Adler map for Darboux q-chain

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1 Continuous case

Spectral problem $\psi'' + (\lambda - u)\psi = 0$ for one-dimensional Schrödinger operator can be rewritten in matrix form $\Psi' = U\Psi$, where

$$\Psi = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}.$$

In their joint work [1], Veselov and Shabat considered the dressing chain for Schrödinger operator, that is, the sequence of second order differential operators L_j on the real line satisfying the following operator relation:

$$L_j = A_j A_j^+ - \alpha_j = A_{j-1}^+ A_{j-1}. \tag{1}$$

Here $A_j = \frac{d}{dx} + f_j$, plus over A_j denotes a formal conjugation and α_j is a constant. In this dressing chain, the operators L_j and $L_{j+1} - \alpha_j$ are connected by first order Darboux transformation

$$A_j^+ L_j = (L_{j+1} - \alpha_j) A_j^+$$

and therefore each A_j shifts the spectrum by α_j:

$$L_j \psi_j = \lambda \psi_j \quad \Longrightarrow \quad L_{j+1}\psi_{j+1} = (\lambda + \alpha_j)\psi_{j+1},$$

where $\psi_{j+1} = A_j^+ \psi_j$. In matrix form this can be rewritten as follows:

$$\Psi_j' = U_j \Psi_j, \quad \Psi_{j+1} = W_j \Psi_j,$$

where

$$U_j = \begin{pmatrix} 0 & 1 \\ f_j' + f_j^2 - \lambda - \beta_j & f_j \end{pmatrix}, \quad W_j = \begin{pmatrix} f_j & -1 \\ -f_j^2 + \lambda + \beta_j & f_j \end{pmatrix}, \quad \beta_j = \alpha_1 + \cdots + \alpha_j. \tag{2}$$

The operator relation (1) is equivalent to the following system of nonlinear differential equations:

$$(f_j + f_{j+1})' = f_j^2 - f_{j+1}^2 + \alpha_{j+1}. \tag{3}$$

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It is well-known that any Darboux transformation for one-dimensional Schrödinger operator can be expressed as a product of Darboux transformations of order 1, but such decomposition is not unique and this ambiguity leads to discrete symmetries of the dressing chain (3). Indeed, any two different factorizations $W_{j+1}W_j = \tilde{W}_{j+1}\tilde{W}_j$ into product of two matrices of the type (2) provide a mapping

$$A: (f_j, f_{j+1}, \beta_j, \beta_{j+1}) \mapsto (\tilde{f}_j, \tilde{f}_{j+1}, \tilde{\beta}_j, \tilde{\beta}_{j+1}), \tag{4}$$

that is, a discrete symmetry of the system (3). This procedure proposed by V. Adler in his PhD thesis for the dressing chain (3) has two important properties: such symmetry is local (i.e. the map (4) acts only on two neighbouring variables and a corresponding pair of parameters leaving the others unchanged) and the overdetermined system

$$W_{j+1}W_j = \tilde{W}_{j+1}\tilde{W}_j \tag{5}$$

is always compatible since it has at least trivial solution $\tilde{f}_k = f_k$, $\tilde{\beta}_k = \beta_k$ for all k.

PROPOSITION. [2] *If $\alpha_k = \beta_{k+1} - \beta_k$ for all k, then for every j the system (3) has unique non-trivial discrete symmetry of the type (5) defined by the formulae*

$$\begin{cases} \tilde{f}_j = f_j - \frac{\beta_{j+1}-\beta_j}{f_j+f_{j+1}}, \quad \tilde{f}_{j+1} = f_{j+1} + \frac{\beta_{j+1}-\beta_j}{f_j+f_{j+1}}, \quad \tilde{\beta}_j = \beta_{j+1}, \quad \tilde{\beta}_{j+1} = \beta_j, \\ \tilde{f}_k = f_k, \quad \tilde{\beta}_k = \beta_k, \quad k \neq j, j+1. \end{cases}.$$

Let X be any set, let $R: X \times X \to X \times X$ be a map from its square into itself. Suppose $R_{ij}: X^n \to X^n$ are the maps acting as R on i-th and j-th factors and acting identically on the others. The map R is called *reversible Yang-Baxter map* if the following holds:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad R_{21}R = \mathrm{Id}, \tag{6}$$

where $R_{21} = PRP$ and $P: X^2 \to X^2$ is a permutation map, $P(x, y) = (y, x)$. According to general Veselov's argument under certain circumstances the above refactorizing procedure always leads to Yang-Baxter maps (see [3] for the details). More precisely, this means that in our case the map $R = PA$ acting on the set of pairs $X = \{(f, \beta)\}$ satisfies the relations (6), although the last claim can be straightforwardly checked. The Adler map A itself satisfies the braid group relation.

Recently V. Adler, Bobenko and Suris [4] classified the so-called quadrirational maps of $\mathbb{C}P^1 \times \mathbb{C}P^1$ into itself. According to their classification any map of that kind is equivalent, under some change of variables from the group $(\mathrm{Mob})^4$, to one of the five maps F_I–F_V, where Mob is the group of projective transformations of $\mathbb{C}P^1$. It is remarkable that all these five maps satisfy the relations (6) though generally not all quadrirational maps satisfy the Yang-Baxter relation and moreover the action of $(\mathrm{Mob})^4$ does not preserve the Yang-Baxter property (see [5]). The map F_V is exactly the Adler map modulo additional permutation (i.e. $F_V = PA$).

