Integral preserving discretization of 2D Toda lattices

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Abstract

There are different methods of discretizing integrable systems. We consider semi-discrete analog of two-dimensional Toda lattices associated to the Cartan matrices of simple Lie algebras that was proposed by Habibullin in 2011. This discretization is based on the notion of Darboux integrability. Generalized Toda lattices are known to be Darboux integrable in the continuous case (that is, they admit complete families of characteristic integrals in both directions). We prove that semi-discrete analogs of Toda lattices associated to the Cartan matrices of all simple Lie algebras are Darboux integrable. By examining the properties of Habibullin's discretization we show that if a function is a characteristic integral for a generalized Toda lattice in the continuous case, then the same function is a characteristic integral in the semi-discrete case as well. We consider characteristic algebras of such integral-preserving discretizations of Toda lattices to prove the existence of complete families of characteristic integrals in the second direction.

1 Introduction

Two-dimensional Toda lattice

$$q_{i,xy} = \exp(q_i - q_{i+1}) - \exp(q_{i-1} - q_i) \tag{1}$$

plays an important role both in classical differential geometry and in mathematical physics. It is known to be Darboux integrable (i.e. it admits complete family of essentially independent characteristic integrals), if the trivial boundary conditions $q_{-1} = -\infty$ and $q_{r+1} = +\infty$ are imposed for some natural r. Toda lattice can be rewritten in the form

$$\left\{ \begin{array}{l} u^1_{xy} = \exp(2u^1 - u^2) \\ u^i_{xy} = \exp(-u^{i-1} + 2u^i - u^{i+1}), \quad i = 2, \dots, r-1 \\ u^r_{xy} = \exp(-u^{r-1} + 2u^r) \end{array} \right. , \tag{2}$$

where $q_i = u^{i+1} - u^i$. This system is a particular case of the so-called *exponential systems*

$$u^i_{xy} = \exp\left(\sum_{j=1}^{r} a_{ij} u^j\right), \quad i = 1, 2, \dots, r, \tag{3}$$

that were introduced in [1] (here a_{ij} are constant coefficients). Such integrable generalizations of the Toda system corresponding to the Cartan matrices $M = (a_{ij})$ of all simple Lie algebras were studied in a number of papers in the beginning of 1980-ies [1]-[4]. Almost at the same time, discrete versions of some of these systems started to appear in literature within the frame of discretizing the

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theory of integrable systems [5, 6]. Due to the fact that generalized Toda systems are related to many relevant mathematical theories, the number of papers on the subject is enormous: continuous Toda systems and their discretizations are addressed from different perspectives and studied using various methods. Not even attempting to make a review, we will mention the papers [7]-[17] that are important for our approach to the problem of discretizeing Toda lattices.

Purely discrete versions of the generalized Toda lattice, associated with the A-series Cartan matrices, were studied in [7, 8, 9]. Purely discrete and semi-discrete versions of the C-series Toda lattice were examined in [11, 15]. Discretizations of generalized Toda lattices introduced in these papers are based mostly on the notion of the Laplace invariants of hyperbolic difference or differential-difference operators and on their specific properties, and therefore this approach is not applicable in the general case of an exponential system associated to arbitrary Cartan matrix.

There are several general approaches that lead to (semi)-discrete versions of exponential systems (3) corresponding to Cartan matrices. Hamiltonian approach was used in [10] to obtain semi-discrete analogs of exponential systems associated to Cartan matrices of all simple Lie algebras. Another approach to the problem of discretization of exponential systems was proposed in [13, 14]. The idea is to look for a semi-discrete system for the functions u_n^i, depending on continuous variable x and discrete variable n, such that its characteristic n-integrals are given by the same formulas as y-integrals of the continuous model. We will call this method *integral preserving discretization.* This approach had appeared to be fruitful earlier in the study of Darboux integrable semi-discrete scalar equations [18], before it was applied to the Toda lattices. The analysis of the case $r = 2$ from the viewpoint of integral preserving method allowed the authors [13] to propose the following discretization for exponential system (3) with the coefficient matrix $M = (a_{ij})$:

$$u^i_{n+1,x} - u^i_{n,x} = \exp\left(\sum_{j=1}^{i-1} a_{ij}u^j_n + \frac{a_{ii}}{2}(u^i_n + u^i_{n+1}) + \sum_{j=i+1}^{r} a_{ij}u^j_{n+1}\right). \tag{4}$$

Similar approach [14] allows to define the analogs of exponential systems in the purely discrete case:

$$\exp(u^i_{n+1,m+1} + u^i_{n,m} - u^i_{n+1,m} - u^i_{n,m+1}) = \\ = 1 + \exp\left(\sum_{j=1}^{i-1} a_{ij}u^j_{n,m+1} + \frac{a_{ii}}{2}(u^j_{n,m+1} + u^j_{n+1,m}) + \sum_{j=i+1}^{r} a_{ij}u^j_{n+1,m}\right). \tag{5}$$

In both cases, Darboux integrability was proved only for discretized systems associated with all Cartan matrices of the rank 2 [13, 14]. In the general case, Darboux integrability for (semi)-discrete versions of the Toda lattice is proven only for the A- and C-series lattice by constructing a generating function for characteristic integrals [19], but this method is not applicable for the B-, D-series lattices and for the systems corresponding to exceptional Cartan matrices E_6–E_8 and F_4.

Although discterizations of A-series Toda lattice given in [10] and in [13] are the same, the methods used there produce different semi-discrete systems for other series of Cartan matrices (continuum limits in both cases are the same). Another version of semi-discrete Toda system of the series B is obtained in [16] by considering a modification of skew-orthogonal polynomials that arise in the random matrix theory. In [17] direct linearization method was used to study semi-discrete analogs of Toda systems corresponding to some series of affine Cartan matrices. In the purely discrete case there also exist different versions of Toda systems corresponding to Cartan matrices, see [14, 12].

Various properties of integrable models are usually taken as a basis for finding discretizations. Discretization of Toda systems proposed in [13] is based on the notion of Darboux integrability and hence it is important to show that systems introduced in [13] are Darboux integrable indeed. In this paper we focus on semi-discrete exponential systems (4) and prove that such systems corresponding to the Cartan matrices of all simple Lie algebras are Darboux integrable. Therefore we justify the integral preserving discretization method for generalized Toda lattices. More precisely, we prove that if exponential system (3) admits y-integral

$$I = I(u_x^1, \dots, u_x^r, u_{xx}^1, \dots, u_{xx}^r, u_{xxx}^1, \dots, u_{xxx}^r, \dots),$$

then the same function

$$I_n = I(u_{n,x}^1, \dots, u_{n,x}^r, u_{n,xx}^1, \dots, u_{n,xx}^r, u_{n,xxx}^1, \dots, u_{n,xxx}^r, \dots),$$

whose arguments are replaced by the dynamical variables for the semi-discrete case, is an n-integral for discretization (4) of this system. Besides this, using *characteristic algebras*, we prove that semi-discrete versions (4) of B-, D-series Toda lattices and the systems associated to exceptional Cartan matrices E_6–E_8 and F_4 admit complete families of essentially independent characteristic x-integrals (Darboux integrability of A- and C-series lattices have been proved earlier). Altogether, this proves Darboux integrability of all semi-discrete Toda lattices (4) corresponding to the Cartan matrices of simple Lie algebras, which was conjectured in [13].

The paper is structured as follows: in Section 2 we review the notions of Darboux integrability, characteristic algebra and the relation between them. In Section 3 we describe Habibullin's method and prove the existence of a complete family of characteristic n-integrals for semi-discrete lattices (4) corresponding to the Cartan matrices of all simple Lie algebras and hence we justify Habibullin's integral preserving discretization method by showing that if a function is a y-integral of discrete exponential system, then the same function defines an n-integral for its semi-discrete analog. Basic properties of characteristic algebras for exponential systems associated to the Cartan matrices of simple Lie algebras are discussed in Section 4. In Section 5 we prove the existence of a complete family of independent x-integrals for semi-discrete exponential systems corresponding to the Cartan matrices of all simple Lie algebras.

