Exactly solvable periodic Darboux $q$-chains *

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Let $L_1, L_2, \ldots$ be selfadjoint differential operators acting on $\mathbb{R}$. They form a Darboux chain if they satisfy the relation

$$L_j = A_j A_{j-1}^+ - \alpha_j = A_{j-1}^+ A_j - \alpha_j,$$

where $A_j = -d/dx + f_j(x)$ are first order differential operators. A Darboux chain is called periodic if $L_{j+r} = L_j$ for some $r$ and for all $j = 1, 2, \ldots$. Number $r$ is called the period of a Darboux chain. The operator $L + q^2$ appears to be the harmonic oscillator in the particular case $r = 1$.

Periodic Darboux chains lead to integrable systems of differential equations for functions $f_j$, which are examined in [1]. The cases $\alpha = 0$ and $\alpha \neq 0$, where $\alpha = \sum_{j=1}^r \alpha_j$, are cardinally different. The operators of a periodic Darboux chain are finite-gap if $\alpha = 0$. If $\alpha \neq 0$, then the equations for $f_j$ lead to the Painlevé equations or their higher analogues; relations (1) define the discrete spectrum of the chain operators $L_j$: the spectrum of each of these operators consists of $r$ arithmetic sequences (see [1]).

Consider the following “$q$-analogue” of a Darboux chain:

$$L_j = A_j A_{j-1}^+ - \alpha_j = q A_{j-1}^+ A_j - \alpha_j,$$

where $A_j = a_j + b_j T$ are difference operators on one-dimensional lattice $\mathbb{Z}$ and $T$ is the shift operator: $(A_j f)(n) = a_j(n) f(n) + b_j(n) f(n+1)$, $a_j(n), b_j(n) \in \mathbb{R} \setminus \{0\}$. Without loss of generality we may assume that $b_j(n) > 0$ for all $j, n$. One of the important features of difference operators in comparison to differential ones is the possibility to define a periodic chain in various ways. We say that a chain (2) is periodic with period $r$ and shift $s$ if the relation

$$L_{j+r} = T^{-s} L_j T^s$$

holds for all $j \geq 1$. In the literature the case $s = 0$ has mostly been considered. However the possibility of factorization in “the reverse order” was mentioned in [2, 4]; operators $A_j = a_j + b_j T$ can be replaced by operators of the form $A_j = a_j T^s + b_j$; such factorization with $s = 0$ is, in fact, equivalent to the factorization “in the right order” for $s = r$ (the operator $L_j$ is replaced by $T^{-s} L_j T^s$). Apparently, the general formulation with an arbitrary $s$ has not been considered before, while the following special cases have been studied in the literature.

1. $\alpha = 0$, $q = 1$, $r$ is arbitrary [2]. Operators $L_j$ are finite-gap.
2. $\alpha > 0$, $q = 1$, $r = 1$ (difference analogue of the harmonic oscillator) [2]. In this case, symmetric operators $L_j$ acting on the space of functions on the lattice $\mathbb{Z}$ do not exist. Nevertheless, there exists a solution on the “half-line” $\mathbb{Z}_{>0}$. The spectrum of the operator $L + q^2$ is exactly the same as that of the harmonic oscillator: $\lambda_k = \alpha(k + \frac{1}{2})$; the eigenfunctions are expressed in terms of the Charlier polynomials, and, therefore, form a complete family in $L_2(\mathbb{Z}_{>0})$.
3. $r = 1$, $\alpha > 0$, $0 < q < 1$ (or $\alpha < 0$, $1 < q$) [2, 5] ($q$-oscillator). The spectrum of the operator $L$ lies in the interval $[0, \frac{\alpha}{q - 1}]$ (or in $[0, \frac{\alpha}{q - 1}]$ if $q > 1$); it forms a “$q$-arithmetic sequence”. It is mentioned in [2] that, in this case, the operator $L$ is unbounded, and conjectured that $L$ has continuous spectrum in the interval $[\frac{\alpha q}{q - 1}, \infty)$.
4. Another version of the $q$-oscillator is considered in [6]. It is presented by a difference operator on the whole “line” $\mathbb{Z}$. In our settings, this version of the $q$-oscillator can be interpreted as the case $s = 1$, $r = 2$, $\alpha_1 = \alpha_2$, ($\alpha$ and $q$ are the same as in 3). A particular solution that is symmetric about the origin

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was found in [6]. This case is specific because here the operator \( L \) is bounded and has no continuous spectrum.

We claim here that the same property holds for a \( q \)-chain of an arbitrary even period \( r \) with the shift \( s = r/2 \): the chain operators are bounded and have no continuous spectrum. We also provide explicitly the general solution of the problem in the case \( s = 1, r = 2 \).

