bun_n(\Sigma).$$

The results of [65] and [61] ensures the correspondence between Hitchin D-modules and flat connections related to $^L\mathfrak{g}$-opers. Due to the construction of the quantum characteristic polynomial for the loop algebra, as well as an explicit construction for the center of $U_{crit}(\widehat{\mathfrak{gl}_n})$ in theorem 4.5 the correspondence for the Lie algebra $\mathfrak{gl}_n$ can be realized in a more effective way. The scheme demonstrate the correspondence

$$\text{Hitchin } D\text{-module } \overset{FF, BD}{\Leftrightarrow} \text{ Character } \chi \text{ on } \mathfrak{z}(U_{crit}(\widehat{\mathfrak{gl}_n})) \overset{CT}{\Leftrightarrow} \chi det(L_{full} - \partial_z).$$

Remark 4.4. *The construction of a character on $\mathfrak{z}(U_{crit}(\widehat{\mathfrak{gl}_n}))$ by a Hitchin D-module is a corollary of the Feigin and Frenkel theorem on existence of the center and the Beilinson and Drinfeld quantization. To obtain the explicit description for the corresponding flat connection [26] one should exploit the identification of commutative algebras: the commutative subalgebra in $U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n[t])$ defined by the coefficients of the quantum characteristic polynomial on one side and the image of the center of $\mathfrak{z}(U_{crit}(\widehat{\mathfrak{g}}))$ by the AKS map on another side.*

4.2 Non-commutative geometry

The main plot of these lectures is relevant to the emerging field of Noncommutative Geometry, substantive issues of which consist in geometric interpretation of algebraic structures in which the commutativity property is weakened. In this context the quantum characteristic polynomial is a natural generalization of classical one. Some properties of this object, in particular the role played by quantum characteristic polynomial in the program of effective solution of the quantum integrable models, suggest it to be a natural noncommutative generalization of an algebraic curve the spectral curve of an integrable system. This section describes some linear algebraic properties of the quantum characteristic polynomial obtained in [27].

4.2.1 The Drinfeld-Sokolov form of the quantum Lax operator

Let $L(z) \in Mat_n \otimes U(\mathfrak{gl}_n)^{\otimes N} \otimes Fun(z)$ be the quantum Lax operator for the Gaudin model (1.14), here and further $Fun(z)$ means the space of rational functions on a parameter z. Let us denote by $L^{[i]}(z)$ *quantum powers* of the Lax operator defined by the formula:

$$\begin{aligned} L^{[0]} &= Id, \\ L^{[i]} &= L^{[i-1]}L + \partial_z L^{[i-1]}. \end{aligned}$$

Theorem 4.6. *The expression $C(z) \in Mat_n \otimes U(\mathfrak{gl}_n)^{\otimes N} \otimes Fun(z)$ defined by the formula*

$$C(z) = \begin{pmatrix} v \\ vL \\ \dots \\ vL^{[n-1]} \end{pmatrix}, \tag{4.5}$$

where $v \in \mathbb{C}^n$ is a generic vector defines a gauge transformation

$$C(z)(L(z) - \partial_z) = \left(\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ QH_n & QH_{n-1} & \dots & QH_2 & QH_1 \end{pmatrix} - \partial_z \right) C(z), \tag{4.6}$$

where the r.h.s. lower line coefficients are determined by the coefficients of the quantum characteristic polynomial

$$\begin{aligned} det(L(z) - \partial_z) &= Tr A_n (L_1(z) - \partial_z) \dots (L_n(z) - \partial_z) \\ &= (-1)^n (\partial_z^n - \sum_i QH_{n-i} \partial_z^i). \end{aligned} \tag{4.7}$$

Knizhnik-Zamolodchikov equation Here and further we denote by V a finite-dimensional representation of $U(\mathfrak{gl}_n)^{\otimes N}$. It was shown in [72] that there exists a relation between solutions of the Knizhnik-Zamolodchikov (KZ) equation [73]

$$(L(z) - \partial_z) S(z) = 0,$$

where $S(z)$ is a function with values in $\mathbb{C}^n \otimes V$, solutions of the Baxter equation

$$det(L(z) - \partial_z) \Psi(z) = 0 \tag{4.8}$$

where $\Psi(z)$ is a function with values in V. To make this relation clear it is sufficient to take the antisymmetric projection of $U(z) = v_1 \otimes \dots \otimes v_{n-1} \otimes S(z)$ where v_i are some vectors in $\mathbb{C}^n$. For special choice of such vectors one obtains that vector components of $S(z)$ solves the equation (4.8).