2 Darboux integrability and characteristic algebras

In the theory of integrable systems there are several different approaches to integrability depending in the class in the systems that are being considered: Liouville integrability, existence of a Lax pair, existence of higher symmetries. *Darboux integrability* is a kind of "very strong" integrability that is defined for hyperbolic systems and that is closely related to explicit integrability. We start this Section with a series of definitions and notation [1, 20] that will be used in this paper.

Function

$$I = I(u^1, \dots, u^r, u_x^1, \dots, u_x^r, u_{xx}^1, \dots, u_{xx}^r, u_{xxx}^1, \dots, u_{xxx}^r, \dots)$$

is called a *y-integral* of hyperbolic system

$$u_{xy}^i = F^i(x, y, u^1, \dots, u^r, u_x^1, \dots, u_x^r, u_y^1, \dots, u_y^r), \quad i = 1, \dots, r, \tag{6}$$

if its total derivative with respect to y by virtue of the system vanishes:

$$0 = D_y(I) = \sum_{i=1}^{r} \left(u_y^i \frac{\partial I}{\partial u^i} + F^i \frac{\partial I}{\partial u_x^i} + D_x(F^i) \frac{\partial I}{\partial u_{xx}^i} + D_x^2(F^i) \frac{\partial I}{\partial u_{xxx}^i} + \dots \right),$$

where D_x is the total derivative with respect to x. The highest order of x-derivative of the functions $u^1, \dots u^r$, on which y-integral I depends, is called *the order* of I; y-integral is called *non-trivial* if it depends not only on the independent variable x. Here and further we will consider only non-trivial integrals; *x-integrals* of hyperbolic system (6) are defined similarly. Both x- and y-integrals are called *characteristic integrals*. Denote

$$u_1^i = u_x^i, \quad u_2^i = u_{xx}^i, \quad u_3^i = u_{xxx}^i, \dots, \quad i = 1, \dots, r.$$

Characteristic y-integrals $I_1, \ldots, I_k$ of orders $d_1, \ldots d_k$ are called *essentially independent* if the rank of the matrix

$$\begin{pmatrix} \frac{\partial I_1}{\partial u^1_{d_1}} & \frac{\partial I_1}{\partial u^2_{d_1}} & \cdots & \frac{\partial I_1}{\partial u^r_{d_1}} \\ \frac{\partial I_2}{\partial u^1_{d_2}} & \frac{\partial I_2}{\partial u^2_{d_2}} & \cdots & \frac{\partial I_2}{\partial u^r_{d_2}} \\ \vdots & & \ddots & \\ \frac{\partial I_k}{\partial u^1_{d_k}} & \frac{\partial I_k}{\partial u^2_{d_k}} & \cdots & \frac{\partial I_k}{\partial u^r_{d_k}} \end{pmatrix}$$

is equal to k.

Definition 1. Hyperbolic system (6) is called *Darboux integrable* if it admits complete families of essentially independent x- and y-integrals.

Example 1. Liouville equation

$$u_{xy} = \exp u \tag{7}$$

is Darboux integrable since it admits characteristic integrals in both directions: functions

$$I = u_{xx} - \frac{1}{2}u_x^2 \quad \text{and} \quad J = u_{yy} - \frac{1}{2}u_y^2 \tag{8}$$

are y- and x-integrals respectively.

Exponential systems associated with the Cartan matrices of simple Lie algebras (i.e. exponential systems (3) such that the coefficient matrix is the Cartan matrix of one of simple Lie algebras) are known to be Darboux integrable [2]. These systems are also known in literature as generalized Toda lattices corresponding to the Cartan matrices of simple Lie algebras. Explicit formulas for characteristic integrals in terms of wronskians were found in [21] for Toda lattices of series A–D. Another approach that allows to obtain generating function for characteristic integrals was developed in [22] for the A-series Toda lattices and in [19] for lattices of the series A–C.

Characteristic integrals are two-dimensional analogs of first integrals for ODEs, but there is an essential difference between these two cases: hyperbolic equations having characteristic integrals are exceptional. If function I is a y-integral of (6), then its x-derivatives $D_x(I), D_x^2(I), \ldots$ are obviously also y-integrals, but these integrals are not essentially independent. The existence of a complete family of characteristic integrals is controlled by an algebraic tool — Lie algebra of differential operators that is called *the characteristic algebra* of hyperbolic system [23, 1, 4]. Characteristic algebra can be defined for arbitrary hyperbolic system of form (6), but in this general case it should be considered as a Lie-Rinehart algebra (see discussion in [24]). In the special case of exponential systems (3) the situation is more simple and the characteristic algebra can be referred to as a Lie algebra generated by differential operators of a certain kind.

Definition 2. Lie algebra generated by operators

$$\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^r}, \quad D_y = \sum_{i=1}^{r} \left(e^{w^i} \frac{\partial}{\partial u^i_1} + D_x\left(e^{w^i}\right) \frac{\partial}{\partial u^i_2} + D_x^2\left(e^{w^i}\right) \frac{\partial}{\partial u^i_3} + \ldots \right),$$

where $w^i = a_{i1}u^1 + \cdots + a_{ir}u^r$, is called *the characteristic algebra* of exponential system (3).

One can easily show that y-integrals of exponential systems (3) cannot depend on $u^1, \ldots u^r$: they depend only on their x-derivatives. Therefore any y-integral annihilates the whole characteristic algebra.

Remark 1. In the general case of hyperbolic systems (6), one has to define characteristic algebra in the direction of the variable x and characteristic algebra in the direction y. For exponential systems (3), these Lie algebras are isomorphic since variables x and y enter the equations symmetrically.

Proposition 1. *Let the matrix $M = (a_{ij})$ of exponential system (3) be non-degenerate. Then its characteristic algebra is generated by operators*

$$\frac{\partial}{\partial u^i}, \quad \tilde{X}_i = e^{w^i}\left(\frac{\partial}{\partial u_1^i} + b_1^i \frac{\partial}{\partial u_2^i} + b_2^i \frac{\partial}{\partial u_3^i} + \dots\right), \quad i = 1, \dots, r,$$

where $b_k^i = b_k^i(w_1^i, w_2^i, \dots w_k^i) = e^{-w^i} D_x^k(e^{w^i})$ is the k-th complete Bell polynomial of the variables $w_1^i, w_2^i, \dots w_k^i$ and $w_k^i = D_x^k(w^i)$.

Proof. Simple calculation shows that for all $i = 1, \dots, r$

$$\left[\frac{\partial}{\partial u^i}, D_y\right] = a_{i1}\tilde{X}_1 + a_{i2}\tilde{X}_2 + \dots + a_{ir}\tilde{X}_r.$$

Hence, all operators of the form $a_{i1}\tilde{X}_1 + a_{i2}\tilde{X}_2 + \dots + a_{ir}\tilde{X}_r$ belong to the characteristic algebra, and it follows from non-degeneracy of the matrix M that operators $\tilde{X}_i$ are linear combinations of these operators. Therefore, they belong to characteristic algebra, and since $D_y = \tilde{X}_1 + \dots + \tilde{X}_r$, they generate the characteristic algebra together with $\frac{\partial}{\partial u^i}$, where $i = 1, \dots, r$. □

Theorem 1. [1] *Exponential system (3) is Darboux integrable if and only if its characteristic algebra is finite-dimensional.*

Example 2. Characteristic algebra of the Liouville equation (7) is two-dimensional: $\left[\frac{\partial}{\partial u}, D_y\right] = D_y$.

Remark 2. Since

$$\left[\frac{\partial}{\partial u^j}, \tilde{X}_i\right] = a_{ij}\tilde{X}_i$$

for all $i, j = 1, \dots, r$ and characteristic y-integrals of an exponential system (3) cannot depend on $u^1, \dots, u^r$, it is sufficient for the study of Darboux integrability to consider *reduced characteristic algebra* generated by $\tilde{X}_1, \dots, \tilde{X}_r$: obviously, exponential system (3) is Darboux integrable if and only if its reduced characteristic algebra is finite-dimensional. Note that this Lie algebra is isomorphic to Lie algebra generated by vector fields $X_1, \dots X_r$ where $X_i = e^{-w^i}\tilde{X}_i$ for all $i = 1, \dots r$.

