Integral preserving discretization of 2D-Toda lattices

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Exponential systems in the continuous case

Exponential system is a system of hyperbolic PDEs

$$u^i_{xy} = \exp\left(\sum_{j=1}^{r} a_{ij} u^j\right), \quad i = 1, 2, \ldots, r, \tag{1}$$

where $M = (a_{ij})$ is a constant matrix. Systems of this type were considered by many authors around 1980. Systematic approach based on the study of *characteristic algebras* was proposed by Shabat and Yamilov (1981).

Important particular cases

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- 2D-Toda lattice $q^i_{xy} = \exp(q^{i+1} - q^i) - \exp(q^i - q^{i-1})$, where $q^0 = +\infty$ and $q^{r+1} = -\infty$, corresponds to matrix

$$\begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{pmatrix},$$

which is the Cartan of A-series simple Lie algebra.

Integrability

All these systems are integrable as well as exponential systems corresponding to the Cartan matrices of all simple Lie algebras (Mikhailov, Olshanetsky, Perelomov; Leznov,...) $\approx$ 1980.

We are interested in *Darboux integrability*, which is a specific kind of integrability of hyperbolic equations.

Characteristic integrals

Function $I = I(x, y, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}, \dots)$ is called a *y-integral* of hyperbolic system

$$\mathbf{u}_{xy} = \mathbf{F}(x, y, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_y)$$

if $D_y(I) = 0$ on solutions of the system.

Example

Liouville equation $u_{xy} = e^u$ admits characteristic integrals

$$I = u_{xx} - \frac{1}{2}u_x^2 \quad \text{and} \quad J = u_{yy} - \frac{1}{2}u_y^2.$$

Darboux integrability

Definition (V.Sokolov, S.Startsev)

Integrals $I_1, I_2, \dots, I_k$ of orders $d_1, d_2, \dots, d_k$ resp. are called *essentially independent*, if

$$\mathrm{rk}\left(\frac{\partial I_i}{\partial u^j_{\underbrace{x\dots x}_{d_i}}}\right) = k.$$

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Exponential systems corresponding to the Cartan matrices of all simple Lie algebras are known to be Darboux integrable.

Semi-discrete exponential systems

Semi-discrete exponential systems were introduced by Habibullin, Zheltukhin, and Yangubaeva (2011):

$$u^i_{n+1,x} - u^i_{n,x} = \exp\left(\sum_{j=1}^{r} (a^-_{ij} u^j_n + a^+_{ij} u^j_{n+1})\right), \quad i = 1, 2, \ldots, r. \tag{2}$$

Here $M^+ = (a^+_{ij})$ and $M^- = (a^-_{ij})$ are upper- and lower-triangular matrices with $\frac{a_{ii}}{2}$ on the principal diagonal and $M = M^- + M^+$ is the Cartan matrix of the corresponding continuous model.

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For the A-series Toda system we have

$$\begin{pmatrix} -2 & 1 & 0 & \ldots \\ 1 & -2 & 1 & \ldots \\ 0 & 1 & -2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & \ldots \\ 1 & -1 & 0 & \ldots \\ 0 & 1 & -1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 & \ldots \\ 0 & -1 & 1 & \ldots \\ 0 & 0 & -1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$u^i_{n+1,x} - u^i_{n,x} = \exp\left(u^{i-1}_n - u^i_n - u^i_{n+1} + u^{i+1}_{n+1}\right). \tag{3}$$

Remark: Laplace series

In the continuous case *A*-series Toda lattice in terms of variables $h^i = \exp(u^{i-1} - 2u^i + u^{i+1})$ describes the *Laplace invariants* of a sequence of hyperbolic operators related by Laplace transformations satisfying the assumption that operators on both edges of this sequence are factorizable: $h^0 = h^{r+1} = 0$. Laplace series can be defined in the semi-discrete case as well. Laplace invariants satisfy a system of semi-discrete hyperbolic equations in this case:

$$\left(\ln \frac{h_n^i}{h_{n+1}^i}\right)_x' = h_{n+1}^{i+1} - h_{n+1}^i - h_n^i + h_n^{i-1}.$$

This system can be reduced to (3) by a change of variables.

Darboux integrability in semi-discrete case

All definitions are similar to the continuous case: a function $I_n = I(\mathbf{u}_{n,x}, \mathbf{u}_{n,xx}, \dots)$ is called an *n-integral*, if its total difference derivative with respect to n vanishes on solutions. Integrals $I_1, I_2, \dots I_k$ are called *essentially independent*, if

$$\mathrm{rk}\left(\frac{\partial I_i}{\partial u^j_{n,\underbrace{x\dots x}_{d_i}}}\right) = k.$$

where d_i is the order of I_i. x-integrals depend on

$$\mathbf{u}_n, \mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \dots,$$

and x-derivatives are replaced by shifts: $\frac{\partial I_i}{\partial u^j_{n,\underbrace{x\dots x}_{d_i}}} \to \frac{\partial J_i}{\partial u^j_{n+d_i}}$.

System is called *Darboux integrable* if it admits complete families of essentially independent n- and x-integrals.

Are semi-discrete exponential systems integrable?

- Semi-discrete Toda lattices corresponding to the Cartan matrices of A- and C-series were proved to be Darboux integrable by using Darboux–Laplace transformations and constructing generating function for characteristic integrals (A-series) and by reduction (C-series) (S., 2015).

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- Another approach for the A-series lattice based on semi-discrete Wronskian formulas was used by Demskoi and Tran (2016).
- Darboux integrability for semi-discrete lattices corresponding to the Cartan matrices of all simple Lie algebras is proved recently (S., 2023).