**Theorem.** Suppose \( r \) is even, \( \alpha_1, \ldots, \alpha_r \) are positive, \( q \) satisfies the inequality \( 0 < q < 1 \), and we have \( s = r/2 \). Then the system (2),(3) has an \( r \)-parametric family of solutions. The operator \( L_j \) is bounded for each \( j \): its spectrum \( \{ \lambda_j, 0, \lambda_{j+1}, \ldots \} \) is discrete and is contained in the interval \([0, \|L_j\|]\). It can be found by using the Darboux scheme:

\[
\lambda_{j,0} = 0, \quad \lambda_{j+1,0} = q(\lambda_j + \alpha_j), \quad \lambda_{j+r,0} = \lambda_{j,k}.
\]

For each \( j \), the eigenfunctions of the operator \( L_j \) also can be obtained by using the Darboux scheme:

\[
A_{j-1}\psi_{j,0} = 0, \quad \psi_{j+1,0} = A_j^+\psi_{j,k};
\]

these eigenfunctions form a complete family in \( L_2(\mathbb{Z}) \).

A similar assertion holds for \( \alpha_1, \ldots, \alpha_r < 0, 1 < q \) (in this case, the point 0 is not included in the spectrum of \( L_j \)).

Now we present an explicit form of the operators \( A_1, A_2 \) for \( r = 2 \).

**Proposition.** For \( r = 2 = 2n, \alpha_1, \alpha_2 > 0, 0 < q < 1 \), the general solution of the problem (2),(3) has the form

\[
a_1(n) = \epsilon \sqrt{q_{2n}}, \quad a_2(n) = \epsilon \sqrt{q_{2n+1}}, \quad b_1(n) = \sqrt{q_{2n+1}}, \quad b_2(n) = \sqrt{q_{2n}},
\]

where \( \epsilon = \pm 1 \),

\[
\xi_n = \frac{1}{2} \left( c_{n+1} - 2k q^{-\varphi} + c_{n+1} q^{-2n-\varphi-1} \right), \quad \eta_n = \frac{1}{2} \left( c_{n+1} - 2k q^{n+\varphi} + c_{n+1} q^{2n+2\varphi+1} \right),
\]

\[
\varphi = \beta - \frac{1}{2} \left( \beta \right) \quad \text{and} \quad \beta \quad \text{stands for the integral part of} \quad \varphi,
\]

\[
c_{\beta}\varphi q^{-\theta} + c_{\beta-1} q^{\theta} < 2k < \min(c_{\beta}\varphi q^{\theta+1} + c_{\beta-1} q^{-\theta-1}, c_{\beta}\varphi q^{\theta-1} + c_{\beta-1} q^{-\theta+1}) \quad \text{if} \quad \varphi \in \mathbb{Z},
\]

and

\[
2k = c_{\varphi} q^{\frac{1}{2}} + c_{\varphi - 1} q^{-\frac{1}{2}} \quad \text{if} \quad \varphi \in \mathbb{Z}.
\]

In the case \( \varphi = 0, \epsilon = -1, q = \exp(-\frac{\pi}{4} h^2) \),

\[
x = nh, \quad T = \exp(h \frac{4}{\pi} k) \quad \text{and assume that} \quad n \quad \text{is real in formulae (4) and that the operator} \quad L_j \quad \text{is a difference operator on} \quad \mathbb{R}.
\]

Then for any \( f \in C^2(\mathbb{R}) \) we have

\[
\left( L_{1,2} + \frac{\alpha_1}{2} \right) f(x) = \left( -\frac{d^2}{dx^2} + \frac{\alpha_1^2}{4} x^2 \right) f(x) + o(h).
\]

If \( \alpha_1 \neq \alpha_2 \), then the operator \( L_j \) converges in the same sense to

\[
-\frac{d^2}{dx^2} + \frac{(\alpha_j + \alpha_{j+1})^2}{16} x^2 - \frac{\alpha_j}{2} x^2 - \frac{(\alpha_j - \alpha_{j+1})^2 (\alpha_j + 3 \alpha_{j+1})}{4(\alpha_j + \alpha_{j+1})^2 x^2},
\]

where \( \alpha_j + 2 = \alpha_j \).

In the cases 2,3 mentioned above there is no link of that kind between the discrete and the continuous models. Thus the considered case \( s = r/2 \) gives, in a certain sense, a proper discretization of a Darboux chain.

We hope to prove that a \( q \)-chain converges in the same way to an ordinary Darboux chain for an arbitrary even \( r \). The numerical experiment confirms that a \( q \)-chain of the period 6 converges to an ordinary Darboux chain of the period 3 if \( \alpha_1 = \alpha_{j+1} \).
References