The notion of Darboux integrability can be extended to the case of (semi)-discrete hyperbolic systems. Function

$$I_n = I(u_n^1, \dots, u_n^r, u_{n,x}^1, \dots, u_{n,x}^r, u_{n,xx}^1, \dots, u_{n,xx}^r, u_{n,xxx}^1, \dots, u_{n,xxx}^r, \dots)$$

is called an *n-integral* of semi-discrete hyperbolic system

$$u_{n+1,x}^i - u_{n,x}^i = F^i(x, n, u_n^1, \dots, u_n^r, u_{n,x}^1, \dots, u_{n,x}^r, u_{n+1}^1, \dots, u_{n+1}^r), \quad i = 1, \dots, r, \tag{9}$$

if its total difference derivative by virtue of the system vanishes: $I_{n+1} - I_n = 0$. One can verify that in the semi-discrete case n-integrals cannot depend on shifted variables u_{n+1}^i and x-integrals cannot depend on the derivatives $u_{n,x}^i$, where $i = 1, \dots, r$. *The order d* of an x-integral is defined as the highest shift u_{n+d} on which it depends. Family of x-integrals $J_1, \dots J_k$ of orders $d_1, \dots d_k$ are called *essentially independent* if the rank of the matrix

$$\begin{pmatrix} \frac{\partial J_1}{\partial u^1_{n+d_1}} & \frac{\partial J_1}{\partial u^2_{n+d_1}} & \dots & \frac{\partial J_1}{\partial u^r_{n+d_1}} \\ \frac{\partial J_2}{\partial u^1_{n+d_2}} & \frac{\partial J_2}{\partial u^2_{n+d_2}} & \dots & \frac{\partial J_2}{\partial u^r_{n+d_2}} \\ \vdots & & \ddots & \\ \frac{\partial J_k}{\partial u^1_{n+d_k}} & \frac{\partial J_k}{\partial u^2_{n+d_k}} & \dots & \frac{\partial J_k}{\partial u^r_{n+d_k}} \end{pmatrix}$$

is equal to k. Similarly to the continuous case, hyperbolic system (9) is called *Darboux integrable* if it admits complete families of essentially independent n- and x- integrals. In the entirely discrete case, Darboux integrability of hyperbolic system

$$u^i_{n+1,m+1} - u^i_{n+1,m} - u^i_{n,m+1} + u^i_{n,m} = F^i(n, m, u^1_{n,m}, \dots, u^r_{n,m}, u^1_{n+1,m}, \dots, u^r_{n+1,m}, u^1_{n,m+1}, \dots, u^r_{n,m+1}),$$

where $i = 1, \dots, r$, requires the existence of essentially independent families of m- and n-integrals.

Darboux integrability of semi-discrete and entirely discrete exponential systems corresponding to the Cartan matrices of the series A and C (i.e. generalized (semi)-discrete Toda lattices of the series A and C) was proved in [19]. Another approach allowing to obtain a complete family of essentially independent x-integrals for semi-discrete A-series Toda lattice in terms of *casoratians* was developed in [25].

3 Integral preserving discretization

There are many different ways to discretize integrable systems. One of the popular methods to discretize a PDE with two independent variables is to consider iterations of its Bäcklund transformations as shinfts in a new discrete variable. Then the formula for Bäcklund transformation that relates the unknown function u with its Bäcklund-image u_1 is a differential-difference equation and it can be referred to as a semi-discrete analog of the initial PDE. In this case, the superposition formula plays the role of entirely discrete analog. Although this approach is widely used in the theory of integrable systems, we will use another approach proposed by Habibullin et al. [18] that is based on the existence of characteristic integrals for generalized Toda lattices, which is a very specific property for this class of systems.

In this section, we describe the integral preserving discretization method for exponential systems associated to the Cartan matrices of simple Lie algebras and justify it by proving that this approach provides Darboux integrable semi-discrete systems.

Goursat [26] had found a complete list of scalar hyperbolic equations having both characteristic integrals of order not greater than 2 (the so-called *Goursat list*) within the frame of the study of hyperbolic equations that admit characteristic integrals. The idea proposed in [18] is to take characteristic y-integral for an equation in the Goursat list and to find semi-discrete hyperbolic equation such that this function is its n-integral. The discretization obtained using this method inherits the main property of its continuous counterpart: by construction, it admits an n-integral of order not greater than 2. Surprisingly, all semi-discrete equations found in [18] appear to be Darboux integrable, i.e. in addition to n-integrals they also admit characteristic x-integrals. Moreover, this method was also applied to these semi-discrete equations in order to get entirely discrete analogs of the equations from the Goursat list [18].

The same method was used in [13] to guess an appropriate formula for semi-discrete analog of exponential systems (3) corresponding to the Cartan matrices of simple Lie algebras. More precisely, careful analysis of the systems corresponding to the Cartan matrices of the rank 2 and their characteristic integrals allowed the authors to find semi-discrete analogs for exponential systems of the rank 2, to obtain formula (4) and to conjecture that systems (4) are Darboux integrable for the Cartan matrices of all simple Lie algebras. The following theorem is the main result of this section and it proves the first part of the conjecture from [13] (that Habilullin's discretization preserves y-integrals); the second part that states the existence of sufficient number of essentially independent x-integrals is proved in section 5.

THEOREM 2. *Let $I = I(u^1_x, \dots, u^r_x, u^1_{xx}, \dots, u^r_{xx}, u^1_{xxx}, \dots, u^r_{xxx}, \dots)$ be a y-integral of exponential system (3) with non-degenerate matrix $M = (a_{ij})$. Then the same function*

$$I_n = I(u^1_{n,x}, \dots, u^r_{n,x}, u^1_{n,xx}, \dots, u^r_{n,xx}, u^1_{n,xxx}, \dots, u^r_{n,xxx}, \dots)$$

is an n-integral of discretization (4) of this system.

Proof.
Let I be a y-integral or order d of the continuous system (3). Similarly to the continuous case, denote $u^i_{n,k} = D^k_x(u^i_n)$ and $w^i_n = a_{i1}u^i_n + \dots + a_{ir}u^i_r$ for all $i = 1, \dots, k$. Using Taylor's expansion, rewrite the difference $I_{n+1} - I_n$, where variables u^i_n satisfy semidiscrete system (4):

$$\begin{aligned} I_{n+1} - I_n &= \sum_{i=1}^{r}\sum_{k=1}^{d} \frac{\partial I}{\partial u^i_k}\left(u^i_{n+1,k} - u^i_{n,k}\right) + \\ &+ \frac{1}{2}\sum_{i_1,i_2=1}^{r}\sum_{k_1,k_2=1}^{d} \frac{\partial^2 I}{\partial u^{i_1}_{k_1}\partial u^{i_2}_{k_2}}\left(u^{i_2}_{n+1,k_2} - u^{i_2}_{n,k_2}\right)\left(u^{i_1}_{n+1,k_1} - u^{i_1}_{n,k_1}\right) + \dots = \\ &= \sum_{s=1}^{\infty}\left(\frac{1}{s!}\sum_{\lambda_{ik}:\ \sum\limits_{i=1}^{r}\sum\limits_{k=1}^{d}\lambda_{ik}=s} \frac{s!}{\prod\limits_{i=1}^{r}\prod\limits_{k=1}^{d}\lambda_{ik}!}\prod_{i=1}^{r}\prod_{k=1}^{d}\left(u^i_{n+1,k} - u^i_{n,k}\right)^{\lambda_{ik}}\frac{\partial^s I}{(\partial u^1_1)^{\lambda_{11}}\dots(\partial u^r_d)^{\lambda_{rd}}}\right) \end{aligned} \tag{10}$$

Denote

$$E_i = \exp(w^i_n), \quad \Delta_i = \exp\left(\frac{a_{ii}}{2}\left(u^i_{n+1} - u^i_n\right) + a_{i,i+1}\left(u^{i+1}_{n+1} - u^{i+1}_n\right) + \dots + a_{ir}\left(u^r_{n+1} - u^r_n\right)\right),$$

$t_i = E_i\Delta_i$, where $i = 1, \dots, r$ (here we omit the dependence on n for simplicity in notation). Hence, the i-th equation of (4) can be rewritten as $u^i_{n+1,x} - u^i_{n,x} = t_i$ and therefore we need to simplify the terms of the form $D^{k-1}_x(t_i)$ in (10) using equations (4).