2 Discrete case

Consider the following difference analog of periodically closed Veselov-Shabat dressing chain (3) that was introduced in [6]: the operator relation (7) is replaced by

$$L_j = A_j A_j^+ - \alpha_j = q A_{j-1}^+ A_{j-1}, \tag{7}$$

where $A_j = a_j T^{-1} + b_j T$, $q > 0$ is a constant and T is a shift operator acting on functions defined on the lattice $\mathbb{Z}$: $(T\psi)(n) = \psi(n+1)$. Such discretization has many properties similar to the ones in the continuous case. In a certain sense is even better than the continuous model from the point of view of spectral theory (see [7] for the details). In this case, q-Darboux transformation A_j^+ acts on eigenvectors and eigenfunctions as follows:

$$L_j\psi_j = \lambda\psi_j \quad \Longrightarrow \quad L_{j+1}\psi_{j+1} = q(\lambda + \alpha_j)\psi_{j+1},$$

where $\psi_{j+1} = A_j^+\psi_j$. The operator relation (7) is equivalent to the following nonlinear system of difference equations:

$$\left\{ \begin{array}{l} \xi_j(n-1) + \eta_j(n) = q(\xi_{j-1}(n) + \eta_{j-1}(n-1)) + \alpha_j \\ \xi_j(n)\eta_j(n-1) = q^2\xi_{j-1}(n-1)\eta_{j-1}(n) \end{array} \right. , \tag{8}$$

where $\xi_j(n) = a_j^2(n+1)$, $\eta_j(n) = b_j^2(n)$. The system (8) will be referred to as the q-Darboux chain. If one consideres lattice $h\mathbb{Z}$ instead of $\mathbb{Z}$ and sets $q = \exp(-Ch^2)$ for certain constant C, then in continuum limit as $h \to 0$ the system (8) tends to (3).

In matrix form the chain (7) can be rewritten as follows:

$$T(\Psi_j) = U_j\Psi_j, \quad \Psi_{j+1} = W_j\Psi_j,$$

where

$$\Psi_j(n) = \begin{pmatrix} \psi_j(n-2) \\ \psi_j(n-1) \\ \psi_j(n) \\ \psi_j(n+1) \end{pmatrix}, \quad U_j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{a_j(n)b_j(n-2)}{a_j(n+2)b_j(n)} & 0 & \frac{\beta_j + q^{j-1}\lambda - a_j^2(n) - b_j^2(n)}{a_j(n+2)b_j(n)} & 0 \end{pmatrix},$$

$$W_j = \begin{pmatrix} 0 & \frac{\beta_j + q^{j-1}\lambda - b_j^2(n-1)}{a_j(n-1)} & 0 & -\frac{a_j(n+1)b_j(n-1)}{a_j(n-1)} \\ b_j(n-2) & 0 & a_j(n) & 0 \\ 0 & b_j(n-1) & 0 & a_j(n+1) \\ -\frac{a_j(n)b_j(n-2)}{b_j(n)} & 0 & \frac{\beta_j + q^{j-1}\lambda - a_j^2(n)}{b_j(n)} & 0 \end{pmatrix}, \tag{9}$$

$\beta_j = \alpha_j + q\alpha_{j-1} + \cdots + q^{j-1}\alpha_1$. Silimarly to the case of the differential dressing, the following proposition holds for discrete symmetries of difference Darboux q-chain.

Proposition. *For every j the system (8) has unique non-trivial discrete symmetry of the type (5),(9) defined by the formulae*

$$\left\{ \begin{array}{l} \tilde{\xi}_j = \xi_j + \frac{(\beta_{j+1} - q\beta_j)\xi_j}{q\xi_j - \eta_{j+1}}, \quad \tilde{\eta}_{j+1} = \eta_{j+1} + \frac{(\beta_{j+1} - q\beta_j)\eta_{j+1}}{q\xi_j - \eta_{j+1}}, \\ \tilde{\eta}_j = \eta_j + \frac{(\beta_{j+1} - q\beta_j)\eta_j}{q\eta_j - \xi_{j+1}}, \quad \tilde{\xi}_{j+1} = \xi_{j+1} + \frac{(\beta_{j+1} - q\beta_j)\xi_{j+1}}{q\eta_j - \xi_{j+1}}, \\ \tilde{\beta}_j = \beta_{j+1}, \quad \tilde{\beta}_{j+1} = \beta_j, \\ \tilde{\xi}_k = f_k, \quad \tilde{\eta}_k = \eta_k, \quad \tilde{\beta}_k = \beta_k, \quad k \neq j, j+1. \end{array} \right. . \tag{10}$$

The mapping

$$(\xi_j, \eta_j, \xi_{j+1}, \eta_{j+1}, \beta_j, \beta_{j+1}) \mapsto (\tilde{\xi}_j, \tilde{\eta}_j, \tilde{\xi}_{j+1}, \tilde{\eta}_{j+1}, \tilde{\beta}_j, \tilde{\beta}_{j+1}) \tag{11}$$

satisfies the Yang-Baxter relation (6) after an additional permutation. The structure of this map is rather simple — it acts independently on pairs of variables (ξ_j, η_{j+1}) and (η_j, ξ_{j+1}) and the action B is the same for both pairs. If $q = 1$, then the map B coincides with the map F_{IV} of the classification list from [4] modulo a permutation.

Another discrete analog of the dressing chain (3) was introduced in [8]: the chain is obtained using the relation (3) and difference operators $A_j = a_j + b_j T$ (or $A = a_j + b_j T^{-1}$). This discretization could possibly seem more natural but it is not as interesting as the previous one from the point of view of spectral theory. In this case, the above procedure provides an invertible Yang-Baxter map as a discrete symmetry of the corresponding system of difference equations as well. The analog of the map (11) is also reduced to two quadrirational maps acting independently on pairs of variables but the formulas are not as symmetric as (10) since the initial choice of A_j already breaks the symmetry.

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