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Exactly solvable periodic Darboux q-chains *

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Let $L_1, L_2, \dots$ be selfadjoint differential operators acting on $\mathbb{R}$. They form *a Darboux chain* if they satisfy the relation

$$L_j = A_j A_j^+ - \alpha_j = A_{j-1}^+ A_{j-1}, \tag{1}$$

where $A_j = -d/dx + f_j(x)$ are first order differential operators. A Darboux chain is called *periodic* if $L_{j+r} = L_j$ for some r and for all $j = 1, 2, \dots$. Number r is called *the period* of a Darboux chain. The operator $L + \frac{\alpha}{2}$ appears to be the harmonic oscillator in the particular case $r = 1$.

Periodic Darboux chains lead to integrable systems of differential equations for functions f_j, which are examined in [1]. The cases $\alpha = 0$ and $\alpha \neq 0$, where $\alpha = \sum_{j=1}^r \alpha_j$, are cardinally different. The operators of a periodic Darboux chain are finite-gap if $\alpha = 0$. If $\alpha \neq 0$, then the equations for f_j lead to the Painlevé equations or their higher analogues; relations (1) define the discrete spectrum of the chain operators L_j: the spectrum of each of these operators consists of r arithmetic sequences (see [1]).

Consider the following "q-analogue" of a Darboux chain:

$$L_j = A_j A_j^+ - \alpha_j = q A_{j-1}^+ A_{j-1}, \tag{2}$$

where $A_j = a_j + b_j T$ are difference operators on one-dimensional lattice $\mathbb{Z}$ and T is the shift operator: $(A_j f)(n) = a_j(n) f(n) + b_j(n) f(n+1)$, $a_j(n), b_j(n) \in \mathbb{R}\backslash\{0\}$. Without loss of generality we may assume that $b_j(n) > 0$ for all j, n. One of the important features of difference operators in comparison to differential ones is the possibility to define a periodic chain in various ways. We say that a chain (2) is *periodic with period r and shift s* if the relation

$$L_{j+r} = T^{-s} L_j T^s \tag{3}$$

holds for all $j \geqslant 1$. In the literature the case $s = 0$ has mostly been considered. However the possibility of factorization in "the reverse order" was mentioned in [2, 4]: operators $A_j = a_j + b_j T$ can be replaced by operators of the form $A_j = a_j T^+ + b_j$; such factorization with $s = 0$ is, in fact, equivalent to the factorization "in the right order" for $s = r$ (the operator L_j is replaced by $T^{-j} L_j T^j$). Apparently, the general formulation with an arbitrary s has not been considered before, while the following special cases have been studied in the literature.

1. $\alpha = 0$, $q = 1$, r is arbitrary [2]. Operators L_j are finite-gap.

2. $\alpha > 0$, $q = 1$, $r = 1$ (difference analogue of the harmonic oscillator) [2]. In this case, symmetric operators L_j acting on the space of functions on the lattice $\mathbb{Z}$ do not exist. Nevertheless, there exists a solution on the "half-line" $\mathbb{Z}_{>0}$. The spectrum of the operator $L + \frac{\alpha}{2}$ is exactly the same as that of the harmonic oscillator: $\lambda_k = \alpha(k + \frac{1}{2})$; the eigenfunctions are expressed in terms of the Charlier polynomials, and, therefore, form a complete family in $\mathcal{L}_2(\mathbb{Z}_{>0})$.

3. $r = 1$, $\alpha > 0$, $0 < q < 1$ (or $\alpha < 0$, $1 < q$) [2, 5] (q-oscillator). The spectrum of the operator L lies in the interval $[0, \frac{q\alpha}{1-q})$ (or in $(0, \frac{q\alpha}{q-1}$ if $q > 1$); it forms a "q-arithmetic sequence". It is mentioned in [2] that, in this case, the operator L is unbounded, and cojectured that L has continuous spectrum in the interval $(\frac{q\alpha}{1-q}, \infty)$.

4. Another version of the q-oscillator is considered in [6]. It is presented by a difference operator on the whole "line" $\mathbb{Z}$. In our settings, this version of the q-oscillator can be interpreted as the case $s = 1$, $r = 2$, $\alpha_1 = \alpha_2$, (α and q are the same as in 3). A particular solution that is symmetric about the origin

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was found in [6]. This case is specific because here the operator L is bounded and has no continuous spectrum.

We claim here that the same property holds for a q-chain of an arbitrary even period r with the shift $s = r/2$: the chain operators are bounded and have no continuous spectrum. We also provide explicitly the general solution of the problem in the case $s = 1$, $r = 2$.

Theorem. *Suppose r is even, $\alpha_1, \dots, \alpha_r$ are positive, q satisfies the inequality $0 < q < 1$, and we have $s = r/2$. Then the system (2),(3) has an r-parametric family of solutions. The operator L_j is bounded for each j; its spectrum $\{\lambda_{j,0}, \lambda_{j,1}, \dots\}$ is discrete and is contained in the interval $[0, \|L_j\|)$. It can be found by using the Darboux scheme:*

$$\lambda_{j,0} = 0, \quad \lambda_{j+1,k+1} = q(\lambda_{j,k} + \alpha_j), \quad \lambda_{j+r,k} = \lambda_{j,k}.$$

For each j, the eigenfunctions of the operator L_j also can be obtained by using the Darboux scheme:

$$A_{j-1}\psi_{j,0} = 0, \quad \psi_{j+1,k+1} = A_j^+ \psi_{j,k};$$

these eigenfunctions form a complete family in $\mathcal{L}_2(\mathbb{Z})$.