The remaining part of the proof of this Theorem is divided into a series of propositions on some specific properties of the the functions t_i and the operators

$$X_i = e^{-w^i}\tilde{X}_i = \frac{\partial}{\partial u^i_1} + b^i_1\frac{\partial}{\partial u^i_2} + b^i_2\frac{\partial}{\partial u^i_3} + \dots$$

PROPOSITION 2. *For all $i = 1, \dots, r$ functions t_i satisfy the relation*

$$D_x(t_i) = t_i\left(b^i_1 + \frac{a_{ii}}{2}t_i + a_{i,i+1}t_{i+1} + \dots + a_{ir}t_r\right). \tag{11}$$

Proof. Differentiate t_i and use relations

$$D_x(E_i) = b^i_1 E_i, \quad D_x(\Delta_i) = \frac{a_{ii}}{2}t_i + a_{i,i+1}t_{i+1} + \dots + a_{ir}t_r. \quad \square$$

PROPOSITION 3. *Operators X_j satisfy the relation*

$$X_j(D_x + b^i_1) = (D_x + b^i_1 + b^j_1)X_j + a_{ij}$$

for all $i, j = 1, \dots, r$.

Proof. Complete Bell polynomials $B_k = B_k(v_1, \dots, v_k)$ are known to satisfy the relation

$$D_x(B_k) = B_{k+1} - B_1 B_k, \quad k = 1, 2, \dots$$

Apply this relation to calculate the commutator:

$$\begin{aligned}
\left[X_j, D_x + b_1^i\right] &= \left(\frac{\partial}{\partial u_1^j} + b_1^j \frac{\partial}{\partial u_2^j} + b_2^j \frac{\partial}{\partial u_3^j} + \dots\right)\left(b_1^i + \sum_{l=1}^{r}\left(u_2^l \frac{\partial}{\partial u_1^l} + u_3^l \frac{\partial}{\partial u_2^l} + \dots\right)\right) - \\
&- \left(b_1^i + D_x\right)\left(\frac{\partial}{\partial u_1^j} + b_1^j \frac{\partial}{\partial u_2^j} + b_2^j \frac{\partial}{\partial u_3^j} + \dots\right) = \\
&= X_j(b_1^i) + X_j(u_2^j)\frac{\partial}{\partial u_1^j} + X_j(u_3^j)\frac{\partial}{\partial u_2^j} + \dots - D_x(b_1^j)\frac{\partial}{\partial u_2^j} - D_x(b_2^j)\frac{\partial}{\partial u_3^j} - \dots = \\
&= a_{ij} + b_1^j \frac{\partial}{\partial u_1^j} + b_2^j \frac{\partial}{\partial u_2^j} + \dots + (b_1^j b_1^j - b_2^j)\frac{\partial}{\partial u_2^j} + (b_1^j b_2^j - b_3^j)\frac{\partial}{\partial u_3^j} + \dots = \\
&= a_{ij} + b_1^j\left(\frac{\partial}{\partial u_1^j} + b_1^j \frac{\partial}{\partial u_2^j} + b_2^j \frac{\partial}{\partial u_3^j} + \dots\right) = a_{ij} + b_1^j X_j. \quad \square
\end{aligned}$$

PROPOSITION 4. *Operators X_j satisfy the relation*

$$X_j^k(D_x + b_1^i) = (D_x + b_1^i + k b_1^j)X_j + \left(k a_{ij} + \frac{k(k-1)}{2} a_{jj}\right) X_j^{k-1}$$

for all $i, j = 1, \dots, r$ and for all natural k.

Proof. Use induction in k. The base of induction follows from Proposition 3. $\square$

PROPOSITION 5. *Let $k_1, \dots, k_s$ be arbitrary natural numbers and assume $1 \leqslant j_1 < j_2 < \dots < j_s \leqslant r$. Then*

$$\begin{aligned}
X_{j_s}^{k_s} X_{j_{s-1}}^{k_{s-1}} \dots X_{j_1}^{k_1}\left(D_x + b_1^i\right) &= \left(D_x + k_s b_1^{j_s} + \dots + k_1 b_1^{j_1} + b_1^i\right) X_{j_s}^{k_s} X_{j_{s-1}}^{k_{s-1}} \dots X_{j_1}^{k_1} + \\
&+ k_s\left(a_{ij_s} + \frac{k_s - 1}{2} a_{j_s j_s} + k_{s-1} a_{j_{s-1} j_s} + k_{s-2} a_{j_{s-2} j_s} + \dots + k_1 a_{j_1 j_s}\right) X_{j_s}^{k_s - 1} X_{j_{s-1}}^{k_{s-1}} \dots X_{j_1}^{k_1} + \\
&+ k_{s-1}\left(a_{ij_{s-1}} + \frac{k_{s-1} - 1}{2} a_{j_{s-1} j_{s-1}} + k_{s-2} a_{j_{s-2} j_{s-1}} + k_{s-3} a_{j_{s-3} j_{s-1}} + \dots + k_1 a_{j_1 j_{s-1}}\right) X_{j_s}^{k_s} X_{j_{s-1}}^{k_{s-1}-1} \dots X_{j_1}^{k_1} + \dots \\
&+ k_3\left(a_{ij_3} + \frac{k_3 - 1}{2} a_{j_3 j_3} + k_2 a_{j_2 j_3} + k_1 a_{j_1 j_3}\right) X_{j_s}^{k_s} \dots X_{j_3}^{k_3 - 1} X_{j_2}^{k_2} X_{j_1}^{k_1} + \\
&+ k_2\left(a_{ij_2} + \frac{k_2 - 1}{2} a_{j_2 j_2} + k_1 a_{j_1 j_2}\right) X_{j_s}^{k_s} \dots X_{j_2}^{k_2 - 1} X_{j_1}^{k_1} + k_1\left(a_{ij_1} + \frac{k_1 - 1}{2} a_{j_1 j_1}\right) X_{j_s}^{k_s} \dots X_{j_2}^{k_2} X_{j_1}^{k_1 - 1}
\end{aligned} \tag{12}$$

for any $i = 1, \dots, r$.

Proof. Use induction in s. The base of induction follows from Proposition 4. $\square$

PROPOSITION 6. *Let $k \in \mathbb{N}$. Then $D_x^k(t_i)$ is a polynomial in $t_i, \dots, t_r$ of the degree $k+1$,*

$$D_x^k(t_i) = \sum_{\varkappa=1}^{k+1} C_{k_i \dots k_r}^k t_i^{k_i} t_{i+1}^{k_{i+1}} \dots t_r^{k_r}, \tag{13}$$

where the summation is taken over all partitions $\varkappa = k_i + \dots + k_r$ into a sum of non-negative whole numbers such that $k_i > 0$. Coefficients are given by

$$C_{k_i \dots k_r}^k = \frac{1}{k_i! \dots k_r!} X_r^{k_r} X_{r-1}^{k_{r-1}} \dots X_{i+1}^{k_{i+1}} X_i^{k_i - 1}(b_i^k). \tag{14}$$