Discretization of the Liouville equation

Rewrite the Liouville equation in the form $u_{xy} = e^{-2u}$. Note that its y-integral $I = u_{xx} + u_x^2$ also defines an n-integral for semi-discrete Liouville equation $u_{n+1,x} - u_{n,x} = \exp(-u_n - u_{n+1})$:

$$(T - I)\left(u_{n,xx} + u_{n,x}^2\right) = u_{n+1,xx} - u_{n,xx} + u_{n+1,x}^2 - u_{n,x}^2 =$$
$$= D_x\left(\exp(-u_n - u_{n+1})\right) + (u_{n+1,x} + u_{n,x})(u_{n+1,x} - u_{n,x}) = 0.$$

Moreover, semi-discrete Liouville equation also admits an x-integral

$$J = \exp(u_n - u_{n+1}) - \exp(u_{n+1} - u_{n+2}).$$

Integral preserving discretization

Theorem (S., J. Phys. A, 2023)

Let $I = I(u_x^1, \dots, u_x^r, u_{xx}^1, \dots, u_{xx}^r, u_{xxx}^1, \dots, u_{xxx}^r, \dots)$ be a y-integral of exponential system (1) with non-degenerate matrix $M = (a_{ij})$ in the continuous case. Then the same function

$$I_n = I(u_{n,x}^1, \dots, u_{n,x}^r, u_{n,xx}^1, \dots, u_{n,xx}^r, u_{n,xxx}^1, \dots, u_{n,xxx}^r, \dots)$$

is an n-integral of its semi-discrete analog (2).

Corollary

Semi-discrete exponential systems (2) corresponding to the Cartan matrices of all simple Lie algebras admit complete families of essentially independent n-integrals.

Characteristic algebra of exponential system

Denote $u^i_k = D^k_x(u^i)$. Operator of total differentiation by virtue of (1) can be represented as $D_y = \sum_{i=1}^r e^{w^i} X_i$, where

$$X_i = \frac{\partial}{\partial u^i_1} + b^i_1 \frac{\partial}{\partial u^i_2} + b^i_2 \frac{\partial}{\partial u^i_3} + \dots, \quad w^i = a_{i1} u^1 + \dots + a_{ir} u^r,$$

and $b^i_k = e^{-w^i} D^k_x(e^{w^i})$ are *complete Bell polynomials*.

Definition (Shabat, Yamilov, 1981)

Lie algebra generated by $X_1, \dots X_r$ is called *characteristic Lie algebra* of (1).

Function I is a y-integral of exponential system (1) with non-degenerate matrix M iff it annihilates the characteristic algebra. Such exponential system is Darboux integrable iff its characteristic algebra is finite-dimensional (Shabat, Yamilov, 1981).

Why integrals are preserved?

Proposition (S., J. Phys. A, 2023)

Let $I = I(\mathbf{u}_x, \mathbf{u}_{xx}, \dots)$ be a y-integral of (1). Then

$$I_{n+1} - I_n = \sum_{m=1}^{\infty} \left(\sum_{k_1+\cdots+k_r=m} \frac{1}{k_1! \dots k_r!} X_r^{k_r} X_{r-1}^{k_{r-1}} \dots X_1^{k_1}(I) t_1^{k_1} \dots t_r^{k_r} \right),$$

where the sum is taken over all partitions $m = k_1 + \dots + k_r$ such that $k_1, \dots, k_r \geqslant 0$ and

$$t_i = \exp\left(\sum_{j=1}^{r} (a_{ij}^- u_n^j + a_{ij}^+ u_{n+1}^j) \right), \quad i = 1, 2, \dots, r$$

Characteristic algebra in the discrete case

The notion of characteristic algebra can be defined for discrete hyperbolic systems as well (Habibullin, 2005, 2007). If the variables enter the equation not symmetrically then one has to consider characteristic algebras along each of the two variables.

Theorem (Habibullin et al.)

(Semi)-discrete hyperbolic system is Darboux integrable iff both of its characteristic algebras are finite-dimensional.

Modified characteristic algebra

In the case of semi-discrete exponential system (2) the operator D_x can be expressed in the form

$$D_x = \sum_{i=1}^{r} e^{w_n^i} Y_i,$$

where $w_n^i = a_{i1} u_n^1 + \cdots + a_{ir} u_n^r$ and

$$Y_i = c_0^i \frac{\partial}{\partial v_n^i} + c_1^i \frac{\partial}{\partial v_{n+1}^i} + c_2^i \frac{\partial}{\partial v_{n+2}^i} + \ldots, \quad v_n^i = u_{n+1}^i - u_n^i.$$

We call Lie algebra generated by vector fields Y_i the *modified characteristic algebra* of (2).

Modified characteristic algebra, continued

Proposition (S., J. Phys. A, 2023)

Let M be non-degenerate. Then function

$$J = J(\mathbf{v}_n, \mathbf{v}_{n+1}, \mathbf{v}_{n+2} \dots)$$

is an x-integral of (2) iff it annihilates operators $Y_1, \dots, Y_r$.

Proposition (S., J. Phys. A, 2023)

Let M be non-degenerate. Then semi-discrete exponential system (2) admits a complete family of essentially independent x-integrals iff its modified characteristic algebra is finite-dimensional.

Existence of x-integrals

Theorem (S., J. Phys. A, 2023)

Modified characteristic algebra for a semi-discrete exponential system (2) associated to the Cartan matrix of any simple Lie algebra is isomorphic to the characteristic algebra of its continuous analog.

Corollary

Semi-discrete exponential system (2) associated to the Cartan matrix of any simple Lie algebra is Darboux integrable.

Thank you!