Proof of theorem 4.6 Let us consider both sides of (4.6)applied to a function $S(z) \in \mathbb{C}^n \otimes V \otimes Fun(z)$,

$$L.H.S = C(L - \partial_z)S = \begin{pmatrix} <v, LS - \partial_z S> \\ <v, L^{[1]}(LS - \partial_z S>) \\ \ldots \\ <v, L^{[n-1]}(LS - \partial_z S)> \end{pmatrix}, \tag{4.9}$$

$$R.H.S = (L_{DS} - \partial_z)CS = \begin{pmatrix} <v, LS - \partial_z S> \\ \ldots \\ <v, L^{[n-1]}S - \partial_z(L^{[n-2]}S)> \\ <v, \sum_{i=0}^{n-1} QH_{n-i}L^{[i]}S - \partial_z(L^{[n-1]}S)> \end{pmatrix}. \tag{4.10}$$

Using the definition for quantum powers we obtain

$$L^{[k]}S - \partial_z(L^{[k-1]}S) = L^{[k-1]}(LS - \partial_z S).$$

The difference (4.10) - (4.9) takes the form

$$\begin{pmatrix} 0 \\ \ldots \\ 0 \\ <v, \sum_{i=0}^{n-1} QH_{n-i}L^{[i]}S - L^{[n]}S> \end{pmatrix}. \tag{4.11}$$

Let us now consider this expression if $S(z)$ is a solution for the KZ equation

$$L(z)S(z) = \partial_z S(z).$$

Let $\Phi(z) = C(z)S(z)$, where $C(z)$ is given by the formula (4.5). Then

$$\begin{aligned} \Phi_1(z) &= <v, S(z)> \\ \Phi_2(z) &= <vL(z), S(z)> = <v, \partial_z S(z)> \\ &\ldots \\ \Phi_k(z) &= <v(L^{[k-1]}L(z) + \partial_z L^{[k-1]}), S(z)> \\ &= <vL^{[k-1]}, \partial_z S(z)> + <v\partial_z L^{[k-1]}, S(z)> = \partial_z \Phi_{k-1}(z) \end{aligned}$$

One of the consequences of [72] is that $\Phi_1(z) = <v, S(z)>$ solves the Baxter equation

$$\sum_{i=0}^{n-1} QH_{n-i}\partial_z^i \Phi_1(z) - \partial_z^n \Phi_1(z) = 0 \tag{4.12}$$

for each solution $S(z)$ of the KZ equation and each vector $v \in \mathbb{C}^n$. The general position argument allows to claim that the n-th element of (4.11) vanishes identically on $S(z) \in \mathbb{C}^n \otimes V \otimes Fun(z)$. Theorem 2.5.7 [74] induces the equality of universal differential operators with values in the quantum algebra. ∎

4.2.2 Caley-Hamilton identity

Corollary 4.6.1. *The quantum powers of the Lax operator satisfy the quantum version of the Caley-Hamilton identity*

$$L^{[n]}(z) = \sum_{i=1}^{n} QH_i(z)L^{[n-i]}(z). \tag{4.13}$$

Proof Let us consider the last line of the equation (4.6)

$$vL^{[n-1]}(z)(L(z) - \partial_z) = \sum_{i=1}^{n} vQH_i(z)L^{[n-i]}(z) - \partial_z vL^{[n-1]}(z).$$

The result follows from the general choice of the vector $v \in \mathbb{C}^n$. □

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