Proof. Use induction in k. The base of the induction follows from (11) since $X_j(b_1^i) = a_{ij}$. Assume the proposition holds for

$$D_x(t_i), \quad D_x^2(t_i), \dots, D_x^k(t_i)$$

and differentiate (13) with respect to x. The total derivative of the coefficient $C^k_{k_i \dots k_s}$ contributes only to the coefficient of $t_i^{k_i} \dots t_r^{k_r}$ in the expansion of D_x^{k+1}. Due to (11), the derivative of $t_i^{k_i} \dots t_r^{k_r}$ contributes to the coefficient of $t_i^{k_i} \dots t_r^{k_r}$ and to all coefficients of the terms of degree $k_i + \dots + k_r + 1$ such that only one power differs from $k_i, \dots, k_r$ by one. First consider one of the leading coefficients $C^{k+1}_{k_i \dots k_r}$ in $D_x^{k+1}(t_i)$. It appears as the result of differentiation of the terms

$$t_i^{k_i-1} t_{i+1}^{k_{i+1}} \dots t_r^{k_r}, \quad t_i^{k_i} t_{i+1}^{k_{i+1}-1} \dots t_r^{k_r}, \dots, t_i^{k_i} t_{i+1}^{k_{i+1}} \dots t_r^{k_r-1},$$

where the corresponding power $k_s - 1$ is non-negative. Hence,

$$\begin{aligned}
C^{k+1}_{k_i k_{i+1} \dots k_r} &= C^k_{k_i-1, k_{i+1} \dots k_r} (k_i - 1) \frac{a_{ii}}{2} + C^k_{k_i k_{i+1}-1 \dots k_r} \left(k_i a_{i,i+1} + (k_{i+1} - 1) \frac{a_{i+1,i+1}}{2} \right) + \\
&+ C^k_{k_i k_{i+1} k_{i+2}-1 \dots k_r} \left(k_i a_{i,i+2} + k_{i+1} a_{i+1,i+2} + (k_{i+2} - 1) \frac{a_{i+2,i+2}}{2} \right) + \dots + \\
&+ C^k_{k_i k_{i+1}-1 \dots k_r-1} \left(k_i a_{i,r} + k_{i+1} a_{i+1,r} + k_{i+2} a_{i+2,r} + \dots + k_{r-1} a_{r-1,r} + (k_r - 1) \frac{a_{rr}}{2} \right) = \\
&= \frac{1}{k_i! \dots k_r!} \left(k_i \frac{k_i - 1}{2} a_{ii} X_r^{k_r} \dots X_{i+1}^{k_{i+1}} X_i^{k_i-2}(b_i^k) + \right. \\
&+ k_{i+1} \left(k_i a_{i,i+1} + \frac{k_{i+1} - 1}{2} a_{i+1,i+1} \right) X_r^{k_r} \dots X_{i+1}^{k_{i+1}-1} X_i^{k_i-1}(b_i^k) + \dots + \\
&\left. + k_r \left(k_i a_{i,r} + k_{i+1} a_{i+1,r} + k_{i+2} a_{i+2,r} + \dots + k_{r-1} a_{r-1,r} + \frac{k_r - 1}{2} a_{rr} \right) X_r^{k_r-1} \dots X_{i+1}^{k_{i+1}} X_i^{k_i-1}(b_i^k) \right).
\end{aligned} \tag{15}$$

Apply now formula (12):

$$\begin{aligned}
X_r^{k_r} X_{r-1}^{k_{r-1}} \dots X_{i+1}^{k_{i+1}} X_i^{k_i-1}(b_i^{k+1}) &= X_r^{k_r} X_{r-1}^{k_{r-1}} \dots X_{i+1}^{k_{i+1}} X_i^{k_i-1}(D_x + b_1^i)(b_i^k) = \\
= k_r \left(a_{ir} + \frac{k_r - 1}{2} a_{rr} + k_{r-1} a_{r-1,r} + \dots + (k_i - 1) a_{ir} \right) & X_r^{k_r-1} X_{r-1}^{k_{r-1}} \dots X_i^{k_i-1}(b_i^k) + \dots + \\
+ k_{i+1} \left(a_{i,i+1} + \frac{k_{i+1} - 1}{2} a_{i+1,i+1} + (k_i - 1) a_{i,i+1} \right) & X_r^{k_r} \dots X_{i+1}^{k_{i+1}-1} X_i^{k_i-1}(b_i^k) + \\
+ k_i \left(a_{ii} + \frac{k_i - 2}{2} a_{ii} \right) & X_r^{k_r} \dots X_{i+1}^{k_{i+1}} X_i^{k_i-2}(b_i^k).
\end{aligned} \tag{16}$$

Here we used relation

$$X_r^{k_r} X_{r-1}^{k_{r-1}} \dots X_{i+1}^{k_{i+1}} X_i^{k_i-1}(b_i^k) = 0,$$

which holds since $k_i + \dots + k_r = k + 2$. Comparing formulas (15) and (16) proves the claim for leading coefficients in the expansion for $D_x^{k+1}(t_i)$.

The formula for non-leading coefficients is proved similarly, although the calculation is more nasty in this case since one has to take into account the terms $b_1^s t_i^{k_i} t_{i+1}^{k_{i+1}} \dots t_r^{k_r}$, where $s = i, \dots r$, that come from the differentiation of $t_i^{k_i} t_{i+1}^{k_{i+1}} \dots t_r^{k_r}$, and the terms that appear from the differentiation of the coefficient $C^k_{k_i \dots k_s}$ which are non-zero if $\varkappa < k + 1$. Note that formula (12) has to be used also to simplify the coefficient that comes from differentiation of $C^k_{k_i \dots k_s}$. $\square$

PROPOSITION 7. *Polynomial $D_x^k(t_i)$ is divisible by t_i for all $i = 1, \dots, r$ and $k \in \mathbb{N}$. The coefficient of t_i in $D_x^k(t_i)$ equals b_i^k:*

$$D_x^k(t_i) = t_i(b_i^k + \dots).$$

Proof. The first claim follows from (11). The second claim immediately follows from (14). □

PROPOSITION 8. *Let I be an analytic y-integral of exponential system (3). Then*

$$I_{n+1} - I_n = \sum_{m=1}^{\infty} \left(\sum_{k_1+\dots+k_r=m} \frac{1}{k_1!\dots k_r!} X_r^{k_r} X_{r-1}^{k_{r-1}} \dots X_1^{k_1}(I) t_1^{k_1} \dots t_r^{k_r} \right), \tag{17}$$

where the sum is taken over all partitions $m = k_1 + \dots + k_r$ such that $k_1, \dots, k_r \geqslant 0$.

Proof. Use induction in m. If $m = 1$, then we need to prove that the coefficient of t_i in the sum (10) equals $X_i(I)$ for all $i = 1, \dots, r$. Since the monomial t_i is contained only in the expansions of form (13) for $t_i, t_{i,x}, t_{i,xx}, \dots$, the coefficient of t_i in (10) equals

$$1 \cdot \frac{\partial I}{\partial u_1^i} + C^1_{1,0,\dots,0} \frac{\partial I}{\partial u_2^i} + C^2_{1,0,\dots,0} \frac{\partial I}{\partial u_3^i} + \dots = \frac{\partial I}{\partial u_1^i} + b_1^i \frac{\partial I}{\partial u_2^i} + b_2^i \frac{\partial I}{\partial u_3^i} + \dots = X_i(I)$$

due to Proposition 7.

Assume now the coefficient B of $t_l^{k_l} \dots t_j^{k_j}$ in (10) has the form

$$\frac{1}{k_l!\dots k_j!} X_j^{k_j} X_{j-1}^{k_{j-1}} \dots X_l^{k_l}(I)$$

for all $l < j$ and $k_l + \dots + k_j \leqslant m$. According to (10), the coefficient B is obtained from the product

$$\prod_{i=1}^{r} \prod_{k=1}^{d} \left(D_x^{k-1}(t_i)\right)^{\lambda_{ik}}$$

by extracting the coefficients of $t_l^{k_l} \dots t_j^{k_j}$ and multiplying them by appropriate multiple derivatives of I. Hence,

$$B = \sum_p \left(\left(\mu_p \prod_{i,k} C^k_{i,q_l\dots q_j} \right) \frac{\partial^s I}{(\partial u_1^1)^{\lambda_{11}} \dots (\partial u_d^r)^{\lambda_{rd}}} \right),$$

where $C^k_{i,q_l\dots q_j}$ are the coefficients of the form (14) in one of the expansions of $D_x^k(t_i)$ in (10), $0 \leqslant q_i \leqslant k_i$ for all $i = l, \dots, j$ (here $q_l \dots q_j$ depend on i and k) and $\mu_p \in \mathbb{R}$ are products of binomial coefficients which appear when expansions of $D_x^k(t_i)$ are raised to powers $\lambda_{i,k}$ in (10). We do not need to specify the set of indices over which the sum and the products are taken — we will only need the general form of B.