A similar assertion holds for $\alpha_1, \dots, \alpha_r < 0$, $1 < q$ (in this case, the point 0 is not included in the spectrum of L_j).

Now we present an explicit form of the operators A_1, A_2 for $r = 2$.

Proposition. *For $r = 2 = 2s$, $\alpha_1, \alpha_2 > 0$, $0 < q < 1$, the general solution of the problem (2),(3) has the form $a_1(n) = \epsilon\sqrt{\xi_{2n}}$, $b_1(n) = \sqrt{\eta_{2n+1}}$, $a_2(n) = \epsilon\sqrt{\xi_{2n-1}}$, $b_2(n) = \sqrt{\eta_{2n}}$, where $\epsilon = \pm 1$,*

$$\xi_n = \frac{1}{2}\frac{c_n - 2\kappa\, q^{-n-\varphi-\frac{1}{2}} + c_{n+1}q^{-2n-2\varphi-1}}{(1-q^{-2(n+\varphi)})(1-q^{-2(n+\varphi+1)})}, \quad \eta_n = \frac{1}{2}\frac{c_{n+1} - 2\kappa\, q^{n+\varphi+\frac{1}{2}} + c_n q^{2n+2\varphi+1}}{(1-q^{2(n+\varphi)})(1-q^{2(n+\varphi+1)})}, \tag{4}$$

$$c_n = \frac{\alpha_1 + \alpha_2}{1-q} + (-1)^n \frac{\alpha_1 - \alpha_2}{1+q},$$

$\varphi \in \mathbb{R}$ is arbitrary, the parameter κ satisfies the restrictions

$$c_{[\varphi]}q^{-\theta} + c_{[\varphi]-1}q^{\theta} < 2\kappa < \min(c_{[\varphi]}q^{\theta+1} + c_{[\varphi]-1}q^{-\theta-1}, c_{[\varphi]}q^{\theta-1} + c_{[\varphi]-1}q^{-\theta+1}) \quad \text{if } \varphi \notin \mathbb{Z},$$

and $2\kappa = c_\varphi q^{\frac{1}{2}} + c_{\varphi-1}q^{-\frac{1}{2}}$ if $\varphi \in \mathbb{Z}$ (in this case, in fractions (4), one has to cancel the factor $(1-q^{2(n+\varphi)})$ for $n \equiv \varphi (\mathrm{mod}\, 2)$ and the factor $(1-q^{2(n+\varphi+1)})$ for $n \equiv \varphi + 1 (\mathrm{mod}\, 2)$). Here, we set $\theta = \varphi - [\varphi] - \frac{1}{2}$ and $[\varphi]$ stands for the integral part of φ.

Observation. In the case $r = 2$, $\alpha_1 = \alpha_2$, operators $L_j + \frac{\alpha_j}{2}$ constructed from the above solutions converge to the harmonic oscillator as $q \to 1$ in the following sense: take $\varphi = 0$, $\epsilon = -1$, $q = \exp(-\frac{\alpha_1}{4}h^2)$, $x = nh$, $T = \exp(h\frac{d}{dx})$ and assume that n is real in formulae (4) and that the operator L_j is a difference operator on $\mathbb{R}$. Then for any $f \in C^2(\mathbb{R})$ we have

$$\Big(L_{1,2} + \frac{\alpha_1}{2}\Big)f(x) = \Big(-\frac{d^2}{dx^2} + \frac{\alpha_1^2}{4}x^2\Big)f(x) + o(h).$$

If $\alpha_1 \neq \alpha_2$, then the operator L_j converges in the same sense to

$$-\frac{d^2}{dx^2} + \frac{(\alpha_j + \alpha_{j+1})^2}{16}x^2 - \frac{\alpha_j}{2} - \frac{(\alpha_j - \alpha_{j+1})(\alpha_j + 3\alpha_{j+1})}{4(\alpha_j + \alpha_{j+1})^2 x^2},$$

where $\alpha_{j+2} = \alpha_j$.

In the cases 2,3 mentioned above there is no link of that kind between the discrete and the continuous models. Thus the considered case $s = r/2$ gives, in a certain sense, a proper discretization of a Darboux chain.

We hope to prove that a q-chain converges in the same way to an ordinary Darboux chain for an arbitrary even r. The numerical experiment confirms that a q-chain of the period 6 converges to an ordinary Darboux chain of the period 3 if $\alpha_j = \alpha_{j+3}$.

References

[1] A. P. Veselov, A. B. Shabat. Dressing chains and the spectral theory of Schrödinger operators. *Functional Anal. Appl.*, **27** (1993), no. 2, 1–21.

[2] S. P. Novikov, I. A. Taimanov. Difference analogs of the harmonic oscillator. Appendix II in [3].

[3] S. P. Novikov, A. P. Veselov. Exactly solvable two-dimensional Schrödinger operators and Laplace transformations. *Solitons, Geometry, and Topology: on the Crossroad*, ed. V. M. Buchstaber, S. P. Novikov. AMS Trans. Ser. 2, **179** (1997), 109–132.

[4] S. P. Novikov, I. A. Dynnikov. Discrete spectral symmetries of low-dimensional differential operators and difference operators on regular lattices and two-dimensional manifolds. *Russian Math. Surveys*, **52** (1997), no. 5(317), 175–234.

[5] V. Spiridonov, L. Vinet, A. Zhedanov. Difference Schrödinger operators with linear and exponential discrete spectra. *Lett. Math. Phys*, **129** (1993), 67–73.

[6] N. Atakishiyev, A. Frank, K. Wolf. A simple difference realization of the Heisenberg q-algebra. *J. Math. Phys*, **35**(7), 1994, 3253–3260.