Consider the coefficient $\tilde{B}$ of $t_l^{k_l} \dots t_j^{k_j} t_{j+1}$ in (10). It is the sum of the terms of two types: the first one is the product of the coefficient of $t_l^{k_l} \dots t_j^{k_j}$ in (10) by the coefficient of t_{j+1} in expansions of $D_x^k(t_{j+1})$ for all $k = 0, 1, \dots$, and the second one appears when t_{j+1} is additionally taken from one of the expansions of $D^k(t_i)$, where $i < j + 1$. For the terms of the first type, the derivative of the function I in (10) is differentiated by u_{k+1}^{j+1} since the whole expression is multiplied by $D_x^k(t_{j+1})$. According to Proposition 7 the coefficient of t_{j+1} in $D_x^k(t_{j+1})$ equals b_k^{j+1}. Therefore, it follows from the inductive assumption that the contribution of the terms of the first type to $\tilde{B}$ has the form

$$\frac{1}{k_l!\dots k_j!} \left(X_j^{k_j} X_{j-1}^{k_{j-1}} \dots X_l^{k_l} \left(\frac{\partial I}{\partial u_1^{j+1}} \right) + b_1^{j+1} X_j^{k_j} X_{j-1}^{k_{j-1}} \dots X_l^{k_l} \left(\frac{\partial I}{\partial u_2^{j+1}} \right) + \right.$$
$$\left. + b_2^{j+1} X_j^{k_j} X_{j-1}^{k_{j-1}} \dots X_l^{k_l} \left(\frac{\partial I}{\partial u_3^{j+1}} \right) + \dots \right). \tag{18}$$

Examine now the contribution of the terms of the second type to $\tilde{B}$. Since t_{j+1} is additionally taken from one of the expansions of $D^k(t_i)$ in this case, the terms contributing to $\tilde{B}$ have the form

$$\left(\mu_p \prod_{i,k} C^k_{i,q_l\ldots q_j q_{j+1}}\right) \frac{\partial^s I}{(\partial u^1_1)^{\lambda_{11}} \ldots (\partial u^r_d)^{\lambda_{rd}}},$$

where $q_{j+1} = 1$ for only one pair (i,k) and is zero for all other pairs (i,k). Note that due to (14)

$$C^k_{i,q_l\ldots q_j q_{j+1}} = X_{j+1}\left(C^k_{i,q_l\ldots q_j}\right) \tag{19}$$

for all $i = 1,\ldots,r$ and for all $k \in \mathbb{N}$.

Since $X_j^{k_j} X_{j-1}^{k_{j-1}} \ldots X_l^{k_l}(I)$ is a linear combination of various multiple derivatives of the function I with certain coefficients and X_{j+1} is a linear first order operator, all the terms in the expression for

$$X_{j+1}\left(\frac{1}{k_l!\ldots k_j!} X_j^{k_j} X_{j-1}^{k_{j-1}} \ldots X_l^{k_l}(I)\right)$$

can be grouped into two families: the operator X_{j+1} is applied to the derivatives of I in the first family and it is applied to the coefficients in the second family. Clearly, it follows from (18,19) and the Leibniz rule that the first family corresponds to the terms of the first type in $\tilde{B}$ and the second family corresponds to the terms of the second type. Therefore, $\tilde{B} = X_{j+1}(B)$.

The proof that the coefficient of $t_l^{k_l} \ldots t_j^{k_j+1}$ in (10) has the form

$$\frac{1}{k_l!\ldots k_{j-1}!(k_j+1)!} X_j^{k_j+1} X_{j-1}^{k_{j-1}} \ldots X_l^{k_l}(I)$$

is conducted in the same way, but one has to take into account additional coefficient $k_j + 1$ that comes from multiplicity in this case. $\square$

Proof of Theorem 2 immediately follows from (17) since every y-integral annihilates the whole characteristic algebra and, in particular, it annihilates the generators $\tilde{X}_1,\ldots \tilde{X}_r$. Since $X_i = e^{-w^i}\tilde{X}_i$, it also annihilates operators $X_1,\ldots X_r$. $\square$

4 Characteristic algebra of exponential system

In this Section we review a number of basic properties of characteristic algebras for exponential systems corresponding to the Cartan matrices of simple Lie algebras in the continuous case.

Lemma 1. *Let*

$$X = \sum_{i=1}^r \sum_{k=1}^\infty P^i_k \frac{\partial}{\partial u^i_k}, \quad D = \sum_{i=1}^r \sum_{k=0}^\infty u^i_{k+1} \frac{\partial}{\partial u^i_k},$$

where $P^i_k = P^i_k(u^1,\ldots,u^r,u^1_1,\ldots,u^r_1,u^1_2,\ldots,u^r_2,\ldots)$ *and* $u^i_0 = u^i$. *If* $[D,X] = 0$, *then* $X = 0$.

This lemma and its various analogs are widely used for explicit description of characteristic algebras. The proof is straightforward and trivial.

Proposition 9. [1] *For any* $1 \leqslant i_1,\ldots,i_k \leqslant r$ *operators* X_i *satisfy following commutation relations:*

$$\begin{aligned}
[D,[X_{i_1},[X_{i_2},\ldots,[X_{i_{k-1}},X_{i_k}]\ldots]]] = -(b^{i_1}_1 + \cdots + b^{i_k}_1)[X_{i_1},[X_{i_2},\ldots,[X_{i_{k-1}},X_{i_k}]\ldots]]-\\
-(a_{i_2 i_1} + a_{i_3 i_1} + \cdots + a_{i_k i_1})[X_{i_2},[X_{i_3},\ldots,[X_{i_{k-1}},X_{i_k}]\ldots]]-\\
-(a_{i_3 i_2} + a_{i_4 i_2} + \cdots + a_{i_k i_2})[X_{i_1},[X_{i_3},\ldots,[X_{i_{k-1}},X_{i_k}]\ldots]] - \ldots\\
-(a_{i_{k-1} i_{k-2}} + a_{i_k i_{k-2}})[X_{i_1},\ldots,[X_{i_{k-3}},[X_{i_{k-1}},X_{i_k}]]\ldots]-\\
-a_{i_k i_{k-1}}[X_{i_1},\ldots,[X_{i_{k-3}},[X_{i_{k-2}},X_{i_k}]]\ldots] + a_{i_{k-1} i_k}[X_{i_1},\ldots,[X_{i_{k-3}},[X_{i_{k-2}},X_{i_{k-1}}]]\ldots].
\end{aligned} \tag{20}$$

Technical Propositions 10–14 can be proved straightforwardly using Lemma 1.

Proposition 10. *Reduced characteristic algebra of exponential system (3) corresponding to the A-series Cartan matrix of the rank r is a linear span of the following vector fields:*

$$
\begin{aligned}
&X_k, \quad k = 1, 2, \dots, r, \\
&[X_k, [X_{k+1}, \dots, [X_{l-1}, X_l] \dots]], \quad 1 \leqslant k < l \leqslant r.
\end{aligned}
$$

Proposition 11. *Reduced characteristic algebra of exponential system (3) corresponding to the B-series Cartan matrix of the rank r is a linear span of the following linearly independent vector fields:*

$$
\begin{aligned}
&X_k, \quad k = 1, 2, \dots, r, \\
&[X_k, [X_{k+1}, \dots, [X_{l-1}, X_l] \dots]], \quad 1 \leqslant k < l \leqslant r, \\
&[X_k, [X_{k-1}, [X_{k-2}, \dots, [X_2, [X_1, [X_1, [X_2, \dots, [X_{l-2}, [X_{l-1}, X_l]] \dots]]]] \dots]]], \quad 1 \leqslant k < l \leqslant r.
\end{aligned}
$$

Proposition 12. *Reduced characteristic algebra of exponential system (3) corresponding to the C-series Cartan matrix of the rank r is a linear span of the following linearly independent vector fields:*

$$
\begin{aligned}
&X_k, \quad k = 1, 2, \dots, r, \\
&[X_k, [X_{k+1}, \dots, [X_{l-1}, X_l] \dots]], \quad 1 \leqslant k < l \leqslant r, \\
&[X_k, [X_{k-1}, [X_{k-2}, \dots, [X_2, [X_1, [X_2, \dots, [X_{l-2}, [X_{l-1}, X_l]] \dots]]] \dots]]], \quad 2 \leqslant k \leqslant l \leqslant r.
\end{aligned}
$$

Proposition 13. *Reduced characteristic algebra of exponential system (3) corresponding to the D-series Cartan matrix of the rank r is a linear span of the following linearly independent vector fields:*

$$
\begin{aligned}
&X_k, \quad k = 1, 2, \dots, r, \\
&[X_k, [X_{k+1}, \dots, [X_{l-1}, X_l] \dots]], \quad 2 \leqslant k < l \leqslant r, \\
&[X_1, [X_2, \dots, [X_{l-1}, X_l] \dots]], \quad 3 \leqslant l \leqslant r, \\
&[X_1, [X_3, \dots, [X_{l-1}, X_l] \dots]], \quad 3 \leqslant l \leqslant r, \\
&[X_k, [X_{k-1}, [X_{k-2}, \dots, [X_3, [X_1, [X_2, [X_3, \dots, [X_{l-2}, [X_{l-1}, X_l]] \dots]]]] \dots]]], \quad 3 \leqslant k < l \leqslant r.
\end{aligned}
$$

Remark 3. The above bases for the series A and D can be found in [1] but with no proofs. Detailed proofs for all these four cases can be found in [35].

Remark 4. One can verify that in characteristic algebras of exponential systems corresponding to the Cartan matrices of the series A–D there are no non-trivial relations between multiple commutators of the form $[X_{i_1}, [X_{i_2}, \dots, X_{i_k}]]$. More precisely, any such non-zero commutator is proportional to some element from the corresponding basis (see Propositions 10–13).

Proposition 14. *Reduced characteristic algebra of exponential systems (3) corresponding to the Cartan matrices of the E-series, of G_2 and F_4 root systems are finite-dimensional, and in each of these cases there exist a basis such that any non-zero multiple commutator $[X_{i_1}, [X_{i_2}, \dots, X_{i_k}]]$ is proportional to some element from this basis*[1].

[1]We do not provide here suitable bases because they cannot be represented in a compact form like in Propositions 10–13.

5 Characteristic x-integrals of semi-discrete exponential systems

Integral preserving discretization (4) of a Darboux integrable exponential system (3) admits a complete family of independent n-integrals. Darboux integrability of a semi-discrete system also requires the existence of a complete family of essentially independent x-integrals, and since the variables x and n do not enter the equations symmetrically (one of them is continuous and the other one is discrete), such family of integrals cannot be obtained just by renaming the variables (as in the continuous case). Complete family of characteristic integrals was obtained explicitly in [19] for semi-discrete A- and C-series lattices, but this approach is not applicable for the lattices associated with the Cartan matrices of other simple Lie algebras since it is not known whether they are reductions of an A-series lattice or not. In this section we are going to prove the existence of complete families of characteristic x-integrals for semi-discrete lattices corresponding to the Cartan matrices of all simple Lie algebras using the concept of characteristic algebra.

Characteristic algebras for (semi)-discrete hyperbolic system were defined in [27]–[29]; these Lie algebras for (semi)-discrete exponential systems (4),(5) associated with the Cartan matrices of rank 2 where described explicitly in terms of generators and relations in [13, 14]. Characteristic algebras are an effective tool in the study of Darboux integrability of hyperbolic equations both in the continuous and in the (semi)-discrete cases (see papers [30]–[34] by Ufa mathematical school).

In this section we define a special Lie algebra of differential operators that controls the existence of complete family of x-integrals for semi-discrete exponential systems (4). Although this Lie algebra is very similar by its properties to characteristic algebra defined in [28] and its construction is based on the same ideas, these Lie algebras are not isomorphic.

We will not describe the construction of characteristic algebra in the semi-discrete case from [28] here. Instead, we introduce a Lie algebra with similar properties that allows to prove Darboux integrability of all exponential systems associated with the Cartan matrices of simple Lie algebras.

Consider semi-discrete exponential system (4) and introduce the following notation:

$$v^i_n = u^i_{n+1} - u^i_n, \quad w^i_n = a_{i1}u^1_n + \dots + a_{ir}u^r_n, \quad z^i_n = w^i_{n+1} - w^i_n, \quad \Delta^i_n = \exp\left(\frac{a_{ii}}{2}v^i_n + a_{i,i+1}v^{i+1}_n + \dots + a_{ir}v^r_n\right),$$

where $i = 1, \dots, r$. Hence, equations (4) can be rewritten as

$$v^i_{n,x} = e^{w^i_n}\Delta^i_n, \quad i = 1, \dots, r. \tag{21}$$

Consider differential operators

$$Y_i = c^i_0 \frac{\partial}{\partial v^i_n} + c^i_1 \frac{\partial}{\partial v^i_{n+1}} + c^i_2 \frac{\partial}{\partial v^i_{n+2}} + \dots, \quad i = 1, \dots, r,$$

where

$$c^i_0 = \Delta^i_n, \quad c^i_1 = \exp\left(z^i_n\right)\Delta^i_{n+1}, \dots, c^i_k = \exp\left(z^i_n + z^i_{n+1} + \dots + z^i_{n+k-1}\right)\Delta^i_{n+k}, \quad i = 1, \dots, r. \tag{22}$$

Proposition 15. *Let the matrix $M = (a_{ij})$ of exponential system (4) be non-degenerate. Then function*

$$J = J(v^1_n, \dots, v^r_n, v^1_{n+1}, \dots, v^r_{n+1}, v^1_{n+2}, \dots, v^r_{n+2}, \dots)$$

is an x-integral of (4) if and only if it annihilates operators $Y_1, \dots, Y_r$.

Proof. Since

$$v^i_{n+1,x} = e^{w^i_{n+1}}\Delta^i_{n+1} = e^{w^i_n}\exp(z^i_n)\Delta^i_{n+1}$$

due to (21), one can easily verify by induction that

$$v^i_{n+k,x} = e^{w^i_n} \exp(z^i_n + z^i_{n+1} + \dots + z^i_{n+k-1})\Delta^i_{n+k}.$$

Hence, apply the total derivative with respect to x:

$$D_x(J) = \sum_{i=1}^{r}\sum_{k=0}^{\infty} v^i_{n+k,x}\frac{\partial J}{\partial v^i_{n+k}} = \sum_{i=1}^{r}\sum_{k=0}^{\infty} e^{w^i_n}\exp(z^i_n + z^i_{n+1} + \dots + z^i_{n+k-1})\Delta^i_{n+k}\frac{\partial J}{\partial v^i_{n+k}} =$$
$$= \sum_{i=1}^{r}\sum_{k=0}^{\infty} e^{w^i_n} c^i_k \frac{\partial J}{\partial v^i_{n+k}} = \sum_{i=1}^{r} e^{w^i_n} Y_i(J).$$

Therefore, if M is non-degenerate, then exponents $e^{w^i_n}$ are linearly independent and J is an x-integral if and only if it annihilates operators $Y_1, \dots, Y_r$. □

Variables

$$u^1_n, \dots, u^r_n, v^1_n, \dots, v^r_n, v^1_{n+1}, \dots, v^r_{n+1}, v^1_{n+2}, \dots, v^r_{n+2}, \dots$$

are independent. Lie algebra $\tilde{\mathcal{L}}$ generated by vector fields

$$\frac{\partial}{\partial u^1_n}, \dots, \frac{\partial}{\partial u^r_n},\ \tilde{Y}_1, \dots, \tilde{Y}_r,$$

where $\tilde{Y}_i = e^{w^i_n} Y_i$ for all $i = 1, \dots, r$, is a discrete analog of the characteristic algebra for exponential system (3). Similarly, Lie algebra $\mathcal{L}$ generated by vector fields $Y_1, \dots, Y_r$ is a discrete analog of the reduced characteristic algebra for (3). Nevertheless, in order to avoid ambiguity in the use of terminology, we will call this Lie algebra *the defining algebra* for semi-discrete system (4). The following theorem is proved exactly as Theorem 1 in the continuous case.

Theorem 3. *Semi-discrete exponential system (4) admits a complete family of essentially independent x-integrals if and only if its defining algebra is finite-dimensional.*

We are going to prove the existence of a complete family of independent x-integrals for semi-discrete exponential systems (4) corresponding to the Cartan matrices of all simple Lie algebras by applying Theorem 3 and by describing the defining algebras $\mathcal{L}$ explicitly.

Proposition 16. *Let T be the shift operator, $T(v_n) = v_{n+1}$. Then*

$$T[Y_{i_1}, [Y_{i_2}, \dots, [Y_{i_{k-1}}, Y_{i_k}]\dots]]T^{-1} = \exp(-(z^{i_1}_n + \dots + z^{i_k}_n))\big([Y_{i_1}, [Y_{i_2}, \dots, [Y_{i_{k-1}}, Y_{i_k}]\dots]] -$$
$$- c^{i_1}_n(a_{i_2 i_1} + a_{i_3 i_1} + \dots + a_{i_k i_1})[Y_{i_2}, [Y_{i_3}, \dots, [Y_{i_{k-1}}, Y_{i_k}]\dots]] -$$
$$- c^{i_2}_n(a_{i_3 i_2} + a_{i_4 i_2} + \dots + a_{i_k i_2})[Y_{i_1}, [Y_{i_3}, \dots, [Y_{i_{k-1}}, Y_{i_k}]\dots]] - \dots$$
$$- c^{i_{k-2}}_n(a_{i_{k-1} i_{k-2}} + a_{i_k i_{k-2}})[Y_{i_1}, \dots, [Y_{i_{k-3}}, [Y_{i_{k-1}}, Y_{i_k}]]\dots] -$$
$$- c^{i_{k-1}}_n a_{i_k i_{k-1}}[Y_{i_1}, \dots, [Y_{i_{k-3}}, [Y_{i_{k-2}}, Y_{i_k}]]\dots] + c^{i_k}_n a_{i_{k-1} i_k}[Y_{i_1}, \dots, [Y_{i_{k-3}}, [Y_{i_{k-2}}, Y_{i_{k-1}}]]\dots]\big) + \dots, \quad (23)$$

where the dots in the end stand for multiple commutators of the degrees $1, \dots, k-2$.

Proof

It follows from relations (22) that

$$TY_iT^{-1} = e^{-z^i_n} Y_i$$

for all $i = 1, \dots, r$. Formula (23) is proved by induction using relation

$$Y_{i_0}\big(\exp(-(z^{i_1}_n + \dots + z^{i_k}_n))\big) = -c^{i_0}_n(a_{i_1 i_0} + a_{i_2 i_0} + \dots + a_{i_k i_0})$$

and representation

$$T[Y_{i_0},[Y_{i_1},[Y_{i_2},\ldots,[Y_{i_{k-1}},Y_{i_k}]\ldots]]]T^{-1} =$$
$$= \left(TY_{i_0}T^{-1}\right)\left(T[Y_{i_1},[Y_{i_2},\ldots,[Y_{i_{k-1}},Y_{i_k}]\ldots]]T^{-1}\right) - \left(T[Y_{i_1},[Y_{i_2},\ldots,[Y_{i_{k-1}},Y_{i_k}]\ldots]]T^{-1}\right)\left(TY_{i_0}T^{-1}\right). \quad \square$$

THEOREM 4. *The defining algebra of semi-discrete exponential system (4) corresponding to the Cartan matrix of any simple Lie algebra is finite-dimensional.*

Proof. We will prove that the defining algebra of semi-discrete exponential system (4) corresponding to the Cartan matrix of any simple Lie algebra is isomorphic to the reduced characteristic algebra of its continuous analog. Since in characteristic algebras corresponding to the Cartan matrices of all simple Lie algebras there are no relations between non-trivial multiple commutators except for the skew-symmetry, the Jacobi identity and their implications (see Propositions 10–14), it follows from Lemma 1 and from formula (20) that commutator

$$[X_{i_1},[X_{i_2},\ldots,[X_{i_{k-1}},X_{i_k}]\ldots]] \tag{24}$$

is trivial if and only if all terms in (20) vanish. Therefore, either commutator

$$[X_{i_1},[X_{i_2},\ldots,[X_{l-1},[X_{l+1},\ldots,[X_{i_{k-1}},X_{i_k}]\ldots]]\ldots]]$$

is trivial, or its coefficient

$$a_{i_{l+1}i_l} + a_{i_{l+2}i_l} + \cdots + a_{i_k i_l}$$

is zero for all $l = 1,\ldots,k$.

For each of the Cartan matrices the isomorphism is established by induction in the degree of multiple commutators. Note that the coefficients of multiple commutators of degree $k-1$ in formula (23) are the same as the coefficients in (20). If commutator (24) is trivial, then all the terms in (20) vanish. Hence, relation (23) for the corresponding multiple commutator of $Y_{i_1},\ldots,Y_{i_k}$ does not contain terms of degree $k-1$. Careful analysis of the relations between coefficients in formula (23) for multiple commutators of degrees l and $l-1$ for each Cartan matrix shows that the absence of the terms of degree $k-1$ yields the absence of all terms of degrees $1,\ldots,k-2$. Hence, commutator (24) satisfies relation

$$T[Y_{i_1},[Y_{i_2},\ldots,[Y_{i_{k-1}},Y_{i_k}]\ldots]]T^{-1} = \exp(-(z_n^{i_1}+\cdots+z_n^{i_k}))[Y_{i_1},[Y_{i_2},\ldots,[Y_{i_{k-1}},Y_{i_k}]\ldots]]$$

which holds only if $k=1$. Therefore, if commutator (24) is trivial in the reduced characteristic algebra of exponential system in the continuous case, then the corresponding multiple commutator of $Y_{i_1},\ldots,Y_{i_k}$ also vanishes. This proves the isomorphism. $\square$

REMARK 5. We used the concept of characteristic algebra in order to prove the existence of a complete family of x-integrals. Although our attempts to find explicit formulas for x-integrals did not give any result in general, some integrals can be found explicitly. For example, semi-discrete Toda lattice (4) corresponding to the B-series Cartan matrix of the rank $r \geqslant 3$ admits x-integral

$$J = \sum_{j=2}^{r-1}\left(\exp(v_{n+j+1}^{j+1} - v_{n+j+1}^{j}) + \exp(v_n^i - v_{n+1}^{j+1})\right) + \exp(v_{n+1}^2 - 2v_{n+1}^1) +$$
$$+ \exp(2v_n^1 - v_{n+1}^2) + 2\exp(v_n^1 - v_{n+1}^1) + \exp(v_n^r) + \exp(-v_{n+r}^r).$$

REMARK 6. It would have been interesting to prove that Habibullin's integral preserving discretization leads to Darboux integrable systems in the purely discrete case as well. Entirely discrete exponential systems were introduced in [14] and it was proved there that for all Cartan matrices of the rank 2 characteristic x-integrals of corresponding semi-discrete lattices appear to be m-integrals of their purely discrete analogs. It was conjectured in [14] that the same property holds for purely discrete exponential systems associated with the Cartan matrices of all simple Lie algebras, but unfortunately our approach appears to be not applicable for proving the integral preserving property in the entirely discrete case due to the form of equations (5).

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